The Requirements on the Non-trivial Roots of the Riemann Zeta via the Dirichlet Eta Sum

William R. Blickos

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Abstract

An explanation of the Riemann Hypothesis is given in 8 parts, with the first being a statement of the problem. In the next 3 parts, the complex valued Dirichlet Eta sum, a known equivalence to Riemann Zeta in the critical strip, is split into 8 real valued sums and 2 constants. Part 5 explains a recursive relationship between the 8 sums. Section 6 shows that the sums must individually equal 0. Part 7 details the ratios of the system when all sums equal 0 at once. Finally, part 8 solves the system in terms of the original Dirichlet Eta sum inputs. The result shows that the only possible solution for the real portion of the complex input, commonly labeled a, is that it must equal 1/2, and thus proves Riemann's suspicion.

1 A Statement of the Problem and the General Approach to the Solution

The explanation begins with a well known version of the hypothesis based on the closely related Dirichlet Eta function. In that version, the Dirichlet Eta sum is given in a functional equation with the Riemann Zeta function, to analytically continue the domain of the Zeta function, and it is shown as equation 1.

$$\eta(s) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \left(1 - 2^{1-s}\right)\zeta(s) \tag{1}$$

Using the Dirichlet Eta sum, the hypothesis is commonly stated as "all the zeros of the Dirichlet eta function, falling in the critical strip $0 < \Re(s) < 1$, lie on the critical line $\Re(s) = 1/2$," where $\Re(s)$ is the real portion of the complex input s. That real portion is often labeled lower case a.

So what is the nature of the zeroes of the Eta function? The Eta function is an infinite sum of fractions, sometimes totaling to zero, where the denominator of that fraction sequence is the changing index of the sum raised to a complex valued power s. Small s is a standard complex number given as a + bi. The numerator of the sum's fraction also contains information, in this case, a negative 1 raised to a power involving the index. This causes the fraction to alternate. The goal, and challenge of the problem, is to explain why the value of a, in the domain between 0 to 1, must be 1/2, and only 1/2, in order for the entire infinite sum of fractions to sum to zero. This is stated as equation 2, still using the short hand complex number s.

$$\eta(s) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 0 \tag{2}$$

As stated in the paper's abstract, an explanation of the Riemann Hypothesis is now given in 8 parts, with the first being this section. In the next 3 parts, the complex valued Dirichlet Eta sum, a known equivalence of the Riemann Zeta within the critical strip, is split into 8 real valued sums and 2 constants. Part 5 explains a self referential functional relationship between the 8 sums. Section 6 shows that the sums must individually equal 0. Part 7 details the ratios of the system when all sums equal 0 at once. Finally, part 8 solves the system in terms of the input of the original Dirichlet Eta sum, and confirms Riemann's suspicion.

The first major step is separating the real and complex portions of the complex Eta sum, so that there is no longer a complex variable inside the sum, but rather 2 real valued sums instead.

2 Separating the Real and Complex Portions of the Complex Sum

Start by expressing s as the complex number a + bi, and then separate the exponents using exponent rules, equation 3.

$$n^s = n^{a+bi} = n^a n^{bi} \tag{3}$$

Then expand the complex exponent n^{bi} with Euler's formula. The result is shown in equation 4.

$$n^s = n^a \left(\cos\left(b \ln n\right) + i \sin\left(b \ln n\right)\right) \tag{4}$$

Put the expanded form back into equation 2, and express the numerator as a complex number, equation 5. Please note, that I also changed the n-1 to n+1 out of personal preference of convention, as I had preferred it while working the problem out on paper. This is allowed as it does not change any of the values indexed by the sum. That is, $(-1)^{n-1} = (-1)^{n+1}$ over the index.

$$\frac{(-1)^{n+1}}{n^s} = \frac{(-1)^{n+1} + 0i}{n^a \cos\left(b \ln n\right) + n^a \sin\left(b \ln n\right) i} \tag{5}$$

Then use the general formula for dividing complex numbers, equation 6, to carry out the division shown in 7.

$$\frac{u + vi}{x + yi} = \frac{(ux + vy) + (vx - uy)\, i}{x^2 + y^2} \tag{6}$$

$$\frac{(-1)^{n+1}+0i}{n^a\cos(b\ln n)+n^a\sin(b\ln n)\,i}=$$
$$\frac{(-1)^{n+1}n^a\cos(b\ln n)}{(n^a\cos(b\ln n))^2+(n^a\sin(b\ln n))^2}+\frac{0-(-1)^{n+1}n^a\sin(b\ln n)}{(n^a\cos(b\ln n))^2+(n^a\sin(b\ln n))^2}i \quad (7)$$

