Elementary Aspects of the Fundamental Theorem of Algebra

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Abstract

We give a sequence of easy inferences from typical topics in high school algebra that relate to the fundamental theorem of algebra (FTA). The sequence builds to an easy proof of FTA. In passing we mention two proofs typically given in complex analysis courses. These proofs, although short, require developing differential and integral calculus for complex variables. The proof given here is leisurely and easy – enough for good high school and typical calculus students.

Introduction

How close can a typical high school algebra student come to understanding the fundamental theorem of algebra? Currently some of the ingredients for a good understanding are present after a typical algebra 1, algebra 2, and pre-calc (or trigonometry) sequence, but the dots aren't connected. Thus students are familiar with quadratics and cubics and general polynomials, as well as Euler's and DeMoivre's formula and theorem; they also are told the fundamental theorem of algebra; but, in no course are they encouraged to see how polynomials might always have roots in the complex numbers, the fundamental theorem of algebra. There are inferences that can be made that suggest that this is a plausible conclusion. Indeed, it is possible to give a proof of this result with just a few unproven assumptions. That's the main trajectory of this article.

We use proofs and results from *Rudin's Principles of Mathematical Analysis* [2] and Spiegel's *Complex Variables* [3]. For references to high school mathematics, we reference Blitzer's *Algebra and Trigonometry* [1].

Review

We know using the quadratic formula [1] that all quadratics can be solved in the complex numbers. So $z^2 + 1 = 0$ is solved by $\pm i$. We also know that some quadratics have graphs that don't show any x-intercepts. z^2 touches the origin and has a double root at $(0, 0)$. It is concave up, meaning it opens up or holds water. When transformed vertically up by 1, its vertex moves away from the origin; it has no real roots. The fundamental theorem of algebra states that all polynomials of degree n with rational coefficients have roots – all n of them in the complex plane. But given that a quadratic formula like formula for the general degree n polynomial's roots is not given in a high school text book, can we explore the situation enough to suspect the truth of the theorem?

Here's a start: suppose we didn't know the quadratic formula. Can we show that all quadratics can still be solved in the complex numbers? If we had an exploratory avenue for this easy case, maybe it would suggest a general approach to higher degree polynomials. There is hope via a simple observation from a chapter on trigonometry, Blitzer's Chapter 7, Section 5: you can solve any nth degree polynomial of the form

$$z^n + a = 0. \tag{1}$$

The n solutions are the nth roots of

$$z = \sqrt[n]{a} = \sqrt[n]{|a|}(\cos\frac{\theta + 2\pi k}{n} + \sin\frac{\theta + 2\pi k}{n}),$$

where $0 \leq k \leq n-1$ and $a = \cos\theta + i\sin\theta$.

Of course polynomials like (1) are not the rule, but we can discern why polynomials might always be solved by complex numbers. The complex numbers, when evaluated by a function, tend to move points around a lot. So with the reals $x^2 + 2$ can't be solved; the function doesn't move a given point around a lot. A related mind experiment is to imagine the $x + iy$ of one complex plane mapping to $u + iv$ of another. Imagine two computer windows and as you drag your mouse around on the far left plane the point your mouse pointer is on is given a corresponding cross-hair on the right window. You would hope you could adjust the mouse pointer so as to find the origin on the right window. The connection between the two windows is the function in question. For $x^2 + 1 = 0$, i and $-i$ map to the right window's origin. For $z^n + a = 0$ tracing a circle on the left generates periodic bulls-eyes at the origin on the right.

Notice that a quadratic like $ax^2+bx+c = 0$ has the standard form $a(x-h)^2 + k$. This is really another $z^2 + k$, only its transformed a little – moved to the right or left and the shape of the parabola legs are squeezed together or spread apart; see Blitzer's Chapter 2, Section 5 on transformations. Relative to a complex number moving this to the origin, the y part of the vertex, it's the same situation. We note that higher powers of the form $(z-a)^3+b=0$ will just be up and over transformation of z^3. Using the binomial theorem, we know the $(z-a)^3$ part will generate all terms in a cubic:

$$(z-a)^3 = z^3 - 3az^2 + 3a^2z - a^3.$$

Are all polynomials just transformations of the base type z^n? and so just as $z^n = 0$ has $z = 0$ a root of multiplicity of n so do all polynomials. Rouche's theorem (see Spiegel, page 128, problem 19 [3]) will get something close to this result.

