Mathematics for Incompletely Predictable Problems depicting Spin-offs from converting Riemann zeta function into its continuous format version: Paper 1 of 2 related papers

John Y. C. Ting

1Mathematics for Incompletely Predictable Problems

Correspondence: Dr. John Yuk Ching Ting, 729 Albany Creek Road, Albany Creek, Queensland 4035, Australia.
Tel: +614-1775-1859. E-mail: jycting@hotmail.com

Received: September 29, 2019 Accepted: September 29, 2019

Abstract Mathematics for Incompletely Predictable Problems makes all mathematical arguments valid and complete in [current] Paper 1 (based on first key step of converting Riemann zeta function into its continuous format version) and [next] Paper 2 (based on second key step of applying Information-Complexity conservation to Sieve of Eratosthenes). Nontrivial zeros and two types of Gram points calculated using this function plus prime and composite numbers computed using this Sieve are defined as Incompletely Predictable entities. Euler product formula alternatively and exactly represents Riemann zeta function but utilizes product over prime numbers (instead of summation over natural numbers). Hence prime numbers are encoded in this function demonstrating deep connection between them. Direct spin-offs from first step consist of proving Riemann hypothesis and explaining manifested properties of both Gram points, and from second step consist of proving Polignac’s and Twin prime conjectures. These mentioned open problems are defined as Incompletely Predictable problems.

Keywords: Completely and Incompletely Predictable entities; Composite numbers; Dirichlet Sigma-Power Laws; Exact and Inexact Dimensional analysis homogeneity; Gram points; Nontrivial zeros; Polignac’s and Twin prime conjectures; Prime numbers; Riemann hypothesis.

Subject Classification: 11A41, 11M26

1. Introduction

Elements of three complete sets constituted by nontrivial zeros and two types of Gram points together with elements of two complete sets constituted by prime and composite numbers are all defined as Incompletely Predictable entities. Riemann hypothesis (1859) proposed all nontrivial zeros in Riemann zeta function are located on its critical line. Defined as Incompletely Predictable problem is sine qua non as direct spin-off for obtaining our novel Dirichlet Sigma-Power Law, which is the continuous format version of [discrete format] Riemann zeta function, to solve this hypothesis. The closely related two types of Gram points, being integral parts of this function, are provided with explanations for their manifested properties – see Appendix A. Defined as Incompletely Predictable problems is sine qua non as direct spin-offs for these explanations to be correct. Involving proposals that prime gaps and associated sets of prime numbers are infinite in magnitude, Twin prime conjecture (1846) deals with even prime gap 2 thus forming a subset of Polignac’s conjecture (1849) which deals with all even prime gaps 2, 4, 6, 8, 10..... Defined as Incompletely Predictable problems is sine qua non as direct spin-offs to solve these conjectures using our novel research method Information-Complexity conservation.

Refined information on Incompletely Predictable entities of Gram and virtual Gram points: These are countable entities dependently calculated using complex equation Riemann zeta function, \( \zeta(s) \), or its proxy Dirichlet eta function, \( \eta(s) \), in the critical strip (denoted by \( 0 < \sigma < 1 \)). In Figure 2 below, Gram[\( y=0 \)], Gram[\( x=0 \)] and Gram[\( x=0,y=0 \)] points respectively refer to geometrical x-axis, y-axis and Origin intercepts at the critical line (denoted by \( \sigma = \frac{1}{2} \)). Gram[\( y=0 \)] and Gram[\( x=0,y=0 \)] points are respectively synonymous with traditional ‘Gram points’ and nontrivial zeros. In Figures 3 and 4 below, virtual Gram[\( y=0 \)] and virtual Gram[\( x=0 \)] points respectively refer to geometrical x-axis and y-axis intercepts at the non-critical lines (denoted by \( \sigma \neq \frac{1}{2} \)). Virtual Gram[\( x=0,y=0 \)] points do not exist.

Refined information on Incompletely Predictable entities of prime and composite numbers: These are countable entities dependently computed (respectively) directly and indirectly using complex algorithm Sieve of Eratosthenes. Denote \( \mathbb{C} \) to be uncountable complex numbers, \( \mathbb{R} \) to be uncountable real numbers, \( \mathbb{Q} \) to be countable rational numbers, \( \mathbb{R}-\mathbb{Q} \) to be uncountable irrational numbers, \( \mathbb{A} \) to be countable algebraic numbers, \( \mathbb{R}-\mathbb{A} \) to be uncountable transcendental numbers, \( \mathbb{Z} \) to be countable integers, \( \mathbb{W} \) to be countable whole numbers, \( \mathbb{N} \) to be countable natural numbers, \( \mathbb{E} \) to be countable even
numbers, \(O\) to be countable odd numbers, \(P\) to be countable prime numbers, and \(C\) to be countable composite numbers. Then (i) Set \(N\) = Set \(E\) + Set \(O\), (ii) Set \(N\) = Set \(P\) + Set \(C\) + Number ‘1’, (iii) Set \(N\) \(\subset\) Set \(W\) \(\subset\) Set \(Z\) \(\subset\) Set \(Q\) \(\subset\) Set \(R\) \(\subset\) Set \(C\), and (iv) Set \(R-Q\) = Set \(A\) + Set \(R-A\).

With increasing size, arbitrary Set \(X\) can be countable finite set (CFS), countable infinite set (CIS) or uncountable infinite set (UIS). Cardinality of Set \(X\), \(|X|\), measures "number of elements" in Set \(X\). E.g. Set negative Gram[y=0] point has CIS of negative Gram[y=0] point with [negative Gram[y=0] point] = 1. Set even \(P\) has CFS of even \(P\) with [even \(P\)] = 1, Set \(N\) has CIS of \(N\) with \(|N| = \aleph_0\), and Set \(R\) has UIS of \(R\) with \(|R| = \pi\) (cardinality of the continuum).

Differentiation of terms "Incompletely Predictable" versus "Completely Predictable": Set \(N = E + O\). The two subsets of even and odd numbers are "Independent" and "Completely Predictable". Example, the next even number after 2,984 (which is the 2,984 \(/2 = 1.492\)nd even number) is [easily] calculated independently using simple algorithm to be 2,984 + 2 = 2,986 which is the 2,986 \(/2 = 1.493\)rd even number. Set \(N = P + C + \text{Number} ‘1’\). The two subsets of prime and composite numbers are "Dependent" and "Incompletely Predictable". Example, the next sixth prime number ‘13’ [after the fifth prime number ‘11’] is [not easily] computed dependently using complex algorithm from scratch via: 2 is 1st prime number, 3 is 2nd prime number, 4 is 1st composite number, 5 is 3rd prime number, 6 is 2nd composite number, 7 is 4th prime number, 8 is 3rd composite number, 9 is 4th composite number, 10 is 5th composite number, 11 is 5th prime number, 12 is 6th composite number, and our desired 13 is 6th prime number.

Indirect spin-offs arising out of solving Riemann hypothesis are often stated as "With this one solution, we have proven five hundred theorems or more at once". This apply to many important theorems in Number theory (mostly about prime numbers) that rely on properties of Riemann zeta function such as where trivial and nontrivial zeros are, and are not, located. A classical example is the resulting absolute and full delineation of prime number theorem, which relates to prime counting function. This function, usually denoted by \(\pi(x)\), is defined as the number of prime numbers less than or equal to \(x\). Public-key cryptography that is widely required for financial security in E-Commerce traditionally depend on solving the difficult problem of factoring prime numbers for astronomically large numbers. The intrinsic "Incompletely Predictable" property present in prime numbers, composite numbers, nontrivial zeros and two types of Gram points cannot be altered to "Completely Predictable" property. For this stated reason, it is a mathematical impossibility that providing rigorous proofs such as for Riemann hypothesis will ever result in crypto-apocalypse. However, fast supercomputers and the far-more-powerful quantum computers that theoretically allow solving difficult factorization problem in quick time will result in less secure encryption and decryption. Then using quantum cryptography that relies on principles of quantum mechanics to encrypt data and transmit it in a way that cannot be hacked will combat this issue.

Formal definitions for Completely Predictable (CP) entities and Incompletely Predictable (IP) entities: We observe the word "number" [singular noun] or "numbers" [plural noun] could easily be used interchangeably with the word "entity" [singular noun] or "entities" [plural noun] throughout this research paper. Respectively, an IP (CP) number is locally defined as a number whose position is dependent (independently) determined by complex (simple) calculations using complex (simple) equation or algorithm with (without) needing to know related positions of all preceding numbers in neighborhood. Simple properties are inferred from a sentence such as "This simple equation or algorithm by itself will intrinsically incorporate actual location [and actual positions] of all CP numbers". Solving CP problems with simple properties amendable to simple treatments using usual mathematical tools such as Calculus result in 'Simple Elementary Fundamental Laws'-based solutions. Complex properties, or "meta-properties", are inferred from a sentence such as "This complex equation or algorithm by itself will intrinsically incorporate actual location [but not actual positions] of all IP numbers". Solving IP problems with complex properties amendable to complex treatments using unusual mathematical tools such as Information-Complexity conservation, and exact and inexact Dimensional analysis homogeneity as well as using usual mathematical tools such as Calculus result in 'Complex Elementary Fundamental Laws'-based solutions.