The result can be simplified via factoring the n^a and the sin squared plus cos squared in the denominator, and the complex input Dirichlet Eta sum can now be expressed as the sum-difference of 2 sums with only real inputs, equation 8.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(b\ln n)}{n^a}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(b\ln n)}{n^a}i=0 \quad (8)$$

Notice that the left sum is real valued and deals with cosines, and that the right sum, though still sitting in front of the imaginary number i, is real valued in magnitude and deals with sines. Since the Dirichlet Eta sum is a sum of complex numbers, the result is also complex, which is expected. Therefore, in order for the original complex Eta sum to equal zero, and thus have a root, both the real and complex parts of its total must be zero. That is, 0 + 0i.

After factoring out and dividing away a constant -1 from equation 8, the results are the 2 sums, equations 9 and 10, labeled A and B as follows.

$$A=\sum_{n=1}^{\infty}\frac{(-1)^n\cos(b\ln n)}{n^a} \quad (9)$$

A is referred to as the real portion of the complex Dirichlet Eta sum.

$$B=\sum_{n=1}^{\infty}\frac{(-1)^n\sin(b\ln n)}{n^a} \quad (10)$$

B is referred to as the complex portion of the complex Dirichlet Eta sum, though its magnitude is real valued.

Now, the task is to determine when these 2 new sums are both zero. To do that, they will need to be broken down further, and the first stage for such, is separating them each into their even and odd parts.

3 Separating the Even and Odd Portions of Both the Real and Complex Sums

Instead of using one sum for each of A and B as they are stated, and letting their indices n run over the full set of integers, use 2 sums for each, separating the even and odd inputs of the indices. Do this by separating n into 2n, for the evens, and into 2n-1, for the odds. This is shown for both A and B in equations 11 and 12.

$$A = \sum_{n=1}^{\infty} \frac{(-1)^n \cos(b \ln n)}{n^a} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} \cos(b \ln(2n-1))}{(2n-1)^a} + \sum_{n=1}^{\infty} \frac{(-1)^{2n} \cos(b \ln 2n)}{(2n)^a} = 0 \quad (11)$$

$$B = \sum_{n=1}^{\infty} \frac{(-1)^n \sin(b \ln n)}{n^a} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} \sin(b \ln(2n-1))}{(2n-1)^a} + \sum_{n=1}^{\infty} \frac{(-1)^{2n} \sin(b \ln 2n)}{(2n)^a} = 0 \quad (12)$$

The behavior of -1 raised to even or odd powers allows the right sides of equations 11 and 12 to be simplified, obtaining 13 and 14 respectively.

$$A = \sum_{n=1}^{\infty} \frac{\cos(b \ln 2n)}{(2n)^a} - \sum_{n=1}^{\infty} \frac{\cos(b \ln(2n-1))}{(2n-1)^a} = 0 \quad (13)$$

$$B = \sum_{n=1}^{\infty} \frac{\sin(b \ln 2n)}{(2n)^a} - \sum_{n=1}^{\infty} \frac{\sin(b \ln(2n-1))}{(2n-1)^a} = 0 \quad (14)$$

The sums involving 2n are known as the even portions, and the sums with 2n-1, the odd portions. Notice that due to how the -1 simplified in this form, that it is now the even sums minus the odd sums. Specifically labeling the 4 sums from equations 13 and 14 gives equations 15 through 18.

$$A_{even} = A_e = \sum_{n=1}^{\infty} \frac{\cos(b \ln 2n)}{(2n)^a} \quad (15)$$

A_e is referred to as the real even portion.

$$A_{odd} = A_o = \sum_{n=1}^{\infty} \frac{\cos(b \ln(2n-1))}{(2n-1)^a} \quad (16)$$

A_o is referred to as the real odd portion.

$$B_{even} = B_e = \sum_{n=1}^{\infty} \frac{\sin(b \ln 2n)}{(2n)^a} \quad (17)$$

B_e is referred to as the complex even portion.

$$B_{odd} = B_o = \sum_{n=1}^{\infty} \frac{\sin(b \ln(2n-1))}{(2n-1)^a} \quad (18)$$

B_o is referred to as the complex odd portion.

This isn't yet broken down far enough, and in order to determine when these new sum-differences in the real and complex sums are equal to zero, they need to be deconstructed further. However, the odd sums do not lend themselves to being broken down easily, if possibly at all. Luckily, the even sums do, and later, functional relationships for the odd sums will be found so that they can be handled. In the mean time, the next main phase of the explanation requires separating the Sine and Cosine portions of the even parts.

