The Proof of Goldbach’s Conjecture

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Abstract

Since the set of $A_{S(+)}$ and $A_{S(\times)}$ is a bijective function, we use the improved theorem of asymptotic density to prove that there exist product of two odd primes in any $A_{S(\times)}$. At the same time, in any $A_{S(+)}$, the sum of two odd primes can be obtained.

Keywords: Goldbach’s conjecture, bijective function, asymptotic density, Inequality transform

Mathematics Subject Classification (2010): Primary 11A41; Secondary 11A05,11A25

1. Introduction

The Goldbach’s conjecture, every even integer greater than 4 can be expressed as the sum of two primes, until now it has not been proven. In this paper we will prove that it is true.

1.1 Concepts and Propositions

Let $M_1$ be the midpoint of any sequence of odd numbers and $M_1$ is an odd number greater than or equal to 3, where $M_1$ assumptions are known. We have the following sequence of $S_1$:

\[1, 3, \cdots, M_1 - 2, M_1, M_1 + 2, \cdots, 2M_1 - 3, 2M_1 - 1;\]  

(1)

Let $M_2$ be the midpoint of any sequence of odd numbers and $M_2$ is an even number greater than or equal to 4 and does not exist in the the sequence, where $M_2$ assumptions are known. We have the following sequence of $S_2$:

\[1, 3, \cdots, M_2 - 1, (M_2), M_2 + 1, \cdots, 2M_2 - 3, 2M_2 - 1;\]  

(2)

By (1), multiply each number in the sequence that is symmetric to the midpoint to get a new sequence of $S_1(\times)$, the first item is the product of 1 and $2M_1 - 1$:

\[1 \times (2M_1 - 1), 3 \times (2M_1 - 3), \cdots, M_1 \times M_1;\]  

(3)

\[1\text{The definitions of } A_{S(+)} \text{ and } A_{S(\times)} \text{ can be found in introduction 1.1} \]
By (2), multiply each number in the sequence that is symmetric to the midpoint to get a new sequence of $S_2(\times)$, the first item is the product of 1 and $2M_2 - 1$:

$$1 \times (2M_2 - 1), 3 \times (M_2 - 3), \ldots, (M_2 - 1) \times (M_2 + 1);$$

(4)

For (3), change sign of “×” into sign of “+” and get sequence of $S_1(\times)$

$$1 + (2M_1 - 1), 3 + (2M_1 - 3), \ldots, M_1 + M_1;$$

(5)

Meanwhile, for (4), change sign of “×” into sign of “+” and get sequence of $S_2(\times)$

$$1 + (2M_2 - 1), 3 + (2M_2 - 3), \ldots, (M_2 - 1) + (M_2 + 1);$$

(6)

Let the $A_{S_1(\times)}$ and $A_{S_1(\times)}$ are the set of $S_1(\times)$ and $S_1(\times)$, then $A_{S_1(\times)}$ and $A_{S_2(\times)}$ is bijective. So if we prove that there have product of two primes in any $A_{S_1(\times)}$ or $A_{S_2(\times)}$, Goldbach’s conjecture will be proved.

1.2. Properties of $S_1(\times), S_2(\times)$ Sequence

In view of the particularity of $S_1(\times), S_2(\times)$ Sequence, we can get the some properties of following:

(I) In $S_1(\times)$ and $S_2(\times)$, all numbers are two or more odd prime factors.

(II) For $S_1(\times)$, $M_1$ must be at least two prime factors. if $M_1$ is a prime number, the proposition hold.

(III) The frequency of all the prime factors appear in the $S_1(\times)$ and $S_2(\times)$ sequence is twice of the prime factor cycle, and if $M_1$ or $M_2$ contains the prime factor, the frequency is once of this prime factor.

(IV) For $S_1(\times)$, $M_1^2$ is the largest odd number. In $S_2(\times), (M_2 - 1) \times (M_2 + 1)$ is the largest odd number.

(V) For $S_1(\times)$, real rows are $(M_1 - 1) \div 2$, and for $S_2(\times)$, real rows are $(M_2 - 2) \div 2$.