Based on Mathematics for Incompletely Predictable Problems, we compare and contrast CP entities (obeying Simple Elementary Fundamental Laws) against IP entities (obeying Complex Elementary Fundamental Laws) using examples: (I) \(E\) are CP entities constituted by CIS of \(Q\) 2, 4, 6, 8, 10, 12.... (II) \(O\) are CP entities constituted by CIS of \(Q\) 1, 3, 5, 7, 9, 11.... (III) \(P\) are IP entities constituted by CIS of \(Q\) 2, 3, 5, 7, 11, 13.... (IV) \(C\) are IP entities constituted by CIS of \(Q\) 4, 6, 8, 9, 10, 12.... (V) With values traditionally given by parameter \(N\) nontrivial zeros in Riemann zeta function are IP entities constituted by CIS of \(R-A\) [rounded off to six decimal places]: 14.134725, 21.022040, 25.010858, 30.424876, 32.935062, 37.586178,... (VI) Traditional 'Gram points' (or Gram[y=0] points) are x-axis intercepts with choice of index 'n' for 'Gram points' historically chosen such that first 'Gram point' [by convention at \(n = 0\)] corresponds to the t value which is larger than (first) nontrivial zero located at \(t = 14.134725\). 'Gram points' are IP entities constituted by CIS of \(R-A\) [rounded off to six decimal places] with the first six given at \(n = -3\), \(t = 0\); at \(n = -2\), \(t = 3.436218\); at \(n = -1\), \(t = 9.666908\); at \(n = 0\), \(t =
17.845599; at n = 1, t = 23.170282; at n = 2, t = 27.670182.

Denoted by parameter t; nontrivial zeros, ‘Gram points’ and Gram[x=0] points all belong to well-defined CIS of \( R-A \) which will twice obey the relevant location definition [in CIS of \( R-A \) themselves and in CIS of numerical digits after decimal point of each \( R-A \)]. First and only negative ‘Gram point’ (at n = -3) is obtained by substituting CP t = 0 resulting in \( \zeta(\frac{1}{2} + it) = \zeta(\frac{1}{2}) = -1.4603545 \), a \( R-A \) number [rounded off to seven decimal places] calculated as a limit similar to limit for Euler-Mascheroni constant or Euler gamma with its precise \((1^i)\) position only determined by computing positions of all preceding (nil) ‘Gram point’ in this case. ‘0’ and ‘1’ are special numbers being neither \( P \) nor \( C \) as they represent nothingness (zero) and wholeness (one). In this setting, the ideas of (i) having factors for ‘0’ and ‘1’, or (ii) treating ‘0’ and ‘1’ as CP or IP numbers, is meaningless. All entities derived from well-defined simple/complex algorithms or equations are "dual numbers" as they can be simultaneously depicted as CP and IP numbers. For instance, \( Q \ '2' \ as \ (P \ (& \ E), \ '97' \ as \ P \ (& \ O), \ '98' \ as \ C \ (& \ E), \ '99' \ as \ C \ (& \ O); \ CP '0' \ values in x=0, \ in y=0 \ & \ in \ simultaneous \ x=0,y=0 \ associated \ with \ various \ IP \ values \ in \ \zeta(s)\). 

\[
\zeta(s) = \frac{e^{\frac{\ln(2\pi)-1}{2}+is}}{(s)^{\frac{1}{2}}(1+\frac{s}{2})} \Gamma(s) \zeta(1-s) = \frac{\Pi_{\rho}(1-\frac{s}{\rho})e^{\Gamma}}{2(s-1)\Gamma(1+\frac{s}{2})}
\]

Proposed by German mathematician Bernhard Riemann (September 17, 1826 – July 20, 1866) in 1859, Riemann hypothesis is mathematical statement on \( \zeta(s) \) that critical line denoted by \( \sigma = \frac{1}{2} \) contains complete Set nontrivial zeros with \( |\text{nontrivial zeros}| = N_0 \). Alternatively, this hypothesis is geometrical statement on \( \zeta(s) \) that generated curves when \( \sigma = \frac{1}{2} \) contain complete Set Origin intercepts with \( |\text{Origin intercepts}| = N_0 \). Depicted in full and abbreviated version, Hadamard product is infinite product expansion of \( \zeta(s) \) based on Weierstrass’s factorization theorem displaying a simple pole at \( s = 1 \). It contains both trivial & nontrivial zeros indicating their common origin from \( \zeta(s) \). Set trivial zeros occurs at \( \sigma = -2, -4, -6, -8, -10, \ldots, \infty \) with \( |\text{trivial zeros}| = N_0 \) due to \( \Gamma \) function term in denominator. Nontrivial zeros occur at \( s = \rho \) with \( \gamma \) denoting Euler-Mascheroni constant.

**Remark 1.1.** Confirming first 10,000,000,000,000 nontrivial zeros location on critical line implies but does not prove Riemann hypothesis to be true.

Locations of first 10,000,000,000,000 nontrivial zeros on critical line have previously been computed to be correct. Hardy in 1914[1], and with Littlewood in 1921[2], showed infinite nontrivial zeros on critical line by considering moments of certain functions related to \( \zeta(s) \). This discovery cannot constitute rigorous proof for Riemann hypothesis because they have not exclude theoretical existence of nontrivial zeros located away from this line.

**Remark 1.2.** We can apply useful concepts from exact and inexact Dimensional analysis homogeneity to well-defined equations and inequations.

Respectively for ‘base quantities’ such as length, mass and time; their fundamental SI ‘units of measurement’ meter (m) is defined as distance travelled by light in vacuum for time interval 1/299 792 458 s with speed of light c = 299,792,458 ms⁻¹, kilogram (kg) is defined by taking fixed numerical value Planck constant h to be 6.626 070 15 X 10⁻³⁴ Joules-second (Js) [whereby Js is equal to kgm²s⁻¹] and second (s) is defined in terms of ΔvCs = Δ(1³³Cs)ħ/Js = 9.192,631,770 s⁻¹. Derived SI units such as J and ms⁻¹ respectively represent ‘base quantities’ energy and velocity. The word ‘dimension’ is commonly used to indicate all those mentioned ‘units of measurement’ in well-defined equations.

Dimensional analysis (DA) is an analytic tool with DA homogeneity and non-homogeneity (respectively) denoting valid and invalid equation occurring when ‘units of measurements’ for ‘base quantities’ are ‘balanced’ and ‘unbalanced’ across both sides of the equation. E.g. equation 2 m + 3 m = 5 m is valid and equation 2 m + 3 kg = 5 mkg is invalid (respectively) manifesting DA homogeneity and non-homogeneity.

Let (2n) and (2n-1) be ‘base quantities’ in Dirichlet Sigma-Power Laws formatted in simplest forms as equations and inequations. E.g. DA on exponent \( i \) in (2n)\( ^{\frac{1}{2}} \) in simplest form is correct but DA on exponent \( \frac{i}{2} \) in equivalent \((2n^2)^{\frac{i}{2}} \) not in simplest form is incorrect. Fractional exponents as ‘units of measurement’ given by \((1 - \sigma)\) for equations and \((\sigma + 1)\) for inequations when \( \sigma = \frac{1}{2} \) coincide with exact DA homogeneity; and \((1 - \sigma)\) for equations and \((\sigma + 1)\) for inequations when \( \sigma \neq \frac{1}{2} \) coincide with inexact DA homogeneity. Respectively for equations and inequations, exact DA homogeneity at \( \sigma = \frac{1}{2} \) denotes \( \Sigma \) (all fractional exponents) as \( 2(1 - \sigma) \) and \( 2(\sigma + 1) \) equates to ["exact"] whole number ‘1’ and ‘3’; and inexact DA homogeneity at \( \sigma \neq \frac{1}{2} \) denotes \( \Sigma \) (all fractional exponents) as \( 2(1 - \sigma) \) and \( 2(\sigma + 1) \) equates to ["inexact"] fractional number ‘≠1’ and ‘≠3’.

**Footnote 1.2:** Exact and inexact DA homogeneity occur in Dirichlet Sigma-Power Laws as equations or inequations for Gram[y=0] points, Gram[x=0] points and nontrivial zeros. Law of Continuity is a heuristic principle whatever succeed.
proof for the finite, also succeed for the infinite. Then these Laws which inherently manifest themselves on finite and infinite time scale should "succeed for the finite, also succeed for the infinite".

Outline of proof for Riemann hypothesis. To simultaneously satisfy two mutually inclusive conditions: I. With rigid manifestation of exact DA homogeneity, Set nontrivial zeros with \([\text{nontrivial zeros}] = \text{N}_0\) is located on critical line (viz. \(\sigma = \frac{1}{2}\)) when \(2(1-\sigma)\) or \(2(\sigma + 1)\) as \(\sum\) (all fractional exponents) = whole number ’1’ [or ’3’] in Dirichlet Sigma-Power Law\(^3\) as equation [or inequation]. II. With rigid manifestation of inexact DA homogeneity, Set nontrivial zeros with \([\text{nontrivial zeros}] = \text{N}_0\) is not located on non-critical lines (viz. \(\sigma \neq \frac{1}{2}\)) when \(2(1-\sigma)\) or \(2(\sigma + 1)\) as \(\sum\) (all fractional exponents) = fractional number ’\neq 1’ [or ’\neq 3’] in Dirichlet Sigma-Power Law\(^3\) as equation [or inequation].

Footnote 3: Derived from original \(\eta(s)\) (proxy for \(\zeta(s)\)) as equation or inequation, this Law symbolizes end-result proof on Riemann hypothesis.

Riemann hypothesis mathematical foot-prints. Six identifiable steps to prove Riemann hypothesis: Step 1 Use \(\eta(s)\), proxy for \(\zeta(s)\), in critical strip. Step 2 Apply Euler formula to \(\eta(s)\). Step 3 Obtain "simplified" Dirichlet eta function which intrinsically incorporates actual location [but not actual positions] of all nontrivial zeros\(^4\). Step 4 Apply Riemann integral to "simplified" Dirichlet eta function in discrete (summation) format. Step 5 Obtain Dirichlet Sigma-Power Law in continuous (integral) format as equation or inequation. Step 6 Note exact and inexact DA homogeneity on their fractional exponents.

Footnote 4: Respectively Gram[y=0] points, Gram[x=0] points and nontrivial zeros are Incompletely Predictable entities with actual positions determined by setting \(\eta(\sigma) = 0\), \(\sum \eta(\sigma) = 0\) and \(\sum \text{ReIm}(\eta(\sigma)) = 0\) to dependently calculate relevant positions of all preceding entities in neighborhood. Respectively actual location of Gram[y=0] points, Gram[x=0] points and nontrivial zeros; and virtual Gram[y=0] points, virtual Gram[x=0] points and “absent” nontrivial zeros occur precisely at \(\sigma = \frac{1}{2}\); and \(\sigma \neq \frac{1}{2}\).