4 Separating the Sin and Cos Portions of the Real Even and Complex Even Sums

To separate the even sums, begin with the ln(2n) using log rules, equation 19, and follow up with the trigonometry formulas for addition within Cosines and Sines, equations 20 and 21. The initial results are then shown in 22 and 23.

$$\ln 2n = \ln 2 + \ln n \tag{19}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{20}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \tag{21}$$

$$A_e = \sum_{n=1}^{\infty} \frac{\cos(b\ln 2)\cos(b\ln n) - \sin(b\ln 2)\sin(b\ln n)}{2^a n^a} \tag{22}$$

$$B_e = \sum_{n=1}^{\infty} \frac{\sin(b\ln 2)\cos(b\ln n) + \cos(b\ln 2)\sin(b\ln n)}{2^a n^a} \tag{23}$$

Because in this case of 22 and 23 having separable addition and subtraction inside them over a common denominator, they can each be split into yet another 2 sums. Because those resulting sums take the form of products of functions of a and b independent of the index, multiplied by a portion of the sum dependent on the index, those independent functions of a and b can be pulled out in front as constants. This is shown as equations 24 and 25.

$$A_e = \left(\frac{\cos(b\ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\cos(b\ln n)}{n^a}\right) - \left(\frac{\sin(b\ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\sin(b\ln n)}{n^a}\right) \tag{24}$$

$$B_e = \left(\frac{\sin(b\ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\cos(b\ln n)}{n^a}\right) + \left(\frac{\cos(b\ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\sin(b\ln n)}{n^a}\right) \tag{25}$$

Next, respectively label K_c and K_s for the new cosine based and sine based constants in equations 24 and 25, shown as 26 and 27.

$$K_c = \frac{\cos(b \ln 2)}{2^a} \tag{26}$$

$$K_s = \frac{\sin(b \ln 2)}{2^a} \tag{27}$$

Also, label the 2 different sums amongst 24 and 25, noting that the K constants are the same for the real even, A_e, and complex even, B_e, sums, only in different positions. These are equations 28 and 29.

$$C = \sum_{n=1}^{\infty} \frac{\cos(b \ln n)}{n^a} \tag{28}$$

This is known as the cosine sum.

$$S = \sum_{n=1}^{\infty} \frac{\sin(b \ln n)}{n^a} \tag{29}$$

This is known as the sine sum.

Now, between the 10 terms A, B, A_e, A_o, B_e, B_o, K_c, K_s, C, and S, there is enough information to determine when the original infinite complex valued Dirichlet Eta sum is equal to zero, and to answer why the real variable a must be 1/2. In order to do that, the next step is to understand what maintains an output of 0 throughout splitting the original Dirichlet Eta sum into 8 other sums and 2 constants.

5 The Partial Recursive Functional Relationships Between the Sums

For the remaining sections, the indices and upper bounds of the sums do not change, and have mostly been omitted for brevity and visual clarity, as they do not affect the relationships.

The self referential relationship amongst the sums is generally stated in words as follows. The real and complex sums are broken into even and odd sums, then the even sums are broken into sine and cosine sums, however, those sine and cosine sums end up being composed in terms of the earlier even and odd sums, and thus include a loop.

Stating the relation from equation 13, using 15 and 16, gives equation 30, which is the even and odd split of the real portion.

$$A = A_e - A_o \tag{30}$$

Similarly, stating the relation from 14, using 17 and 18, gives 31, which is the even and odd split of the complex portion.

$$B = B_e - B_o \tag{31}$$

With equation 8, it was noted that the sums A, eq.9, and B, eq.10, must both be 0, and this is stated again with eq.30 and eq.31 as equation 32.

$$A = A_e - A_o = 0 \quad \text{AND} \quad B = B_e - B_o = 0 \tag{32}$$

This leads to the requirement in 33.

$$A_e = A_o \quad \text{AND} \quad B_e = B_o \tag{33}$$

Using the labels from equations 26-29, 24 and 25 are written as 34 and 35.

$$A_e = K_c C - K_s S \tag{34}$$

$$B_e = K_s C + K_c S \tag{35}$$

Now, review and more closely examine equations 28 and 29. Do the cosine and sine sums look familiar? They sure look like the real sum A, eq.9, and the complex sum B, eq.10, except for the -1 raised to the power, that is, except for the alternating part. In fact though, that is exactly what they are! The alternating real and complex sums, eqs. 9 and 10, subtract out every other term, while the sine and cosine sums, eqs. 28 and 29, add all the terms, of the otherwise same sum. What are those other terms, which are being subtracted in the case of the real and complex sums, but added in the case of the sine and cosine sums? Equations 13 and 14 show that those terms are the odd function sums! That is, the real and complex sums are the difference of their respective even and odd sums, while the cosine and sine sums are the sum of their respective even and odd sums. This gives equations 36 and 37.

