Trigonometric and hyperbolic functions for general solutions of Duffing equation

Kolawolé Kêgnidé Damien Adjaï[1], Jean Akande[1], Pravanjan Mallick[2],

Biswanath Rath[2+] and Marc Delphin Monsia[1*]

1- Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01. BP.526, Cotonou, BENIN (*Corresponding author: E-mail: monsiadelphin@yahoo.fr)
2- Department of Physics, North Orissa University, Takatpur, Baripada-757003, Odisha, INDIA, (+E-mail: biswanathrath10@gmail.com)

Abstract

It is well established that the cubic Duffing equation exhibits each of the Jacobi elliptic functions as solution. However, in this paper it is shown for the first time that the general solutions of such an equation may be computed as a trigonometric function and also as a hyperbolic function in a direct and straightforward manner by first integral and Lagrangian analysis following the sign of parameters.

Keywords: Cubic Duffing equation, exact trigonometric function solution, exact hyperbolic function solution, Lagrangian and first integral analysis.

Introduction

The cubic Duffing differential equation

$$\ddot{x} + \alpha x + \beta x^3 = 0 \tag{1}$$

where overdot stands for a differentiation with respect to t, is widely used to account for many phenomena in physics and applied mathematics. For instance equation (1) is applied to model the frequency dependent amplitude response of nonlinear dynamical systems. The forced response of (1) is used to describe sub and superharmonic resonances, and jumps in nonlinear oscillators [1-3]. Equation (1) arises also in traveling wave solution method for nonlinear partial differential equations. The exact solution of (1) is well known and established so that each of the Jacobi elliptic functions satisfies (1) [1, 4]. Such a knowledge on the cubic Duffing equation is very precise so that it would be almost unnatural to suspect the existence of other types of general periodic solutions. However, the objective of this work is to show the existence of trigonometric function as general solution computed directly and straightforward fashion to the cubic Duffing equation for the first time. Also hyperbolic function may be obtained as a general solution following the sign of parameters. To this end the Lagrangian analysis introduced recently by our group is reviewed (section 2) and used to compute easily the expected exact and explicit general solutions (section 3). Finally a conclusion of this work is carried out.

2-**Review of the Lagrangian analysis**

Let [5]

$$a(x,\dot{x}) = \dot{x}g(x) + x^{\ell} f(x) \tag{2}$$

be a time independent first integral. The Lagrangian may then be computed as [5]

$$L(x,\dot{x}) = \dot{x}g(x)\ln(\dot{x}) - x^{\ell} f(x) + K\dot{x} \tag{3}$$

where ℓ and K are arbitrary parameters, and $f(x)$ and $g(x)$ are arbitrary functions of x. From (3) the Euler-Lagrange equation becomes

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + x^{\ell}\frac{f'(x)}{g(x)}\dot{x} + a\ell x^{\ell-1}\frac{f(x)}{g^2(x)} - \ell x^{2\ell-1}\frac{f^2(x)}{g^2(x)} = 0 \tag{4}$$

where prime means differentiation with respect to x.

Now the purpose is to show that for appropriate choice of functions $f(x)$, $g(x)$ and parameter ℓ, equation (4) may be reduced to the cubic Duffing equation (1), which secures its integrability.

3-**Exact and explicit general solution of (1)**

In this part the criteria of integrability of (1) as a trigonometric function is established (subsection 3.1) such that the exact and explicit general solution may be computed in a straightforward manner and discussed (subsection 3.2).

3.1 **Conditions of integrability**

Substituting

$$\dot{x} = \frac{a - x^{\ell} f(x)}{g(x)} \tag{5}$$

where $g(x) \neq 0$, into (4), yields the general equation

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+\frac{1}{g^2(x)}\left[a\,x^{\ell} f'(x)-x^{2\ell} f(x) f'(x)+a\ell x^{\ell-1} f(x)-\ell x^{2\ell-1} f^2(x)\right]=0 \tag{6}$$

Applying the functional choice $g(x)=1$, reduces (6) to

$$\ddot{x}+a\,x^{\ell} f'(x)-x^{2\ell} f(x) f'(x)+a\ell x^{\ell-1} f(x)-\ell x^{2\ell-1} f^2(x)=0 \tag{7}$$

such that for $f(x)=a_1x^2$, (7) becomes

$$\ddot{x}+aa_1(2+\ell)\,x^{\ell+1}-a_1^2(2+\ell)\,x^{2\ell+3}=0 \tag{8}$$

Making $\ell=0$, leads to obtain as cubic Duffing equation

$$\ddot{x}+2a\,a_1x-2a_1{}^2x^3=0 \tag{9}$$

In this context a first integral of equation (9) may, using (2), read

$$a=\dot{x}+a_1x^2 \tag{10}$$

So that the integrability condition of equation (1) becomes

$$\alpha=2a\,a_1 \tag{11}$$

and

$$\beta=-2a_1{}^2 \tag{12}$$

Now one may use (10) to find the exact and explicit general solution of (1).

3.2 **General solution of (1)**

3.2.1 **Case** $a\,a_1 \prec 0$

From (10) one may write

$$\frac{dx}{a-a_1x^2}=dt \tag{13}$$

Letting $a_1=c^2$, and $a=-b^2$ into (13), yields

$$\frac{dx}{1+\dfrac{c^2}{b^2}x^2}=-b^2dt \tag{14}$$

The integration of (14) is immediate and gives [6]

$$\arctan\left(\frac{c}{b}x\right) = -bc(t + a_2) \tag{15}$$

From which one may obtain, as a result, the general solution of the cubic Duffing differential equation in the form

$$x(t) = \frac{b}{c}\tan[-bc(t + a_2)] \tag{16}$$

where a_2 is a constant of integration. Using the previous definition of $b \neq 0,$ and $c \neq 0,$ the solution (16) takes the definitive form

$$x(t) = -\sqrt{-\frac{a}{a_1}}\tan\left[\sqrt{-aa_1}(t + a_2)\right] \tag{17}$$

where $a_1 \neq 0.$ It is worth to note that the solution (17) may be obtained by a direct integration of (9). Hence, multiliplying (9) by $\dot{x},$ and integrating once, yields

$$(\dot{x})^2 = a_1{}^2 x^4 - 2a\,a_1 x^2 + 2K_1 \tag{18}$$

where K_1 is an integration constant. Choosing $2K_1 = a^2,$ one may arrive at the equation

$$(\dot{x})^2 = (a - a_1 x^2)^2 \tag{19}$$

from which the solution (17) may be obtained by a simple integration. The equation (13) or (19) allows also one to get non periodic solution for the cubic Duffing equation (1).

3.2.2 **Case** $a\,a_1 \succ 0$

In this case, the general solution (17) may be written as

$$x(t) = \frac{1}{i}\sqrt{\frac{a}{a_1}}\tan\left[i\sqrt{a\,a_1}(t + a_2)\right] \tag{20}$$

which may definitively read as [6]

$$x(t) = \sqrt{\frac{a}{a_1}}\tanh\left[\sqrt{aa_1}(t + a_2)\right] \tag{21}$$

Such a result may also be directly obtained from equation (19). In view the above, a conclusion can be drawn for the work.

Conclusion

It is well known from the literature that the cubic Duffing differential equation has as general solution each of the Jacobi elliptic functions. This work shows however, for the first time, that the cubic Duffing may exhibit trigonometric and hyperbolic functions as general solutions. This has been obtained within the framework of first integral and Lagrangian analysis.

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