2. Riemann zeta and Dirichlet eta functions

Figure 1: INPUT for \(\sigma = \frac{1}{2}, \frac{3}{2}\), and \(\frac{5}{2}\). \(\zeta(s)\) has countable infinite set of Completely Predictable trivial zeros at \(\sigma = \text{all negative even numbers and countable infinite set of Incompletely Predictable nontrivial zeros at } \sigma = \frac{1}{2}\) for various \(t\) values.

L-functions form an integral part of ‘L-functions and Modular Forms Database’ (LMFDB) with far-reaching implications. In perspective, \(\zeta(s)\) is simplest example of an L-function. \(\zeta(s)\) is a function of complex variable \(s = \sigma + it\) that analytically continues sum of infinite series \(\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\). The common convention is to write \(s\) as \(\sigma + it\) with \(i = \sqrt{-1}\), and \(\sigma\) and \(t\) real. Valid for \(\sigma > 0\), we write \(\zeta(s)\) as \(\text{Re}(\zeta(s)) + i \cdot \text{Im}(\zeta(s))\) and note that \(\zeta(\sigma + it)\) when \(0 < t < +\infty\) is the complex conjugate of \(\zeta(\sigma - it)\) when \(-\infty < t < 0\).

Also known as alternating zeta function, \(\eta(s)\) must act as proxy for \(\zeta(s)\) in critical strip (viz. \(0 < \sigma < 1\)) containing critical line (viz. \(\sigma = \frac{1}{2}\)) because \(\zeta(s)\) only converges when \(\sigma > 1\). This implies \(\zeta(s)\) is undefined to left of this region in critical strip which then requires \(\eta(s)\) representation instead. They are related to each other as \(\zeta(s) = \gamma \cdot \eta(s)\) with proportionality factor \(\gamma = \frac{1}{1 - 2^{1-s}}\) and \(\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots\).
Proof for Riemann hypothesis and Explanations for Gram points

Figure 2: OUTPUT for $\sigma = \frac{1}{2}$. Schematically depicted polar graph of $\zeta\left(\frac{1}{2} + it\right)$ plotted along critical line for real values of $t$ running from 0 to 34, horizontal axis: $\text{Re}\{\zeta\left(\frac{1}{2} + it\right)\}$, and vertical axis: $\text{Im}\{\zeta\left(\frac{1}{2} + it\right)\}$. There are presence of Origin intercepts which are totally absent in Figures 3 and 4 [with identical axes definitions].

Figure 3: OUTPUT for $\sigma = \frac{2}{5}$.

Figure 4: OUTPUT for $\sigma = \frac{3}{5}$.

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]
\[ = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \]
\[ = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \]
\[ = \frac{1}{(1 - 2^{-s}) \cdot (1 - 3^{-s}) \cdot (1 - 5^{-s}) \cdot (1 - 7^{-s}) \cdot (1 - 11^{-s}) \cdots} \]
\[ \cdots \]

Eq. (1) is defined for only $1 < \sigma < \infty$ region where $\zeta(s)$ is absolutely convergent. There are no zeros located here. In Eq. (1), equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] can also represent $\zeta(s)$.

\[ \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \cdot \Gamma(1 - s) \cdot \zeta(1 - s) \]  

Eq. (2) is Riemann’s functional equation valid for $-\infty < \sigma < \infty$. It can be used to find all trivial zeros on horizontal line at $t = 0$ occurring when $\sigma = -2, -4, -6, -8, -10, \ldots$, whereby $\zeta(s) = 0$ because factor $\sin\left(\frac{\pi s}{2}\right)$ vanishes. $\Gamma$ is gamma function, an extension of factorial function [a product function denoted by ! notation whereby $n! = n(n-1)(n-2)\ldots(n-(n-1))$ with its argument shifted down by 1, to real and complex numbers. That is, if $n$ is a positive integer, $\Gamma(n) = (n-1)!$]

\[ \zeta(s) = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \]
\[ = \frac{1}{1 - 2^{-s}} \left( \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \right) \]
Equation (3) is defined for all $\sigma > 0$ values except for simple pole at $\sigma = 1$. As alluded to above, $\zeta(s)$ without $\frac{1}{(1 - 2^{1-s})}$ is $\eta(s)$. It is a holomorphic function of $s$ defined by analytic continuation and is mathematically defined at $\sigma = 1$ whereby analogous trivial zeros with presence only for $\eta(s)$ [but not for $\zeta(s)$] on vertical straight line $\sigma = 1$ are found at $s = 1 \pm i \cdot \frac{2\pi k}{\ln(2)}$ where $k = 1, 2, 3, 4, \ldots, \infty$.

Figure 1 depict complex variable $s (= \sigma \pm it)$ as INPUT with x-axis denoting real part Re($s$) equating to $\sigma$; and y-axis denoting imaginary part Im($s$) equating to $t$. Figures 2, 3 and 4 respectively depict $\zeta(s)$ as OUTPUT for real values of $t$ running from 0 to 34 at $\sigma = \frac{1}{2}$ (critical line), $\sigma = \frac{3}{4}$ (non-critical line), and $\sigma = \frac{5}{4}$ (non-critical line) with x-axis denoting real part Re($\zeta(s)$) and y-axis denoting imaginary part Im($\zeta(s)$). There are infinite types-of-spirals possibilities associated with each $\sigma$ value arising from all infinite $\sigma$ values in critical strip. Mathematically proving all nontrivial zeros location on critical line as denoted by solitary $\sigma = \frac{3}{2}$ value equates to geometrically proving all Origin intercepts occurrence at solitary $\sigma = \frac{3}{2}$ value. Both result in rigorous proof for Riemann hypothesis.

3. Prerequisite lemma, corollary and propositions for Riemann hypothesis

Original equation $\eta(s)$, proxy for $\zeta(s)$, is treated as unique mathematical object with key properties and behaviors. Containing all x-axis, y-axis and Origin intercepts, it will intrinsically incorporate actual location [but not actual positions] of all Gram[$y=0$] points, Gram[$x=0$] points and nontrivial zeros. Proofs on lemma, corollary and propositions on nontrivial zeros depict exact and inexact DA homogeneity in both derived equation and inequality. Parallel procedure on Gram[$y=0$] and Gram[$x=0$] points below depict exact and inexact DA homogeneity in similarly derived equations and inequalities.

Lemma 3.1. "Simplified" Dirichlet eta function is derived directly from Dirichlet eta function with Euler formula application and it will intrinsically incorporate actual location [but not actual positions] of all nontrivial zeros.

Proof. Denote complex number $(C)$ as $z = x + iy$. Then $z = \text{Re}(z) + i \cdot \text{Im}(z)$ with $\text{Re}(z) = x$ and $\text{Im}(z) = y$; modulus of $z$, $|z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2} = \sqrt{x^2 + y^2}$; and $|z|^2 = x^2 + y^2$.

Euler formula is commonly stated as $e^{it} = \text{cos} x + i \cdot \text{sin} x$. Euler identity (where $x = \pi$) is $e^{it} = \text{cos} \pi + i \cdot \text{sin} \pi = -1 + 0$ [or stated as $e^{it} + 1 = 0$]. The $n^t$ of $\zeta(s)$ is expanded to $n^t = n^{(t+\sigma)} = n^\sigma e^{i\text{ln}(n)}$ since $n^\sigma = e^{i\text{ln}(n)}$. Apply Euler formula to $n^t$ result in $n^t = n^{(t+\sigma)} \text{cos}(t \text{ln}(n)) + i \cdot \text{sin}(t \text{ln}(n))$. This is written in trigonometric form [designated by short-hand notation $n^t(Euler)$] whereby $n^\sigma$ is modulus and $t \text{ln}(n)$ is polar angle (argument).

Apply $n^t(Euler)$ to Eq. (1). Then $\zeta(s) = \text{Re}(\zeta(s)) + i \text{Im}(\zeta(s))$ with $\text{Re}(\zeta(s)) = \sum_{n=1}^{\infty} n^{\sigma-1} \cos(t \text{ln}(n))$ and $\text{Im}(\zeta(s)) = \sum_{n=1}^{\infty} n^{\sigma-1} \sin(t \text{ln}(n))$. As Eq. (1) is defined only for $\sigma > 1$ where zeros never occur, we will not carry out further treatment here.

Apply $n^t(Euler)$ to Eq. (3). Then $\zeta(s) = \gamma \cdot \eta(s) = \gamma \cdot [\text{Re}(\eta(s)) + i \text{Im}(\eta(s))]$ with $\text{Re}(\eta(s)) = \sum_{n=1}^{\infty} ((2n-1)^{\sigma-1} \cos(t \text{ln}(2n-1)) - (2n)^{\sigma-1} \cos(t \text{ln}(2n)));$

$\text{Im}(\eta(s)) = \sum_{n=1}^{\infty} ((2n)^{\sigma-1} \sin(t \text{ln}(2n)) - (2n-1)^{\sigma-1} \sin(t \text{ln}(2n-1)));

and proportionality factor $\gamma = \frac{1}{(1 - 2^{1-s})}$.

