Coordinate Transformation and Static Charged Sphere in General Relativity

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Abstract
We consider a static charged sphere in general relativity. We make a coordinate transformation of a specific form. The electromagnetic energy-momentum tensor in the transformed coordinates is shown to be zero contrary to what is expected.

1 Electromagnetic potential and field

Let \( A_\mu(t, x, y, z) \) and \( g_{\mu\nu}(t, x, y, z) \) be the electromagnetic potential and metric tensor respectively. The electromagnetic field is

\[
F_{\mu\nu}(t, x, y, z) = A_{\nu,\mu}(t, x, y, z) - A_{\mu,\nu}(t, x, y, z)
\]

For a scalar function \( \phi(t, x, y, z) \) define

\[
\hat{A}_\mu(t, x, y, z) = A_\mu(t, x, y, z) + (g_{\mu\alpha}\phi,\alpha)(t, x, y, z)
\]

We have by (1) and (2)

\[
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = (A_\nu + \phi,\nu)_\mu - (A_\mu + \phi,\mu)_\nu = (g_{\nu\alpha}[A^\alpha + g^{\alpha\beta}\phi,\beta])_\mu - (g_{\mu\alpha}[A^\alpha + g^{\alpha\beta}\phi,\beta])_\nu = (g_{\nu\alpha}\hat{A}^\alpha)_\mu - (g_{\mu\alpha}\hat{A}^\alpha)_\nu
\]

2 Static charged sphere and Einstein field equations

Let there be a static charged sphere of total charge \( Q \) and mass \( M \) centred at the origin. Let the charge and mass densities be spherically symmetric. For this charged sphere let the metric \( g_{\mu\nu}(r) \) of isotropic coordinate form

\[
-a(r)dt^2 + b(r)(dx^2 + dy^2 + dz^2)
\]

satisfy the Einstein field equations

\[
G_{\mu\nu} = 8\pi \left[ g^{\sigma\tau}F_{\mu\sigma}F_{\nu\tau} - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}F_{\sigma\tau}F_{\alpha\beta} \right] + 8\pi T_{\mu\nu}
\]

where \( T_{\mu\nu}(r) \) is the energy-momentum tensor of matter. Require the electromagnetic energy-momentum tensor is not zero and

\[
A_0(r) = A_1(r) = A_2(r) = A_3(r) = 0
\]

Define \( h_{\mu\nu}(r) = g_{\mu\nu}(r) - \eta_{\mu\nu} \). Require \( rA_\mu(r) \) and \( rh_{\mu\nu}(r) \) have finite limits as \( r \) goes to infinity. Consequently \( r[a^{-1}(r) - 1] \) and \( r[b^{-1}(r) - 1] \) have finite limits as \( r \) goes to infinity. Require also for small \( Q \) and \( M \) that

\[
|A_0(r)| <<< 1 \quad |h_{\mu\nu}(r)| <<< 1
\]

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3 Coordinate transformation

Let
\[ \phi(t, x, y, z) = x \] (8)

hence by (2), (6), and (8)
\[ \hat{A}^0(r) = -(a^{-1}A_0)(r) \quad \hat{A}^1(r) = b^{-1}(r) \quad \hat{A}^2(r) = \hat{A}^3(r) = 0 \] (9)

Let \( Q \) and \( M \) be small so that \( b(r) \) is approximately one. Consider the transformation from \( x, y, z \) coordinates to \( x', y', z' \) coordinates given by
\[ x' = \int_0^x b(\sqrt{u^2 + y^2 + z^2})du \quad y' = y \quad z' = z \] (10)

From the inverse of this transformation define the function \( \varphi \) by \( x = \varphi(x', y', z') \). Define the coordinate transformation from \( t', x', y', z' \) coordinates to \( t, x, y, z \) coordinates by
\[ t = t' - \int_0^{x'} (a^{-1}A_0)(\sqrt{\varphi^2(u', y', z') + y'^2 + z'^2})du' \quad x = \varphi(x', y', z') \quad y = y' \quad z = z' \] (11)

The inverse of this transformation transforms \( \hat{A}^\mu(r) \) of (9) to \( \hat{A}^\mu(x', y', z') \) so that
\[ \hat{A}^{\alpha}(x', y', z') = 0 \quad \hat{A}^1(x', y', z') = 1 \quad \hat{A}^2(x', y', z') = 0 \quad \hat{A}^3(x', y', z') = 0 \] (12)