1.3. An Example of $S_1(\times)$ Sequence

In $S_1(\times)$, take $M_1 = 25$ as an example.(see the table below). Let $P_i$ denote the $i$-th prime, and the numbers greater than or equal to three prime factors is Multi-Factor Number(MFN). For each MFN, rank its prime factors from the smallest to the largest and name it with first smallest prime factor, even if there are two prime factors, in order to prove simplification, we also treat it as the MFN. For example, in $S_1(\times)$, when $H_1 = \{2, 3, 5, 6, 8, 9, 11, 12\}$, all numbers are MFN of $p_1$. At the same time, let MFN of $p_2$ denote the biggest of first digit of MFN in $S_1(\times)$, when $H_1 = 1$, MFN of $p_3$ is $5 \times 5 \times 5 \times 5$. 


Let $\alpha_i$ denote the frequency of occupied by the $i$-th prime, that there always have $\alpha_i = 1$ (see 1.2(III)) or $\alpha_i = 2$.

Let $H_1$ is real row, then

$$H_1 = \frac{M_i - 1}{2} = 12$$  \hspace{1cm} (7)

<table>
<thead>
<tr>
<th>$S_1(\times)$</th>
<th>$S_1(\times)$</th>
<th>$H_1$</th>
<th>$MFNo f p_1$</th>
<th>$MFNo f p_2$</th>
<th>$\alpha_1$</th>
<th>$\alpha_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25+25</td>
<td>$25 \times 25 = 5 \times 5 \times 5 \times 5$</td>
<td>1</td>
<td>$\checkmark$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>23+27</td>
<td>$23 \times 27 = 3 \times 3 \times 3 \times 23$</td>
<td>2</td>
<td>$\checkmark$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21+29</td>
<td>$21 \times 29 = 3 \times 7 \times 29$</td>
<td>3</td>
<td>$\checkmark$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19+31</td>
<td>$19 \times 31$</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17+33</td>
<td>$17 \times 33 = 3 \times 11 \times 27$</td>
<td>5</td>
<td>$\checkmark$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15+35</td>
<td>$15 \times 35 = 3 \times 5 \times 5 \times 27$</td>
<td>6</td>
<td>$\checkmark$</td>
<td>1 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13+37</td>
<td>$13 \times 37$</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>11+39</td>
<td>$11 \times 39 = 3 \times 11 \times 13$</td>
<td>8</td>
<td>$\checkmark$</td>
<td>1</td>
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<td></td>
</tr>
<tr>
<td>9+41</td>
<td>$9 \times 41 = 3 \times 3 \times 41$</td>
<td>9</td>
<td>$\checkmark$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7+43</td>
<td>$7 \times 43$</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5+45</td>
<td>$5 \times 45 = 3 \times 3 \times 5 \times 5$</td>
<td>11</td>
<td>$\checkmark$</td>
<td>1 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3+47</td>
<td>$3 \times 47$</td>
<td>12</td>
<td>$\checkmark$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1+49</td>
<td>$1 \times 49$</td>
<td>13</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
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</tr>
</tbody>
</table>

1.4. Theorem

The following are two important theorems involved in the proof.

**Theorem 1.** For asymptotic density of sequence, the set $A_s$ of integers not divisible by any of the prime number $p_1, p_2, \ldots, p_X$, has density

$$\delta(A_s) = 1 - \sum_{i=1}^{X} \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i p_j} - \sum_{i<j<k} \frac{1}{p_i p_j p_k} + \cdots + (-1)^i \frac{1}{p_1} \times \frac{1}{p_2} \times \cdots \times \frac{1}{p_X},$$  \hspace{1cm} (8)

Heilbronn\(^2\) and Rohrbach\(^3\) proved that

$$\delta(A_s) \geq \prod_{i=1}^{X} \left(1 - \frac{1}{p_i}\right).$$  \hspace{1cm} (9)

**Theorem 2.** Let $\delta(A_{S_1(\times)})$ or $\delta(A_{S_2(\times)})$ is asymptotic density of the product of two primes in the $S_1(\times)$ or $S_2(\times)$, and let $\delta(A_{S_1(+)})$ or $\delta(A_{S_2(+)})$ is asymptotic density of the sum of two primes in the $S_1(+) \text{ or } S_2(+)$. According to bijective principle, then, must have

$$\delta(A_{S_1(\times)}) = \delta(A_{S_1(+)})$$  \hspace{1cm} (10)$$

and

$$\delta(A_{S_2(\times)}) = \delta(A_{S_2(+)})$$  \hspace{1cm} (11)


2. Lemmas

In this section, through the proofs of several Lemmas, we can obtain all conditions for proof of Goldbach’s conjecture.