Complex number $s$ in critical strip is designated by $s = \sigma + it$ for $0 < t < +\infty$ and $s = \sigma - it$ for $-\infty < t < 0$. Nontrivial zeros equating to $\zeta(s) = 0$ give rise to our desired $\eta(s) = 0$. Modulus of $\eta(s)$, $|\eta(s)|$, is defined as $\sqrt{[\text{Re}(\eta(s))]^2 + [\text{Im}(\eta(s))]^2}$ with $|\eta(s)|^2 = (\text{Re}(\eta(s)))^2 + (\text{Im}(\eta(s)))^2$. Mathematically $|\eta(s)|^2 = |\eta(s)|^2 = 0$ is an unique condition giving rise to $\eta(s) = 0$ occurring only when $\text{Re}(\eta(s)) = \text{Im}(\eta(s)) = 0$ as any non-zero values for $\text{Re}(\eta(s))$ and/or $\text{Im}(\eta(s))$ will always result in $|\eta(s)|$ and $|\eta(s)|^2$ having non-zero values. Important implication is that sum of $\text{Re}(\eta(s))$ and $\text{Im}(\eta(s))$ equating to zero [given by Eq. (4)] must always hold when $|\eta(s)| = |\eta(s)|^2 = 0$ and consequently $\eta(s) = 0$.

$$\sum \text{Re}(\eta(s)) = \text{Re}(\eta(s)) + \text{Im}(\eta(s)) = 0 \quad (4)$$

In principle, advocating for existence of theoretical $s$ values leading to non-zero values in $\text{Re}(\eta(s))$ and $\text{Im}(\eta(s))$ depicted as possibility $+\text{Re}(\eta(s)) = -\text{Im}(\eta(s))$ or $-\text{Re}(\eta(s)) = +\text{Im}(\eta(s))$ could satisfy Eq. (4). This reverse implication is not
necessarily true as these s values will not result in \(|\eta(s)| = |\eta(s)|^2 = 0\). In any event, we need not consider these two possibilities since solving Riemann hypothesis involves nontrivial zeros defined by \(\eta(s) = 0\) with non-zero values in \(\text{Re}(\eta(s))\) and/or \(\text{Im}(\eta(s))\) being not compatible with \(\eta(s) = 0\).

Riemann hypothesis proposed all nontrivial zeros to be located on critical line. This location is conjectured to be uniquely associated with presence of exact DA homogeneity in derived equation and inequation of Dirichlet Sigma-Power Law with Eq. (4) intrinsically incorporated into this Law as the \(\eta(s) = 0\) definition for nontrivial zeros equates to Eq. (4).

Apply trigonometry identity \(\cos(x) - \sin(x) = \sqrt{2} \sin\left(x + \frac{3}{4}\pi\right)\) to \(\text{Re}(\eta(s)) + \text{Im}(\eta(s))\) to get Eq. (5) with terms in last line built by mixture of terms from \(\text{Re}(\eta(s))\) and \(\text{Im}(\eta(s))\).

\[
\sum_{n=1}^{\infty} \left((2n-1) - \sigma \sin(\ln(2n-1)) - (2n-1) - \sigma \sin(\ln(2n-1))\right)
- (2n)^{-\sigma} \cos(\ln(2n)) + (2n)^{-\sigma} \sin(\ln(2n))
= \sum_{n=1}^{\infty} \left((2n-1)^{-\sigma} \sqrt{2} \sin(\ln(2n-1) + \frac{3}{4}\pi) - (2n)^{-\sigma} \sqrt{2} \sin(\ln(2n) + \frac{3}{4}\pi)\right)
\]

When depicted in terms of Eq. (4), Eq. (5) becomes

\[
\sum_{n=1}^{\infty} (2n)^{-\sigma} \sqrt{2} \sin(\ln(2n) + \frac{3}{4}\pi) = \sum_{n=1}^{\infty} (2n-1)^{-\sigma} \sqrt{2} \sin(\ln(2n-1) + \frac{3}{4}\pi)
- \sum_{n=1}^{\infty} (2n-1)^{-\sigma} \sqrt{2} \sin(\ln(2n-1) + \frac{3}{4}\pi) = 0
\]

Eq. (6) in discrete (summation) format is a non-Hybrid integer sequence equation – see Appendix C. \(\eta(s)\) calculations for all \(\sigma\) values result in infinitely many non-Hybrid integer sequence equations for \(0 < \sigma < 1\) critical strip region of interest with \(n = 1, 2, 3, 4, 5, \ldots, \infty\) as discrete integer number values, or \(n = 1\) to \(\infty\) as continuous real numbers values with Riemann integral application. These equations will geometrically represent entire plane of critical strip, thus (at least) allowing our proposed proof to be of a complete nature.

Eq. (6) being the "simplified" Dirichlet eta function derived directly from \(\eta(s)\) will intrinsically incorporate actual location [but not actual positions] of all nontrivial zeros. The proof is now complete for Lemma 3.1 □.

**Proposition 3.2.** Dirichlet Sigma-Power Law in continuous (integral) format given as equation and inequation can both be derived directly from "simplified" Dirichlet eta function in discrete (summation) format with Riemann integral application. [Note: Dirichlet Sigma-Power Law in continuous (integral) format here refers to the end-product obtained from "first key step of converting Riemann zeta function into its continuous format version".]

**Proof.** In Calculus, integration is reverse process of differentiation viewed geometrically as area enclosed by curve of function and x-axis. Apply definite integral \(I\) between points a and b is to compute its value when \(\Delta x \rightarrow 0\), i.e. \(I = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^{\infty} f(x_i) \Delta x = \int_a^b f(x) dx\). This is Riemann integral of function \(f(x)\) in interval [a, b] where \(a < b\). Apply Riemann integral to "simplified" Dirichlet eta function in \([\Delta x \rightarrow 1]\) discrete (summation) format which intrinsically incorporates actual location [but not actual positions] of all nontrivial zeros criterion to obtain Dirichlet Sigma-Power Law in \([\Delta x \rightarrow 0]\) continuous (integral) format with the later validly representing the former. Then Dirichlet Sigma-Power Law will also fulfill this criterion. Due to resemblance to power law functions in \(\sigma\) from \(s = \sigma + it\) being exponent of a power function \(n^\sigma\), logarithm scale use, and harmonic \(\zeta(s)\) series connection in Zipf’s law; we elect to call this Law by its given name. A characteristic and crucial step of this Law is its exact formula expression in usual mathematical language \(y = f(x_1, x_2)\) format description for a 2-variable function with \((2n)\) and \((2n - 1)\) parameters consist of \(y = f(t, \sigma)\) with discrete \(n = 1, 2, 3, 4, 5, \ldots, \infty\) or continuous \(n = 1\) to \(\infty\); \(-\infty < t < +\infty\); and \(0 < \sigma < 1\).

With steps of manual integration shown using indefinite integrals [for simplicity], solve definite integral below based on numerator portion of R1 with \((2n)\) parameter in Eq. (6):

\[
\int_1^\infty \frac{2^{-\sigma} \sin(\ln(2n) + \frac{3}{4}\pi)}{n^\sigma} dn = \int_1^\infty \frac{-\cos(\ln(2n))}{2^\sigma n^\sigma} dn. \text{ We deduce most other important integrals to be}
\]
“variations” of this particular integral containing (i) deletion of $(2n)^{-\sigma}$, $\sqrt{2}$ or $\frac{3}{4}\pi$ terms, and/or (ii) interchange of sine and cosine function. We check all derived antiderivatives to be correct using computer algebra system Maxima.

Simplifying and applying linearity, we obtain $2^{1-\sigma}\int \sin \left( t \ln (2n) + \frac{3\pi}{4} \right) \frac{dn}{n^\sigma}$.

Now solving $\int \sin \left( t \ln (2n) + \frac{3\pi}{4} \right) \frac{dn}{n^\sigma}$. Substitute $u = t \ln (2n) + \frac{3\pi}{4}$, $du = \frac{n}{t} \, dn$, 

use $n^{1-\sigma} = e^{(\frac{1}{1-\sigma})(u - t \ln (2n) + \frac{3\pi}{4})}$, $n^{1-\sigma} = e^{\frac{e^{(\frac{1}{1-\sigma})(u - t \ln (2n) + \frac{3\pi}{4})}}{\sigma}} \int e^{\frac{1}{1-\sigma}} \sin (u) \, du$.

Now solving $\int e^{\frac{1}{1-\sigma}} \sin (u) \, du$. We integrate by parts twice in a row: $\int f'g = fg - \int fg'$. First time: $f = \sin (u)$, $g' = e^{\frac{1}{1-\sigma}}$.

Then $f' = -\sin (u)$, $g = \frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}}}{\sigma^2 - 2\sigma + 1}$, 

$= \frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}} \sin (u)}{\sigma^2 - 2\sigma + 1} - \int \frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}} \cos (u)}{\sigma^2 - 2\sigma + 1} \, du$.

Second time: $f = \cos (u)$, $g' = \frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}}}{\sigma^2 - 2\sigma + 1}$.

Then $f' = -\sin (u)$, $g = \frac{t^2 e^{\frac{1}{1-\sigma}} \cos (u)}{\sigma^2 - 2\sigma + 1}$, 

$= \frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}} \sin (u)}{\sigma^2 - 2\sigma + 1} - \int \frac{t^2 e^{\frac{1}{1-\sigma}} \sin (u)}{\sigma^2 - 2\sigma + 1} \, du$.

Apply linearity: 

$= \frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}} \sin (u)}{\sigma^2 - 2\sigma + 1} - \int \frac{t^2 e^{\frac{1}{1-\sigma}} \cos (u)}{\sigma^2 - 2\sigma + 1} \, du$.

As integral $\int e^{\frac{1}{1-\sigma}} \sin (u) \, du$ appears again on Right Hand Side, we solve for it:

$= \frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}} \sin (u)}{\sigma^2 - 2\sigma + 1} - \int e^{\frac{1}{1-\sigma}} \sin (u) \, du$.

Plug in solved integrals: $\int e^{\frac{1}{1-\sigma}} \sin (u) \, du$ 

$= e^{\frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}} \sin (u)}{\sigma^2 - 2\sigma + 1}} - e^{\frac{1}{1-\sigma}} \cos (u)$.

Undo substitution $u = t \ln (2n) + \frac{3\pi}{4}$ and simplifying:

$= \left( \frac{(\sigma^2 - 2\sigma + 1) t}{t} \right) \left( e^{\frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}} \sin (t \ln (2n) + \frac{3\pi}{4})}{\sigma^2 - 2\sigma + 1}} - e^{\frac{1}{1-\sigma}} \cos (t \ln (2n) + \frac{3\pi}{4}) \right)$.