4 Size of metric perturbation

We have by (10) that
\[ \frac{\partial x}{\partial y'} = -y'b^{-1}(\sqrt{\varphi^2(x', y', z') + y'^2 + z'^2}) \int_0^{\varphi(x', y', z')} \frac{db}{db}(\sqrt{u'^2 + y'^2 + z'^2})du' \] (13)

Now \( r[b(r) - 1] \) and \( r[b^{-1}(r) - 1] \) have finite limits as \( r \) goes to infinity hence \( r^2(db/dr)(r) \) has finite limit as \( r \) goes to infinity. Consequently the integral is finite as \( x' \) goes to infinity and goes to zero as \( \sqrt{y'^2 + z'^2} \) goes to infinity. For small \( Q \) and \( M \) since \( b(r) - 1 \) is small we then have \( \partial x/\partial y' \) is small. We have by (11) that
\[ \frac{\partial t}{\partial y'} = -\int_0^{x'} \frac{d(a^{-1}A_0)}{dr}(\sqrt{\varphi^2(u', y', z') + y'^2 + z'^2})\varphi(u', y', z')\partial y' \] (14)

Now \( r(a^{-1}A_0)(r) \) has finite limit as \( r \) goes to infinity. Consequently \( r^2(d(a^{-1}A_0)/dr)(r) \) has a finite limit as \( r \) goes to infinity. Also we just showed \( \partial \varphi/\partial y' = \partial x/\partial y' \) is small for small \( Q \) and \( M \). Consequently the integral is finite as \( x' \) goes to infinity. Also \( \partial t/\partial y' \) will go to zero as \( \sqrt{y'^2 + z'^2} \) goes to infinity. For small \( Q \) and \( M \) we then have \( \partial t/\partial y' \) is small. Also we have
\[ \frac{\partial t}{\partial t'} = 1 \quad \frac{\partial t}{\partial x'} = -(a^{-1}A_0)(\sqrt{\varphi^2(x', y', z') + y'^2 + z'^2}) \quad \frac{\partial y}{\partial y'} = 1 \] (15)

We can then conclude for small \( Q \) and \( M \) that
\[ \left| \frac{\partial x^\mu}{\partial x'^\mu} - \delta^\mu_\nu \right| << 1 \] (16)
Now
\[ g'_{\mu\nu}(x',y',z') = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g_{\alpha\beta}(\sqrt{\varphi^2(x',y',z') + y'^2 + z'^2}) \] (17)
and define \( h'_{\mu\nu}(x',y',z') = g'_{\mu\nu}(x',y',z') - \eta_{\mu\nu} \). By (7) and (16) we have for small \( Q \) and \( M \) that
\[ |h'_{\mu\nu}(x',y',z')| << 1 \] (18)

5 Contradiction

We have by (3) transformed to \( t', x', y', z' \) coordinates and (12) that
\[ F'_{\mu\nu} = A'_{\nu,\mu} - A'_{\mu,\nu} = (g'_{\mu\alpha} \hat{A}^{\alpha}_{\nu})_{,\mu} - (g'_{\nu\alpha} \hat{A}^{\alpha}_{\mu})_{,\nu} = g'_{\nu\mu} - g'_{\mu\nu} = h'_{\nu\mu,\nu} - h'_{\mu\nu,\nu} \] (19)
Assuming the Principal of General Covariance and transforming (5) to \( t', x', y', z' \) coordinates and using (19) we have \( h'_{\mu\nu}(x',y',z') \) satisfies
\[ G'_{\mu\nu} = 8\pi g'^{\sigma\tau}[h'_{\sigma1,\mu} - h'_{\mu1,\sigma}][h'_{\tau1,\nu} - h'_{\nu1,\tau}] - 2\pi g'_{\mu\nu} g'^{\alpha\sigma} g'^{\beta\tau}[h'_{\tau1,\sigma} - h'_{\sigma1,\tau}][h'_{\beta1,\alpha} - h'_{\alpha1,\beta}] + 8\pi T'_{\mu\nu} \] (20)
By (18) and (20) we have \( h'_{\mu\nu}(x',y',z') \) approximately satisfies
\[ G'_{\mu\nu}(x',y',z') = 8\pi T'_{\mu\nu}(r') \] (21)
From (21) we can conclude that the electromagnetic energy-momentum tensor in \( t', x', y', z' \) coordinates is zero. This is a contradiction since we started with a charged sphere with nonzero electromagnetic energy-momentum tensor.

References