2.1. lemma 1.

For $S_1(\times)$ or $S_2(\times)$, let $\alpha_i$ is occurrence frequency of $p_i$ in the sequence, the set $A_{S_1(\times)}$ or $A_{S_2(\times)}$ of integers not divisible by any of the prime number $p_1, p_2, \ldots, p_X$, has density

$$
\delta(A_{S_1(\times)}) = \prod_{i=1}^{x}(1 - \frac{\alpha_i}{p_i}) \quad (12)
$$

$$
\delta(A_{S_2(\times)}) = \prod_{i=1}^{x}(1 - \frac{\alpha_i}{p_i}) \quad (13)
$$

**Proof.** By Theorem 1, for $S_1(\times)$ or $S_2(\times)$, and $\alpha_i$ is occurrence frequency of $p_i$, we can get

$$
\delta(A_{S_1(\times)}) = 1 - \sum_{i} \frac{\alpha_i}{p_i} + \sum_{i<j} \frac{\alpha_i}{p_i} \times \frac{\alpha_j}{p_j} - \sum_{i<j<k} \frac{\alpha_i}{p_i} \times \frac{\alpha_j}{p_j} \times \frac{\alpha_k}{p_k} + \cdots + (-1)^i \frac{\alpha_i}{p_i} \times \frac{\alpha_2}{p_2} \times \cdots \times \frac{\alpha_x}{p_x}, \quad (14)
$$

the same reason

$$
\delta(A_{S_2(\times)}) = 1 - \sum_{i} \frac{\alpha_i}{p_i} + \sum_{i<j} \frac{\alpha_i}{p_i} \times \frac{\alpha_j}{p_j} - \sum_{i<j<k} \frac{\alpha_i}{p_i} \times \frac{\alpha_j}{p_j} \times \frac{\alpha_k}{p_k} + \cdots + (-1)^i \frac{\alpha_i}{p_i} \times \frac{\alpha_2}{p_2} \times \cdots \times \frac{\alpha_x}{p_x}, \quad (16)
$$

lemma 1 is proved.

2.2. lemma 2.

By (15) (17), we can obtain a simplified inequality

$$
\delta(A_{S_1(\times)}) \geq \frac{p_1 - \alpha_1}{p_x} \quad (18)
$$

$$
\delta(A_{S_2(\times)}) \geq \frac{p_1 - \alpha_1}{p_x} \quad (19)
$$

**Proof.** From any two continuous prime numbers $p_i$ and $p_{i+1}$ must have the relationship $p_{i+1} \geq p_i + 2$. when $\alpha_i = 1$, or $\alpha_i = 2$, therefore, we could perform reduction using $p_{i+1} \geq p_i + 2$ on the formula (15) (17).

Change
\[
\prod_{i=1}^{x} \left( \frac{p_i - \alpha_i}{p_i} \right) = \left( \frac{p_1 - \alpha_1}{p_1} \right) \times \left( \frac{p_2 - \alpha_2}{p_2} \right) \times \cdots \times \left( \frac{p_x - \alpha_x}{p_x} \right)
\] (20)

Into
\[
\prod_{i=1}^{x} \left( \frac{p_i - \alpha_i}{p_i} \right) \geq \frac{p_1 - \alpha_1}{p_x}
\] (21)

Get
\[
\delta(A_{S_1(\times)}) \geq \frac{p_1 - \alpha_1}{p_x}
\]
\[
\delta(A_{S_2(\times)}) \geq \frac{p_1 - \alpha_1}{p_x}
\]

Lemma 2 is proved.

2.3. Lemma 3.

For any \(S_1(\times)\) or \(S_2(\times)\), when \(p_x \geq p_2\), \(p_2 = 5\), \(M_1\) or \(M_2\) with \(p_x\) have the following relationship

\[
\frac{M_1 - 1}{2} \geq p_x
\] (22)

\[
\frac{M_2 - 2}{2} \geq p_x
\] (23)

Proof.

