Periodic sequences of progressions of the same type

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Abstract. A few progressions of the same type and their periodic sequences.

Keywords. periodic sequence, progression, prime nubmer, Fermat's little theorem

0. Introduction.

We define some progressions of the same type, and study their periodic sequences to find the rule related to them.

1. Periodicity of a progression(1).

Now we define a progression as follows.

Let k(>1) and n be also a positive integer, then

$$\begin{aligned} a_{\mathrm{n,k}} &= 1 && \text{(when n = 1)} \\ &= (a_{\mathrm{n-1,k}}\text{+n})^{\mathrm{k-1}} \pmod{\mathrm{k}} && \text{(when n > 1)} \end{aligned}$$

One by one we survey the shortest periods of the progressions of this kind, for some cases of k.

(e.q.) When k=2, then $\{a_{\mathrm{n,2}}\}=\{1, 1, 0, 0, 1, 1, 0, 0, 1, 1, \ldots\}$.
This progression seems periodic and its shortest period is assumed 4.
When k=3, then $\{a_{\mathrm{n,3}}\}=\{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots\}$.
This progression seems periodic and its shortest period is assumed 3.
When k=4, then $\{a_{\mathrm{n,4}}\}=\{1, 3, 0, 0, 1, 3, 0, 0, 1, 3, 0, \ldots\}$.
This progression seems periodic and its shortest period is assumed 4.
Periodicity of progressions is easily found for now (See Table 1).

Table 1: (A.S.P. means the assumed shortest period.)

k\n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	A.S.P.
2	1	1	0	0	1	1	0	0	1	1	0	0	1	1	4
3	1	0	0	1	0	0	1	0	0	1	0	0	1	0	3
4	1	3	0	0	1	3	0	0	1	3	0	0	1	3	4
5	1	1	1	0	0	1	1	1	0	0	1	1	1	0	5
6	1	3	0	4	3	3	4	0	3	1	0	0	6	1	12
7	1	1	1	1	1	0	0	1	1	1	1	1	0	0	7
8	1	3	0	0	5	3	0	0	1	3	0	0	5	3	8
9	1	0	0	7	0	0	4	0	0	1	0	0	7	0	9

Theorem 1

Let l be a positive integer. If $a_{\mathrm{n,k}}$=$a_{\mathrm{n+l,k}}$ and k|l (i.e. l is divisible by k.) for the above-mentioned progression $\{a_{\mathrm{n,k}}\}$, then $\{a_{\mathrm{n,k}}\}$ has a period equal to l.

Proof.

We will prove deductively, that if $a_{\mathrm{n+m,k}}$=$a_{\mathrm{n+m+l,k}}$ then $a_{\mathrm{n+m+1,k}}$=$a_{\mathrm{n+m+l+1,k}}$ where m is a non-negative integer.

When m=0 evidently $a_{\mathrm{n,k}}$=$a_{\mathrm{n+l,k}}$.

Furthermore if $a_{\mathrm{n+m,k}}$=$a_{\mathrm{n+m+l,k}}$ then $a_{\mathrm{n+m+1,k}}$≡$(a_{\mathrm{n+m,k}}$+n+m+1$)^{\mathrm{k}-1}$ (mod k)≡$(a_{\mathrm{n+m+l,k}}$+n+m+l+1$)^{\mathrm{k}-1}$(mod k)= $a_{\mathrm{n+m+l+1,k}}$, for l≡0(mod k).

This completes Theorem 1.

□

Theorem 2

Suppose k is a prime number larger than 2.

If n≡0 or n≡k−1 (mod k) then $a_{\mathrm{n,k}}$=0, otherwise $a_{\mathrm{n,k}}$=1.

Proof.

When k=3 then $a_{1,3}$=1, $a_{2,3}$=$(a_{1,3}+2)^2$(mod 3)=0, $a_{3,3}$=$(a_{2,3}+3)^2$(mod 3)=9(mod 3)=0, $a_{4,3}$=$(a_{3,3}+4)^2$(mod 3)=1(mod 3)=1.

Therefore $a_{1,3}$=1=$a_{4,3}$, so 3 is a period of this progression.

This completes Theorem 2 for k=3.