What is an example of a function that doesn't seem to move a complex point all over the u-v plane. It turns out, in a lot of ways, only the constant function behaves in this way. From a different angle, suppose a polynomial doesn't have a root. This implies that there is a point that never is reached in the range of the polynomial. This means that $1/p(z)$ is defined for all z – there is no division by zero. Using Liouville's theorem (Spiegel, page 125, problem 10), this forces our polynomial to be a constant, something we know that it is not. The theorems of Rouche and Liouville are covered in courses in complex variables and require evolving complex differentiation and integration. We seek an easier approach that is almost within the range of high school algebra – no calculus.

Problems

Here are a few problems which will help you become familiar with the ideas of a proof of the fundamental theorem of algebra, FTA. Do the following for linear, quadratic, and cubic polynomials $p(z)$. Assume coefficients can be complex numbers.

1. Show that $|p(z)|$ values go to infinity.
2. Show that $|p(z)|$ values can be made less than the absolute value of $p(z)$'s constant term.

Lemma 1.

$$|A + B| \geq |A| - |B| \tag{2}$$

Proof. By the triangle inequality,

$$|A + B + (-B)| \leq |A + B| + |B|$$

and this gives (2). ☐

Lemma 2.

$$|A + B + C| \geq |A| - |B| - |C| \tag{3}$$

Proof. By the triangle inequality,

$$|A + B + C| \geq |A| - |B + C|$$

$$\geq |A| - (|B| - |C|) = |A| - |B| + |C|$$

$$\geq |A| - |B| - |C|$$

and this gives (3). ☐

Clearly, an induction proof will yield the general result. One can, of course, simply say that if you start with A and subtract rather than add potentially positive numbers you will decrease its value. I.e. it's kind of obvious.

Theorem 1. *$|p(z)|$ can be made as large as we please.*

Proof. Quadratic case: Let $|z| = R$ and suppose

$$p(z) = a_2 z^2 + a_1 z + a_0.$$

Then

$$|p(R)| = |a_2 R^2 + a_1 R + a_0| \geq \tag{4}$$

$$|a_2|R^2 - |a_1|R - |a_0| = R^2[|a_2| - |a_1|R^{-1} - |a_0|R^{-2}].$$

We've used our lemma. The factor in brackets shrinks to $|a_2|$ with growing R and this implies $|p(z)|$ can be made as large as we please. ☐

Of course one could use the end-behavior of real polynomials to make the same conclusion: the left and right tails of all absolute values of real polynomials will go to infinity; see Blitzer, Chapter 3, Section 2. Positive headed to the x-axis bounce off of it and head north, for example. But in this theorem we allow for complex coefficients, so this image can't necessarily be relied upon. The complication of allowing complex coefficients forces the constant reliance on conversion to statements with absolute values. We see this in the next theorem.

We need that the function $re^{i\theta}$ has as its range all of $\mathbb{C}$. This is not hard to imagine. Using

$$re^{i\theta} = r(\cos\theta + i \sin\theta),$$

we see that any point in the complex plane can be expressed in polar coordinates – that's it. To anticipate complex variable ideas, notice that $i\theta$ is like a linear function going up and down the imaginary i-axis; this generates a circle. Thus a line in the x-y plane goes to a circle in the u-v. This type of transformation broadens modeling opportunities. Back to our main thread.

Lemma 3. *For every real non-zero number a there exists a real θ such that $ae^{i\theta} = -|a|$. This is also true for complex a.*

Proof. If $a < 0$ then let $\theta = 0$ and $ae^{0i} = -|a|$. If $a > 0$ then let $\theta = \pi$ and $ae^{\pi i} = -a = -|a|$.