Plug in solved integrals: $2^{1-\sigma} \int \sin \left( t \ln (2n) + \frac{3\pi}{4} \right) \frac{dn}{n^\sigma}$ 

$= 2^{1-\sigma} \left( \frac{(\sigma^2 - 2\sigma + 1) t}{t} \right) \left( e^{\frac{(1 - \sigma) t e^{\frac{1}{1-\sigma}} \sin (t \ln (2n) + \frac{3\pi}{4})}{\sigma^2 - 2\sigma + 1}} - e^{\frac{1}{1-\sigma}} \cos (t \ln (2n) + \frac{3\pi}{4}) \right)$.

By rewriting and simplifying, $\int_{1}^{\infty} 2^{1-\sigma} \sin \left( t \ln (2n) + \frac{3\pi}{4} \right) \frac{dn}{n^\sigma}$ is finally solved as 

$$\left[ (2n)^{1-\sigma} ((t + \sigma - 1) \sin (t \ln (2n)) + (t - \sigma + 1) \cos (t \ln (2n))) \right]_{1}^{\infty}$$

For denominator portion of R1 with $(2n - 1)$ parameter in Eq. (6), Eq. (7) equates to
We illustrate this for Eq. (12) as expanded antiderivative [depicted as linear combination of sine and cosine waves:

\[
\left[ (2n - 1)^{1-\sigma} \left( (t + \sigma - 1) \sin(\ln(2n - 1)) + (t - \sigma + 1) \cos(\ln(2n - 1)) \right) \right]_1^\infty + C
\]

Dirichlet Sigma-Power Law as equation derived from Eq. (6) is given by:

\[
\frac{1}{2 \left( t^2 + (\sigma - 1)^2 \right)} \cdot [(2n)^{1-\sigma} \left( (t + \sigma - 1) \sin(\ln(2n)) + (t - \sigma + 1) \cos(\ln(2n)) \right) - (2n - 1)^{1-\sigma} \left( (t + \sigma - 1) \sin(\ln(2n - 1)) + (t - \sigma + 1) \cos(\ln(2n - 1)) \right) ]_1^\infty = 0
\]  

(9)

Apply Ratio Study to Eq. (6) – see Appendix B. This involves [intentional] incorrect but "balanced" rearrangement of terms in Eq. (6) giving rise to Eq. (10) which is a non-Hybrid integer sequence inequation. Left-hand side contains 'cyclical' sine function in first term (Ratio R1) and 'non-cyclical' power function in second term (Ratio R2).

\[
\sum_{n=1}^\infty \frac{\sqrt{2} \sin(t \ln(2n) + \frac{3\pi}{4})}{(2n)^{\sigma}} - \sum_{n=1}^\infty \frac{(2n)^{\sigma}}{(2n - 1)^{\sigma}} \neq 0
\]  

(10)

Apply Riemann integral to selected parts of Eq. (10) without depicting steps of calculation:

\[
\int_1^\infty \frac{\sqrt{2} \sin(t \ln(2n) + \frac{3\pi}{4})}{(2n)^{\sigma}} \, dn = \frac{(2n)^{(t-1)} \sin(t \ln(2n)) + (t + 1) \cos(t \ln(2n)))}{(2(t^2 + 1))^{\sigma} + C} \bigg|_1^\infty
\]

and

\[
\int_1^\infty \frac{\sqrt{2} \sin(t \ln(2n - 1) + \frac{3\pi}{4})}{(2n - 1)^{\sigma}} \, dn = \frac{(2n - 1)^{(t-1)} \sin(t \ln(2n - 1)) + (t + 1) \cos(t \ln(2n - 1)))}{(2(t^2 + 1))^{\sigma} + C} \bigg|_1^\infty
\]

Dirichlet Sigma-Power Law as inequation derived from Eq. (10) is given by:

\[
\int_1^\infty \frac{(2n)^{\sigma}}{(2n)^{\sigma + 1}} \, dn = \frac{(2n - 1)^{(t-1)} \sin(t \ln(2n - 1)) + (t + 1) \cos(t \ln(2n - 1)))}{(2(t^2 + 1))^{\sigma} + C} \bigg|_1^\infty
\]

and

\[
\int_1^\infty \frac{(2n - 1)^{\sigma}}{(2n - 1)^{\sigma + 1}} \, dn = \frac{(2n - 1)^{(t-1)} \sin(t \ln(2n - 1)) + (t + 1) \cos(t \ln(2n - 1)))}{(2(t^2 + 1))^{\sigma} + C} \bigg|_1^\infty
\]

(11)

Intended derivation of Dirichlet Sigma-Power Law as equation and inequation have been successful. The proof is now complete for Proposition 3.2.

**Proposition 3.3.** Exact Dimensional analysis homogeneity at \( \sigma = \frac{1}{2} \) in Dirichlet Sigma-Power Law as equation and inequation is (respectively) indicated by \( \sum (all\ fractional\ exponents) = whole\ number\ '1' \ and '3'.

**Proof.** Dirichlet Sigma-Power Law as equation for \( \sigma = \frac{1}{2} \) value is given by:

\[
\frac{1}{2t^2 + \frac{1}{2}} \cdot [(2n)^{\frac{1}{2}} \left( (t - \frac{1}{2}) \sin(\ln(2n)) + (t + \frac{1}{2}) \cos(\ln(2n)) \right) - (2n^{\frac{1}{2}}) \left( (t - \frac{1}{2}) \sin(\ln(2n - 1)) + (t + \frac{1}{2}) \cos(\ln(2n - 1)) \right) ]_1^\infty = 0
\]  

(12)

Respectively evaluation of definite integrals Eq. (12), Eq. (24) and Eq. (26) using limit as \( n \to +\infty \) for \( 0 < t < +\infty \) enable countless computations resulting in \( t \) values for CIS of nontrivial zeros, Gram[y=0] points and Gram[x=0] points. We illustrate this for Eq. (12) as expanded antiderivative [depicted as linear combination of sine and cosine waves: 

\[
a \sin x + b \cos x = c \sin(x + \varphi) \text{ with } c = \sqrt{a^2 + b^2} \text{ and } \varphi = \arctan\left(\frac{b}{a}\right) \text{ for } a > 0.
\]

\[
(2\infty)^{\frac{1}{2}} \sin\left( t \ln(2\infty) + \tan^{-1}\left(\frac{t + \frac{1}{2}}{t - \frac{1}{2}}\right) \right) - (2\infty - 1)^{\frac{1}{2}} \sin\left( t \ln(2\infty - 1) + \tan^{-1}\left(\frac{t + \frac{1}{2}}{t - \frac{1}{2}}\right) \right)
\]
\[ -2^{\frac{1}{2}} \sin \left( (t \ln 2 + \tan^{-1} \left( \frac{t + \frac{1}{2}}{t - \frac{1}{2}} \right) \right) + \frac{t + \frac{1}{2}}{2t^2 + \frac{1}{2}} = 0 \]

(2\infty) and (2\infty-1) involve \( \frac{1}{2} \), sin and ln functions. At relevant t values for all nontrivial zeros, (first term - second term) = (- third term + fourth term).

Dirichlet Sigma-Power Law as inequation for \( \sigma = \frac{1}{2} \) value is given by:

\[
\sum \frac{(2n) \left( (t - 1) \sin (t \ln (2n)) + (t + 1) \cos (t \ln (2n)) \right)}{(2n - 1) \left( (t - 1) \sin (t \ln (2n - 1)) + (t + 1) \cos (t \ln (2n - 1)) \right)} - \frac{(2n)^2}{(2n - 1)^2} \bigg|_{1}^{\infty} \neq 0 \tag{13}
\]

\( \sum \) (all fractional exponents) as 2(1 - \( \sigma \)) = whole number '1' for Eq. (12) and 2(\( \sigma + 1 \)) = whole number '3' for Eq. (13). These findings signify presence of complete set nontrivial zeros for Eq. (12) and Eq. (13). The proof is now complete for Proposition 3.3.]

Corollary 3.4. Inexact Dimensional analysis homogeneity at \( \sigma \neq \frac{1}{2} \) [illustrated using \( \sigma = \frac{3}{5} \)] in Dirichlet Sigma-Power Law as equation and inequation is (respectively) indicated by \( \sum \) (all fractional exponents) = fractional number '1' and '3'.

Proof. Dirichlet Sigma-Power Law as equation for \( \sigma = \frac{3}{5} \) value is given by:

\[
\frac{1}{2t^2 + \frac{18}{25}} \cdot \left( (t - \frac{3}{5}) \sin (t \ln (2n)) + (t + \frac{3}{5}) \cos (t \ln (2n)) \right) - (2n - 1)^2 \left( (t - \frac{3}{5}) \sin (t \ln (2n - 1)) + (t + \frac{3}{5}) \cos (t \ln (2n - 1)) \right) \bigg|_{1}^{\infty} = 0 \tag{14}
\]

Dirichlet Sigma-Power Law as inequation for \( \sigma = \frac{3}{5} \) value is given by:

\[
\sum \frac{(2n) \left( (t - 1) \sin (t \ln (2n)) + (t + 1) \cos (t \ln (2n)) \right)}{(2n - 1) \left( (t - 1) \sin (t \ln (2n - 1)) + (t + 1) \cos (t \ln (2n - 1)) \right)} - \frac{(2n)^2}{(2n - 1)^2} \bigg|_{1}^{\infty} \neq 0 \tag{15}
\]

\( \sum \) (all fractional exponents) as 2(1 - \( \sigma \)) = fractional number '1' for Eq. (14) and 2(\( \sigma + 1 \)) = fractional number '3' for Eq. (15). These findings signify absence of complete set nontrivial zeros for Eq. (14) and Eq. (15). The proof is now complete for Corollary 3.4.