(I) According to the definition of \(S_1(\times)\), \(S_1(+)\) at the same time, if \(S_1(\times)\) exist MFN of \(p_x\), then \(p_x \times p_x \times p_x\) must be minimal form and that can be written as only one form in \(S_1(+)\)

\[
2M_1 = p_x + p_x \times p_x
\] (24)

Converting the formula (24) to

\[
M_1 - 1 = \frac{p_x + p_x \times p_x - 2}{2}
\] (25)

For (25), when \(p_x \geq p_2\), \(p_2 = 5\), obviously there always have

\[
M_1 - 1 > 2p_x
\] .

(II) Similarly, since \(S_2(\times)\) and \(S_2(+)\) at the same time, we can get

\[
M_2 - 2 = \frac{p_x + p_x \times p_x - 4}{2}
\] (26)

For (26), when \(p_x \geq p_2\), \(p_2 = 5\), obviously there always have

\[
M_2 - 2 > 2p_x
\]

For \(S_1(\times)\) and \(S_2(\times)\), if there appears any MFN of \(p_x\) larger than \(p_x \times p_x \times p_x\), by the reason of (25) and (26), when \(p_x \geq p_2\), always able to suitable (22) (23).

Lemma 2 is proved.
2.4. Lemma 4.

For any $S_1(\times)$ or $S_2(\times)$, when $p_x = p_1$, $p_1 = 3$, there always exists the product of two prime numbers.

Proof. According to the definition of $a_i = 1$ or $a_i = 2$, if the real row of $S_1(\times)$ or $S_2(\times)$ greater than or equal to 3, that is $H_1 \geq 3$ or $H_2 \geq 3$, by (18) (19), there always have

$$\delta(A_{S_1(\times)}) \geq \frac{p_1 - \alpha_1}{p_x} \geq \frac{1}{3}$$

(27)

$$\delta(A_{S_2(\times)}) \geq \frac{p_1 - \alpha_1}{p_x} \geq \frac{1}{3}$$

(28)

If the real row of $S_1(\times)$ and $S_2(\times)$ equal to 2, that is $H_1 = 2$ or $H_2 = 2$, at this time, only have following form

- when $M_1 = 5$, have $3 \times 7, 5 \times 5$, and when $M_2 = 6$, have $3 \times 9, 5 \times 7$, always have the product of two prime numbers.
- when the real row of $S_1(\times)$ or $S_2(\times)$ is one, then only exist following form, $M_1 = 3$, has $3 \times 3$, and $M_2 = 4$, has $3 \times 5$, also has the product of two prime numbers.

Lemma 4 is proved.

3. Conclusion

Through the above proof, if we get any

$$\delta(A_{S_1(\times)}) \geq \frac{1}{H_1}$$

(29)

and

$$\delta(A_{S_2(\times)}) \geq \frac{1}{H_2}$$

(30)

By the reason of Theorem 2, Goldbach’s conjecture must be true.

Proof. Since (18) (19), can get

$$\delta(A_{S_1(\times)}) \geq \frac{H_1 \times \frac{p_1 - \alpha_1}{p_x}}{H_1}$$

(31)

$$\delta(A_{S_2(\times)}) \geq \frac{H_2 \times \frac{p_1 - \alpha_1}{p_x}}{H_2}$$

(32)

By the introduction 1.2 (V)

$$H_1 = \frac{M_1 - 1}{2}$$

(33)

$$H_2 = \frac{M_2 - 2}{2}$$

(34)
Put (33) into (31), and (34) into (32), due to (22) (23) of lemma 3, and $p_1 - \alpha_i \geq 1$, when $p_x \geq p_2$, $p_2 = 5$, always have
\[
\delta(A_{S_1(+)}) \geq \frac{1}{H_1}
\] (35)
and
\[
\delta(A_{S_2(+)}) \geq \frac{1}{H_2}
\] (36)

When $p_x = p_1$, $p_1 = 3$, the reason for the establishment of (29) (30), see lemma 4.

Through the above proof, we get the conclusion that Goldbach’s conjecture was established.

References


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