For complex a, we just note $ae^{i\theta} = -|a|$ implies

$$ei\theta = \frac{-|a|}{a}.$$

Not that it is necessary to note, we note $|-|a|/a| = 1$, so r in the polar coordinate representation of this number is 1. □

Theorem 2. *$|p(z)|$ can be made less than the absolute value of its constant.*

Proof. Cubic case: Let $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$. We always will have a non-zero constant and here we assume a_1 is the first non-zero coefficient. It could be a_2 or a_3. The argument won't change. Then

$$p(re^{i\theta}) = a_0 + a_1 re^{i\theta} + a_2 r^2 e^{2i\theta} + a_3 r^3 e^{3i\theta}.$$

Using Lemma 3, there exists θ such that $a_1 e^{i\theta} = -|a_1|$. So now we have

$$|p(re^{i\theta})| = |a_0 - |a_1|r + a_2 r^2 e^{2i\theta} + a_3 r^3 e^{3i\theta}|$$

and taking the absolute value of the first constant term and the terms after $-|a_1|r$ increases the value of the right hand side. So

$$|p(re^{i\theta})| \le |a_0| - |a_1|r + |a_2|r^2 + |a_3|r^3. \tag{5}$$

We've used $r > 0$ and $|e^{ik\theta}| = 1$. Rearranging (5),

$$|p(re^{i\theta})| \le |a_0| - r\{|a_1| - |a_2|r^1 - |a_3|r^2\}.$$

Now for small enough r the value in the braces is positive, so the right hand side drops below $|a_0|$, as needed. □

We are now in a position to prove, with a couple of assumptions from advanced mathematics, the FTA. The assumptions are not difficult to intuitively understand. First, polynomials are continuous on any disc in the complex plane and will reach their maximum and minimum values with a large enough disc (think circle of sufficient radius with its interior). Absolute values are bounded by a minimum value of 0. This is all to say polynomials are not rational functions. Rational functions, like $r(z) = 1/x$, do not reach their minimum value of 0. This function asymptotically approaches it (see Blitzer Chapter 3, Section 5); there is no z_0 such that $r(z_0) = 0$. In contrast, $|p(z)|$ will have a value z_0 such that $|p(z_0)|$ is its minimum value. Certainly one sees this with the real quadratics with concave up graphs. Consider $a(x-h)^2 + k$ with $k > 0$; see Blitzer on this the standard form of a quadratic, Chapter 3, Section 1. If $a > 0$, its minimum is k. Apart from the horizontal shift, this function is $x^2 + k$. We have shown that using complex numbers its value can fall below k, its constant. It reaches zero when evaluated at $\pm i\sqrt{k}$. For a proof by contradiction we assume there is a polynomial such that the minimum of its absolute value is not 0.

The following proof of the FTA is based on that given in Rudin [2].

Theorem 3. *If $p(z)$ is a polynomial, then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. We assume $|p(z)|$ is a continuous function and that its minimum is $z = z_0$. To derive a contradiction, we will assume $|p(z_0)| = \mu \neq 0$. Consider the polynomial defined by

$$Q(z) = \frac{p(z + z_0)}{p(z_0)}.$$

Then the constant of $Q(z)$ is, as it is with all polynomials, given by $Q(0)$; $Q(0) = 1$. As $p(z_0)$ is the minimum value, all other z values make $|Q(z)| > 1$. But this contradicts Theorem 2. We can't get below this polynomial's constant. □

Conclusion

The fastest avenue to believing and proving the FTA is to notice that $p(z) = z^n + 1 = 0$ can be solved in $\mathbb{C}$ and this means that $\mathbb{C}$ values drop this function's absolute value below its constant. Show that for all absolute values of polynomials there are values in $\mathbb{C}$ that are less than the absolute value of the polynomial's constant. Note: if a polynomial has no constant, then its terms have a common z factor; $z = 0$ is a root; done. Next, show that with the assumption that the minimum value of $|p(z)| > 0$, there is a polynomial that never goes below its constant, a contradiction.

References

[1] R. Blitzer, *Algebra and Trigonometry*, 4th ed., Pearson, Upper Saddle, NJ, 2010.

[2] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, 1976.

[3] M. R. Spiegel and J. Liu, *Complex Variables*, 2nd ed., McGraw-Hill, New York, 1999.