4. Rigorous proof for Riemann hypothesis summarized as Theorem Riemann I – IV

\[ \zeta(s) = \frac{1}{s - 1} + \frac{1}{s + 2} + 2 \int_{0}^{\infty} \frac{\sin(s \arctan t)}{(1 + t^2) (e^{2\pi t} - 1)} \, dt \]

is integral relation (cf. Abel-Plana summation formula [3][4]) for all \( s \in \mathbb{C} \) and \( s \neq 1 \). This integral is insufficient for our purpose as it involves integration with respect to \( t \) instead of \( n \) for \( \zeta(s) \) [instead of \( \eta(s) \)]. Rigorous proof for Riemann hypothesis is summarized by Theorem Riemann I – IV. One could obtain this proof with only using Dirichlet Sigma-Power Law [solely] as equation. For completeness and clarification of this proof, we supply following important mathematical arguments.

For \( 0 < \sigma < 1 \), then \( 0 < 2(1 - \sigma) < 2 \). The only whole number between 0 and 2 is '1' which coincide with \( \sigma = \frac{1}{2} \). When \( 0 < \sigma < \frac{1}{2} \) and \( \frac{1}{2} < \sigma < 1 \), then \( 0 < 2(1 - \sigma) < 1 \) and \( 1 < 2(1 - \sigma) < 2 \).

For \( 0 < \sigma < 1 \), \( 2 < (\sigma + 1) < 4 \). The only whole number between 2 and 4 is '3' which coincide with \( \sigma = \frac{1}{2} \). When \( 0 < \sigma < \frac{1}{2} \) and \( \frac{1}{2} < \sigma < 1 \), then \( 2 < (\sigma + 1) < 3 \) and \( 3 < (\sigma + 1) < 4 \).

Legend: \( R = \) all real numbers. For \( 0 < \sigma < 1 \), \( \sigma \) consist of \( 0 < R < 1 \). For \( 0 < 2(1 - \sigma) < 2 \) and \( 2 < (\sigma + 1) < 2 \) and \( 2 < 4 < 4 \) and \( 2 < 1 - \sigma < 1 \). An important caveat is that previously used phrases such as "fractional exponent \( \sigma \)" and "\( \sum \) (all fractional exponents) = whole number '1' [or '3'] and fractional number '1' [or '3']", although not incorrect per se, should respectively be replaced by "real number exponent \( \sigma \)" and "\( \sum \) (all real number exponents) = whole number '1' [or '3'] and real number '1' [or '3']" for complete accuracy. We apply this caveat to Theorem Riemann I – IV.

Footnote 5: As whole numbers \( c \) real numbers, one could also depict this phrase as "\( \sum \) (all real number exponents) = real number '1' [or '3'] and real number '1' [or '3']".
Theorem Riemann I. Derived from proxy Dirichlet eta function, "simplified" Dirichlet eta function will exclusively contain de novo property for actual location [but not actual positions] of all nontrivial zeros.

**Proof.** The phrase "actual location [but not actual positions] of all nontrivial zeros" can be validly shortened to "actual location of all nontrivial zeros" as used in Theorem Riemann II, III and IV. The proof for Theorem Riemann I is now complete as it successfully incorporates proof for Lemma 3.1.\(\Box\).

Theorem Riemann II. Dirichlet Sigma-Power Law [in continuous (integral) format] as equation and inequation which are both derived from "simplified" Dirichlet eta function [in discrete (summation) format] will exclusively manifest exact DA homogeneity in equation and inequation only when real number exponent \(\sigma = \frac{1}{2}\).

**Proof.** The proof for Theorem Riemann II is now complete as it successfully incorporates proofs from Proposition 3.2 on derivation for equation and inequation of Dirichlet Sigma-Power Law [with both containing de novo property for "actual location of all nontrivial zeros"] and Proposition 3.3 on manifestation of exact DA homogeneity in Dirichlet Sigma-Power Law as equation and inequation when real number exponent \(\sigma = \frac{1}{2}\).\(\Box\).

Theorem Riemann III. Real number exponent \(\sigma = \frac{1}{2}\) in Dirichlet Sigma-Power Law as equation and inequation satisfying exact DA homogeneity is identical to \(\sigma\) variable in Riemann hypothesis which propose \(\sigma\) to also have exclusive value of \(\frac{1}{2}\) (representing critical line) for "actual location of all nontrivial zeros", thus fully supporting Riemann hypothesis to be true with further clarification by Theorem Riemann IV.

**Proof.** Since \(s = \sigma \pm it\), complete set of nontrivial zeros which is defined by \(\eta(s) = 0\) is exclusively associated with one (and only one) particular \(\eta(\sigma \pm it) = 0\) value solution, and by default one (and only one) particular \(\sigma\) [conjecturally] \(= \frac{1}{2}\) value solution. When performing exact DA homogeneity on Dirichlet Sigma-Power Law as equation and inequation [with both containing de novo property for "actual location of all nontrivial zeros"], the phrase "If real number exponent \(\sigma\) has exclusively \(\frac{1}{2}\) value, only then will exact DA homogeneity be satisfied" implies one (and only one) possible mathematical solution. Theorem Riemann III reflect Theorem Riemann II on presence of exact DA homogeneity for \(\sigma = \frac{1}{2}\) in Dirichlet Sigma-Power Law as equation and inequation. This Law has identical \(\sigma\) variable as that referred to by Riemann hypothesis [whereby \(\sigma\) here uniquely refer to critical line]. The proof for Theorem Riemann III is now complete as it independently refers to simultaneous association of confirmed (i) solitary \(\sigma = \frac{1}{2}\) value in Dirichlet Sigma-Power Law as equation and inequation satisfying exact DA homogeneity and (ii) critical line defined by solitary \(\sigma = \frac{1}{2}\) value being the "actual location [but with no request to determine actual positions]" of all nontrivial zeros as proposed in original Riemann hypothesis.\(\Box\).

Theorem Riemann IV. Condition 1. All \(\sigma \neq \frac{1}{2}\) values (non-critical lines), viz. \(0 < \sigma < \frac{1}{2}\) and \(\frac{1}{2} < \sigma < 1\) values, exclusively does not contain "actual location of all nontrivial zeros" [manifesting de novo inexact DA homogeneity in equation and inequation], together with Condition 2. One (and only one) \(\sigma = \frac{1}{2}\) value (critical line) exclusively contains "actual location of all nontrivial zeros" [manifesting de novo exact DA homogeneity in equation and inequation], fully support Riemann hypothesis to be true when these two mutually inclusive conditions are met.

**Proof.** Condition 2 Theorem Riemann IV simply reflect proof from Theorem Riemann III [incorporating Proposition 3.3] for "actual location of all nontrivial zeros" exclusively on critical line manifesting de novo exact DA homogeneity \(\Sigma\)(all real number exponents) \(=\) whole number ‘1’ for equation [or ‘3’ for inequation]. The proof for Condition 2 Theorem Riemann IV is now complete.\(\Box\). Corollary 3.4 confirms de novo inexact DA homogeneity manifested as \(\Sigma\)(all real number exponents) \(=\) real number ‘\(\neq1\)’ for equation [or ‘\(\neq3\)’ for inequation] by all \(\sigma \neq \frac{1}{2}\) values (non-critical lines) that are exclusively not associated with "actual location of all nontrivial zeros". Applying inclusion-exclusion principle: Exclusive presence of nontrivial zeros on critical line for Condition 2 Theorem Riemann IV implies exclusive absence of nontrivial zeros on non-critical lines for Condition 1 Theorem Riemann IV. The proof for Condition 1 Theorem Riemann IV is now complete.\(\Box\).

We logically deduce that explicit mathematical explanation why presence and absence of nontrivial zeros should (respectively) coincide precisely with \(\sigma = \frac{1}{2}\) and \(\sigma \neq \frac{1}{2}\) [literally the Completely Predictable meta-properties (’overall’ complex properties)] will require "complex" mathematical arguments. Attempting to provide explicit mathematical explanation with "simple" mathematical arguments would intuitively mean nontrivial zeros have to be (incorrectly and impossibly) treated as Completely Predictable entities.

**Footnote 6:** Completely Predictable meta-properties for Gram and virtual Gram points equating to "Presence of Gram[y=0] and Gram[x=0] points, and virtual Gram[y=0] and virtual Gram[x=0] points (respectively) coincide precisely with \(\sigma = \frac{1}{2}\), and \(\sigma \neq \frac{1}{2}\)."
5. Prerequisite lemma, corollary and propositions for Gram[x=0] and Gram[y=0] conjectures

For Gram[y=0] and Gram[x=0] points (and corresponding virtual Gram[y=0] and virtual Gram[x=0] points with totally different values), we apply a parallel procedure carried out on nontrivial zeros but only depict abbreviated treatments and discussions.

Lemma 5.1. "Simplified" Gram[y=0] and Gram[x=0] points-Dirichlet eta functions are derived directly from Dirichlet eta function with Euler formula application and (respectively) they will intrinsically incorporate actual location [but not actual positions] of all Gram[y=0] and Gram[x=0] points.

Proof. For Gram[y=0] points, the equivalent of Eq. (4) and Eq. (6) are respectively given by Eq. (16) and Eq. (17) below.

\[ \sum_{n=1}^{\infty} (2n)^{-\sigma} \sin(t \ln(2n)) = \sum_{n=1}^{\infty} (2n - 1)^{-\sigma} \sin(t \ln(2n - 1)) \]

\[ \sum_{n=1}^{\infty} (2n)^{-\sigma} \sin(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\sigma} \sin(t \ln(2n - 1)) = 0 \]

For Gram[x=0] points, the equivalent of Eq. (4) and Eq. (6) are respectively given by Eq. (18) and Eq. (19) below.

\[ \sum_{n=1}^{\infty} (2n)^{-\sigma} \cos(t \ln(2n)) = \sum_{n=1}^{\infty} (2n - 1)^{-\sigma} \cos(t \ln(2n - 1)) \]

\[ \sum_{n=1}^{\infty} (2n)^{-\sigma} \cos(t \ln(2n)) - \sum_{n=1}^{\infty} (2n - 1)^{-\sigma} \cos(t \ln(2n - 1)) = 0 \]

Eq. (17) and Eq. (19) being the "simplified" Gram[y=0] and Gram[x=0] points-Dirichlet eta functions derived directly from \( \eta(s) \) will intrinsically incorporate actual location [but not actual positions] of (respectively) all Gram[y=0] and Gram[x=0] points. The proof is now complete for Lemma 5.1. \( \Box \).