When k is larger than 3 then, applying Fermat's little theorem[1], $a_{1,\mathrm{k}}$=1, $a_{2,\mathrm{k}}$=$(a_{1,\mathrm{k}}+2)^{\mathrm{k}-1}$(mod k)=$3^{\mathrm{k}-1}$(mod k)=1, $a_{3,\mathrm{k}}$=$(a_{2,\mathrm{k}}+3)^{\mathrm{k}-1}$(mod k)=$4^{\mathrm{k}-1}$ (mod k) =1,. . .,$a_{\mathrm{k}-1,\mathrm{k}}$=$(a_{\mathrm{k}-2,\mathrm{k}}+\mathrm{k}-1)^2$(mod k)=0(mod k)=0, . . ., $a_{\mathrm{k,k}}$=$(a_{\mathrm{k}-1,\mathrm{k}}+\mathrm{k})^2$(mod k)=0(mod k)=0.

Also $a_{k+1,k}$=$(a_{k,k}$+k+1$)^2$(mod k)=1(mod k)=1, so k is a period of this progression.

This completes Theorem 2 for k is larger than 3.

□

2. Periodicity of a progression(2).

Now we define another progression as follows.

Let k(>1) and n be also a positive integer, then

$$\begin{aligned} b_{n,k} &= 1 && \text{(when n = 1)} \\ &= (b_{n-1,k} - n)^{k-1} \text{ (mod k)} && \text{(when n > 1)} \end{aligned}$$

Periodicity of progressions is easily found for now (See Table 2).

Table 2: (A.S.P. means the assumed shortest period.)

k\n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	A.S.P.
2	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	4
3	1	1	1	0	1	1	0	1	1	0	1	1	0	1	1	3(*)
4	1	3	0	0	3	1	0	0	3	1	0	0	3	1	0	4(*)
5	1	1	1	1	1	0	1	1	1	1	0	1	1	1	1	5(*)
6	1	5	2	4	5	5	4	2	5	1	2	2	1	5	2	12
7	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	7(*)

On Table 2, * indicates that the period for each k does not start from the first term.

Theorem 3

Let l be a positive integer. If $b_{n,k}$=$b_{n+l,k}$ and k|l (i.e. l is divisible by k.) for the above-mentioned progression $\{b_{n,k}\}$, then $\{b_{n,k}\}$ has a period equal to l.

Proof.

We will prove deductively, that if $b_{n+m,k}$=$b_{n+m+l,k}$ then $b_{n+m+1,k}$=$b_{n+m+l+1,k}$ where m is a non-negative integer.

When m=0 evidently $b_{n,k}$=$b_{n+l,k}$.

Furthermore if $b_{n+m,k}$=$b_{n+m+l,k}$ then $b_{n+m+1,k}$≡$(b_{n+m,k}$−n−m−1$)^{k-1}$ (mod k)≡$(b_{n+m+l,k}$−n−m−l+1$)^{k-1}$(mod k)= $b_{n+m+l+1,k}$, for l≡0(mod k).

This completes Theorem 3, similarly as Theorem 1.

□

3. Periodicity of a progression(3).

Now we define another progression again and again as follows.

Let k(>1) and n be also a positive integer, then

$$\begin{aligned} c_{\mathrm{n,k}} &= 1 & \text{(when n} = 1) \\ &= (c_{\mathrm{n-1,k}}+(-1)^{\mathrm{n}}\mathrm{n})^{\mathrm{k}-1} \pmod{\mathrm{k}} & \text{(when n} > 1) \end{aligned}$$

Periodicity of progressions is easily found for now (See Table 3).

Table 3: (A.S.P. means the assumed shortest period.)

k\n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	A.S.P.
2	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	4
3	1	0	0	1	1	0	1	1	1	1	1	0	1	1	1	1	6(*)
4	1	3	0	0	3	1	0	0	3	1	0	0	3	1	0	0	4(*)
5	1	1	1	0	0	1	1	1	1	1	0	1	1	0	0	1	10(*)
6	1	3	0	4	5	5	4	0	3	1	2	2	1	3	0	4	12
7	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	1	14(*)

On Table 3, * indicates that the period for each k does not start from the first term.

references

[1] Patrick St-Amant, International Journal of Algebra, Vol.4, 2010, no.17-20, 959-994