Proposition 5.2. Gram[y=0] and Gram[x=0] points-Dirichlet Sigma-Power Laws in continuous (integral) format given as equations and inequations can both be (respectively) derived directly from "simplified" Gram[y=0] and Gram[x=0] points-Dirichlet eta functions in discrete (summation) format with Riemann integral application. [Note: Gram[y=0] and Gram[x=0] points-Dirichlet Sigma-Power Laws in continuous (integral) format here refers to the end-products obtained from "first key step of converting Riemann zeta function into its continuous format version".]

Proof. Antiderivatives below using \( (2n) \) parameter help obtain all subsequent equations: first two for Gram[y=0] points and second two for Gram[x=0] points.

\[ \int_{1}^{\infty} (2n)^{-\sigma} \sin(t \ln(2n)) \, dn = \left[ \frac{(2n)^{1-\sigma} (t \sin(t \ln(2n)) + t \cos(t \ln(2n)))}{2(t^2 + (\sigma - 1)^2)} + C \right]_{1}^{\infty} \]

\[ \int_{1}^{\infty} \sin(t \ln(2n)) \, dn = \left[ \frac{(2n)(t \sin(t \ln(2n)) - t \cos(t \ln(2n)))}{2(t^2 + 1)} + C \right]_{1}^{\infty} \]

\[ \int_{1}^{\infty} (2n)^{-\sigma} \cos(t \ln(2n)) \, dn = \left[ \frac{(2n)^{1-\sigma} (t \sin(t \ln(2n)) - (\sigma - 1) \cos(t \ln(2n)))}{2(t^2 + (\sigma - 1)^2)} + C \right]_{1}^{\infty} \]

\[ \int_{1}^{\infty} \cos(t \ln(2n)) \, dn = \left[ \frac{(2n)(t \sin(t \ln(2n)) + \cos(t \ln(2n)))}{2(t^2 + 1)} + C \right]_{1}^{\infty} \]

For Gram[y=0] points-Dirichlet Sigma-Power Law, the equivalent of Eq. (9) and Eq. (11) are respectively given by Eq.
Proof for Riemann hypothesis and Explanations for Gram points

(20) as equation and Eq. (21) as inequation.

\[
- \frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot [(2n)^{1-\sigma} ((\sigma - 1) \sin (t \ln (2n)) + t \cos (t \ln (2n))) - \\
(2n - 1)^{1-\sigma} ((\sigma - 1) \sin (t \ln (2n - 1)) + t \cos (t \ln (2n - 1))))]_1^\infty = 0 \tag{20}
\]

(21)

\[
\left[ \frac{(2n) (t \sin (t \ln (2n))) - t \cos (t \ln (2n)))}{(2n - 1)(t \sin (t \ln (2n - 1)) - t \cos (t \ln (2n - 1)))} - \frac{(2n)^{\sigma + 1}}{(2n - 1)^{\sigma + 1}} \right]_1^\infty \neq 0
\]

For Gram[x=0] points-Dirichlet Sigma-Power Law, the equivalent of Eq. (9) and Eq. (11) are respectively given by Eq. (22) as equation and Eq. (23) as inequation.

\[
- \frac{1}{2(t^2 + (\sigma - 1)^2)} \cdot [(2n)^{1-\sigma} (t \sin (t \ln (2n))) - (\sigma - 1) \cos (t \ln (2n))) - \\
(2n - 1)^{1-\sigma} (t \sin (t \ln (2n - 1))) - (\sigma - 1) \cos (t \ln (2n - 1)])]_1^\infty = 0 \tag{22}
\]

\[
\left[ \frac{(2n) (t \sin (t \ln (2n))) + \cos (t \ln (2n)))}{(2n - 1)(t \sin (t \ln (2n - 1))) + \cos (t \ln (2n - 1)))} - \frac{(2n)^{\sigma + 1}}{(2n - 1)^{\sigma + 1}} \right]_1^\infty \neq 0 \tag{23}
\]

Intended derivation of Gram[y=0] and Gram[x=0] points-Dirichlet Sigma-Power Laws as equations and inequations is successful. The proof is now complete for Lemma 5.2.\(\Box\)

**Proposition 5.3.** Exact Dimensional analysis homogeneity at \(\sigma = \frac{1}{2}\) in Gram[y=0] and Gram[x=0] points-Dirichlet Sigma-Power Laws as equations and inequations are (respectively) indicated by \(\sum\) (all fractional exponents) = whole number ‘1’ and ‘3’.

**Proof.** Gram[y=0] points-Dirichlet Sigma-Power Law as equation for \(\sigma = \frac{1}{2}\) value is given by:

\[
- \frac{1}{2t^2 + \frac{1}{2}} \cdot [(2n)^{\frac{1}{2}} (t \cos (t \ln (2n)) - \frac{1}{2} \sin (t \ln (2n))) - \\
(2n - 1)^{\frac{1}{2}} (t \cos (t \ln (2n - 1)) - \frac{1}{2} \sin (t \ln (2n - 1))))]_1^\infty = 0 \tag{24}
\]

Gram[y=0] points-Dirichlet Sigma-Power Law as inequation for \(\sigma = \frac{1}{2}\) value is given by:

\[
\left[ \frac{(2n)(t \sin (t \ln (2n))) - t \cos (t \ln (2n)))}{(2n - 1)(t \sin (t \ln (2n - 1))) - t \cos (t \ln (2n - 1)))} - \frac{(2n)^{\frac{3}{2}}}{(2n - 1)^{\frac{3}{2}}}]_1^\infty \neq 0 \tag{25}
\]

Gram[x=0] points-Dirichlet Sigma-Power Law as equation for \(\sigma = \frac{1}{2}\) value is given by:

\[
\frac{1}{2t^2 + \frac{1}{2}} \cdot [(2n)^{\frac{1}{2}} (t \sin (t \ln (2n)) + \frac{1}{2} \cos (t \ln (2n))) - \\
(2n - 1)^{\frac{1}{2}} (t \sin (t \ln (2n - 1)) + \frac{1}{2} \cos (t \ln (2n - 1))))]_1^\infty = 0 \tag{26}
\]

Gram[x=0] points-Dirichlet Sigma-Power Law as inequation for \(\sigma = \frac{1}{2}\) value is given by:

\[
\left[ \frac{(2n)(t \sin (t \ln (2n))) + \cos (t \ln (2n)))}{(2n - 1)(t \sin (t \ln (2n - 1))) + \cos (t \ln (2n - 1)))} - \frac{(2n)^{\frac{3}{2}}}{(2n - 1)^{\frac{3}{2}}} \right]_1^\infty \neq 0 \tag{27}
\]
\[ \Sigma \text{(all fractional exponents)} = 2(1 - \sigma) = \text{whole number '1'} \text{ for Eqs. (24) and (26), and } 2(\sigma + 1) = \text{whole number '3'} \text{ for Eqs. (25) and (27). These findings signify presence of complete sets Gram[y=0] points for Eqs. (24) and (25) and Gram[x=0] points for Eqs. (26) and (27). The proof is now complete for Proposition 5.3.} \]

**Corollary 5.4.** Inexact Dimensional analysis homogeneity at \( \sigma \neq \frac{1}{7} \) for Eqs. (25) and (27). These findings signify presence of complete sets Gram[y=0] points-Dirichlet Sigma-Power Laws as equations and inequations are (respectively) indicated by \( \Sigma \text{(all fractional exponents)} = \text{fractional number '1' and '3'.} \)

**Proof.** Gram[y=0] points-Dirichlet Sigma-Power Law as equation for \( \sigma = \frac{3}{3} \) value is given by:

\[
- \frac{1}{2t^2 + \frac{18}{25}} \cdot [(2n)\left(t \cos(t \ln(2n)) - \frac{3}{5} \sin(t \ln(2n)) \right] - \\
(2n - 1)\left(t \cos(t \ln(2n - 1)) - \frac{3}{5} \sin(t \ln(2n - 1)) \right) \right]_1^{\infty} = 0
\]

(28)

Gram[y=0] points-Dirichlet Sigma-Power Law as inequation for \( \sigma = \frac{3}{3} \) value is given by:

\[
\left[ \frac{(2n)(\sin(t \ln(2n)) - t \cos(t \ln(2n)))}{(2n - 1)(\sin(t \ln(2n - 1)) - t \cos(t \ln(2n - 1)))} \right]_1^{\infty} \neq 0
\]

(29)

Gram[x=0] points-Dirichlet Sigma-Power Law as equation for \( \sigma = \frac{3}{3} \) value is given by:

\[
\frac{1}{2t^2 + \frac{18}{25}} \cdot [(2n)\left(t \sin(t \ln(2n)) + \frac{3}{5} \cos(t \ln(2n)) \right] - \\
(2n - 1)\left(t \sin(t \ln(2n - 1)) + \frac{3}{5} \cos(t \ln(2n - 1)) \right) \right]_1^{\infty} = 0
\]

(30)

Gram[x=0] points-Dirichlet Sigma-Power Law as inequation for \( \sigma = \frac{3}{3} \) value is given by:

\[
\left[ \frac{(2n)(t \sin(t \ln(2n)) + \cos(t \ln(2n)))}{(2n - 1)(t \sin(t \ln(2n - 1)) + \cos(t \ln(2n - 1)))} \right]_1^{\infty} \neq 0
\]

(31)

\( \Sigma \text{(all fractional exponents)} = 2(1 - \sigma) = \text{fractional number '1' for Eqs. (28) and (30), and } 2(\sigma + 1) = \text{fractional number '3' for Eqs. (29) and (31). These findings signify presence of complete sets virtual Gram[y=0] points for Eqs. (28) and (29) and virtual Gram[x=0] points for Eqs. (30) and (31). The proof is now complete for Corollary 5.4.} \]

6. Conclusions

We have introduce a new branch of mathematics coined "Mathematics for Incompletely Predictable Problems". In our Hybrid method of Integer Sequence classification – see Appendix C, a formula is either non-Hybrid or Hybrid integer sequence. Inequation with two 'necessary' Ratio (R) or equation with one 'unnecessary' R contains non-Hybrid integer sequence. Equation with one 'necessary' R contains Hybrid integer sequence. "In the limit" Hybrid integer sequence approach unique Position X, it becomes non-Hybrid integer sequence for all Positions \( \geq X \).

Consider kinetic energy (KE) in MJ with \( m_0 = \text{rest mass in kg and } v = \text{velocity in } \text{m/s}. \) In classical mechanics concerning low velocity with \( v << c \). Newtonian KE = \( \frac{1}{2}m_0v^2 \). In relativistic mechanics concerning high velocity with \( v \geq 0.01c \), Relativistic KE = \( \frac{m_0c^2}{\sqrt{1 - (v^2/c^2)}} - m_0c^2 \). Obtained from the later by binomial approximation or by taking first two terms of Taylor expansion for reciprocal square root, the former approximates the later well at low speed. We arbitrarily divide DA homogeneity into inexact DA homogeneity for ["<100% accuracy"] Newtonian KE and exact DA homogeneity for ["100% accuracy"] Relativistic KE. "In the limit" ["<100% accuracy"] Newtonian KE at low speed approach ["100% accuracy"] Relativistic KE at high speed, we achieve perfection.

Analogy: "In the limit" all three version of Dirichlet Sigma-Power Laws for Gram[y=0] points, Gram[x=0] points and nontrivial zeros as 'less than 100% accuracy' inequations approach perfection as '100% accuracy' equations, compliance with
inexact DA homogeneity becomes compliance with exact DA homogeneity. We note R1 terms in all inequations contain (2n) and (2n-1) 'base quantities' but these are not endowed with fractional exponent (σ+1) as relevant 'unit of measurement'. As Incompletely Predictable problems, we gave relatively elementary proof of Riemann hypothesis and explain two types of Gram points whereby various "meta-properties" such as exact and inexact DA homogeneity occur in (respectively) equations and inequations of relevant Dirichlet Sigma-Power Laws.

Acknowledgements The author is indebted to Mr. Rodney Williams (Civil Engineer & Mathematician), Mr. Tony O’Hagan (Software Engineer & Mathematician) for their expert review, constructive criticism and feedback on this paper. To the loving memory of Jasmine (and Grace) who had provided deep inspirations to many in 2015 (and 2016) and was a caring auntie (and grandmother) to Jelena, the author’s 27-weeker premature daughter born in 2012.

References

Appendix A Gram’s Law and traditional ’Gram points’

Named after Danish mathematician Jørgen Pedersen Gram (June 27, 1850 – April 29, 1916), traditional 'Gram points' (Gram[y=0] points) are other conjugate pairs values on critical line defined by \( \text{Im}[\zeta(\frac{1}{2} \pm it)] = 0 \). Belonging to Incompletely Predictable entities, they obey Gram’s Rule and Rosser’s Rule with interesting characteristic properties as outlined by our brief exposition below.

Z function is used to study Riemann zeta function on critical line. Defined in terms of Riemann-Siegel theta function & Riemann zeta function by \( Z(t) = e^{\theta(t)} \zeta(\frac{1}{2} + it) \) whereby \( \theta(t) = \arg(\Gamma(\frac{1+(2r+1)i}{2})) - \frac{\ln \pi}{2}t \); it is also called Riemann-Siegel Z function, Riemann-Siegel zeta function, Hardy function, Hardy Z function, & Hardy zeta function.

The algorithm to compute \( Z(t) \) is called Riemann-Siegel formula. Riemann zeta function on critical line, \( \zeta(\frac{1}{2} + it) \), will be real when \( \sin(\theta(t)) = 0 \). Positive real values of \( t \) where this occurs are called 'Gram points' and can also be described as points where \( \frac{\theta(t)}{\pi} \) is an integer. Real part of this function on critical line tends to be positive, while imaginary part alternates more regularly between positive & negative values. That means sign of \( Z(t) \) must be opposite to that of sine function most of the time, so one would expect nontrivial zeros of \( Z(t) \) to alternate with zeros of sine term, i.e. when \( \theta \) takes on integer multiples of \( \pi \). This turns out to hold most of the time and is known as Gram’s Rule (Law) – a law which is violated infinitely often though. Thus Gram’s Law is statement that nontrivial zeros of \( Z(t) \) alternate with 'Gram points'. 'Gram points' which satisfy Gram’s Law are called 'good', while those that do not are called 'bad'. A Gram block is an interval such that its very first & last points are good 'Gram points' and all 'Gram points' inside this interval are bad. Counting nontrivial zeros then reduces to counting all 'Gram points' where Gram’s Law is satisfied and adding the count of nontrivial zeros inside each Gram block. With this process we do not have to locate nontrivial zeros, and we just have to accurately compute \( Z(t) \) to show that it changes sign.

Appendix B Ratio Study and Inequations

A mathematical equation, containing one or more variables, is a statement that values of two ['left-hand side’ (LHS) and ‘right-hand side’ (RHS)] mathematical expressions is related as equality: LHS = RHS; or as inequalities: LHS < RHS, LHS > RHS, LHS ≤ RHS, or LHS ≥ RHS. A ratio is one mathematical expression divided by another. The term ‘unnecessary’ Ratio (R) for any given equation is explained by two examples: (1) LHS = RHS and with rearrangement, ‘unnecessary’ R is given by \( \frac{\text{LHS}}{\text{RHS}} = 1 \) or \( \frac{\text{RHS}}{\text{LHS}} = 1 \); and (2) LHS > RHS and with rearrangement, ’unnecessary’ R is given by \( \frac{\text{LHS}}{\text{RHS}} > 1 \) or \( \frac{\text{RHS}}{\text{LHS}} < 1 \).
Consider exponent $y \in \mathbb{R}$ values & base $x \in \mathbb{R}$ values for mathematical expression $x^y$. Equations such as $x^1 = x$, $x^0 = 1$ & $0^y = 0$ are all valid. Simultaneously letting both $x$ & $y = 0$ is an incorrect mathematical action because $x^y$ as function of two-variables is not continuous & is thus undefined at Origin. But if we elect to intentionally carry out this "balanced" action [equally] on $x$ & $y$, we obtain (simple) inequation $0^0 \neq 1$ with associated perpetual obeysance of '=' equality symbol in $x^y$ for all applicable $\mathbb{R}$ values except when both $x$ & $y = 0$. The Number '1' value in this inequation is justified by two arguments: I. Limit of $x^y$ value as both $x$ & $y$ tend to zero (from right) is 1 [thus fully satisfying criterion "$x^y$ is right continuous at the Origin"]; and II. Expression $x^y$ is product of $x$ with itself $y$ times [and thus $x^0$, the "empty product", should be 1 (no matter what value is given to $x$)].

Mathematical operator 'summation' must obey the law: We can break up a summation across a sum or difference but not across a product or quotient viz, factoring a sum of quotients into a corresponding quotient of sums is a incorrect mathematical action. But if we elect to carry out this action equally on LHS & RHS products or quotients in a suitable equation, we obtain two (unique) 'necessary' $R$ denoted by $R1$ for LHS and $R2$ for RHS whereby $R1 \neq R2$ relationship will always hold. We define 'Ratio Study' as intentionally performing this incorrect [but "balanced"] mathematical action on suitable equation [equivalent to one (non-unique) 'unnecessary' $R$] to obtain its inequation [equivalent to two (unique) 'necessary' $R$]. Set $\mathbb{C}$ is a field (but not an ordered field). Thus it is not possible to define a relation between two given $(z_1 & z_2) \in \mathbb{C}$ as $z_1 < z_2$ since inequality operation here is not compatible with addition and multiplication. But performing Ratio Study to obtain inequations involving $\mathbb{C}$ does not involve defining a relation between two $\mathbb{C}$.

**Appendix C Hybrid method of Integer Sequence classification**

Hybrid method of Integer Sequence classification enables meaningful division of all integer sequences into either Hybrid or non-Hybrid integer sequences. Our exotic A228186 integer sequence[5] was published on The On-line Encyclopedia of Integer Sequences website in 2013. It is the first ever [infinite length] Hybrid integer sequence synthesized from Combinatorics Ratio. In 'Position i' notation, let $i = 0, 1, 2, 3, 4, 5,..., \infty$ be complete set of natural numbers. A228186 "Greatest $k > n$ such that ratio $R < 2$ is a maximum rational number with $R = \frac{\text{CombinationsWithRepetition}}{\text{CombinationsWithoutRepetition}}$" is equal to [infinite length] non-Hybrid (usual garden-variety) integer sequence A100967[6] except for finite 21 'exceptional' terms at Positions 0, 11, 19, 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with their values given by relevant A100967 terms plus 1. The first 49 terms [from Position 0 to Position 48] of A100967 "Least $k$ such that $\text{binomial}(2k+1, k-n) \geq \text{binomial}(2k, k)$" are listed below: 3, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 225, 263, 303, 347, 393, 442, 494, 549, 606, 667, 730, 797, 866, 938, 1013, 1091, 1172, 1255, 1342, 1431, 1524, 1619, 1717, 1818, 1922, 2029, 2138, 2251, 2366, 2485, 2606, 2730, 2857, 2987, 3119, 3255, 3394, and 3535. For those 21 'exceptional' terms: at Position 0, A228186 (= 4) is given by A100967 (= 3) + 1; at Position 11, A228186 (= 226) is given by A100967 (= 225) + 1; at Position 13, A228186 (= 304) is given by A100967 (= 303) + 1; at Position 19, A228186 (= 607) is given by A100967 (= 606) + 1; etc. Here is a useful concept: Commencing from Position 0 onwards "in the limit" that this Position approaches 82, A228186 Hybrid integer sequence becomes (and is identical to) A100967 non-Hybrid integer sequence for all Positions $\geq 82$. 