$$C = A_e + A_o \tag{36}$$

$$S = B_e + B_o \tag{37}$$

Adding 2 copies of the corresponding odd function to each side of equations in 32, and then substituting with equations 36 and 37 respectively, gives the 2 equations shown in 38.

$$C = A_e + A_o = 2A_o \quad \text{AND} \quad S = B_e + B_o = 2B_o \tag{38}$$

However, eq.33 requires $A_e = A_o$ and $B_e = B_o$, which leads to 39.

$$C = 2A_e = 2A_o \quad \text{AND} \quad S = 2B_e = 2B_o \tag{39}$$

Equation 33 can be split into 2 cases. At a minimum, it shows that corresponding even and odd sums have the same value. Case one is a shared value of 0. Case 2 is sharing any value other than 0. The next section shows that it must be case one, and that the 4 even and odd sums, as well then as the other 4 sums, must all individually be 0.

6 Showing that the Even and Odd Sums Must Individually be 0

Dividing the twos over in eq.39, and substituting into 34 and 35, gives 40 and 41.

$$\frac{1}{2}C = K_cC - K_sS \tag{40}$$

$$\frac{1}{2}S = K_sC + K_cS \tag{41}$$

Since the Ks are constants to the sums, this can now be treated as 2 equations and 2 unknowns. Solving for S in eq.40 gives the following.

$$S = \frac{\left(K_c - \frac{1}{2}\right)}{K_s}C \tag{42}$$

Substituting 42 into 41, and simplifying, leaves 43.

$$\left(K_c^2 - K_c + K_s^2 + \frac{1}{4}\right)C = 0 \tag{43}$$

This shows that either C is 0, the portion in the parentheses is 0, or both parts are 0. In any case where C is 0, it means from eq.39 and eq.42 that case 1 must be true. Using the quadratic equation on the parentheses gives eq.44.

$$K_c = \frac{1 \pm \sqrt{-4K_s^2}}{2} \tag{44}$$

From eq.27 it is known that K_s is real valued, and therefore its square will be positive. Similarly, from eq.26, K_c is real valued. Because of the -4 inside the square root, the only possible solution is for $K_s = 0$ and $K_c = 1/2$.

Then, if $K_s = 0$, eq.27 gives that $b \ln 2 = n\pi$ for an integer n. However, if $b \ln 2 = n\pi$, then from eq.26, and the results from eq.44, you get the following.

$$\frac{1}{2} = K_c = \frac{\pm 1}{2^a} \tag{45}$$

This says that a is either complex valued or 1, which places it outside the domain of a, and therefore that the solutions within the domain of a occur in case one when C is 0.

$$A = B = A_e = A_o = B_e = B_o = C = S = 0 \tag{46}$$

So now that it's been determined that all 8 sums must be 0 in order to make the Dirichlet Eta 0, a system of equations can be formed in terms of ratios of the sums. That system allows the irreducible sums to be canceled out, and places conditions on K_c and K_s such that they can be solved for the requirements on a.

7 The Ratio Conditions Generated by the Sums

The method used at the start of this section on the real and complex sums is also duplicated on the even and odd sums. If 2 values are both 0, then the sum and difference of those values are also 0. From 46 and 32 you get the following.

$$A + B = A_e - A_o + B_e - B_o = 0 \quad \text{AND} \quad A - B = A_e - A_o - B_e + B_o = 0 \tag{47}$$

Substitute in using 34 and 35 for the even sums, and in C and S using eq.39 for the corresponding odd sums, to get 48 and 49.

$$K_c C - K_s S - \frac{1}{2}C + K_s C + K_c S - \frac{1}{2}S = 0 \tag{48}$$

$$K_c C - K_s S - \frac{1}{2}C - K_s C - K_c S + \frac{1}{2}S = 0 \tag{49}$$

Rearranging each for the ratio C/S gives 50 and 51.

$$\frac{C}{S} = \frac{-K_c + K_s + \frac{1}{2}}{K_c + K_s - \frac{1}{2}} \tag{50}$$

$$\frac{C}{S} = \frac{K_c + K_s - \frac{1}{2}}{K_c - K_s - \frac{1}{2}} \tag{51}$$

Setting 50 and 51 equal, cross multiplying, and simplifying, gives 52.

$$4K_c^2 - 4K_c + 4K_s^2 + 1 = 0 \tag{52}$$

Next, repeat the process for the odd sums. From eq.38 you get the following.

$$A_o + B_o = C - A_e + S - B_e = 0 \quad \text{AND} \quad A_o - B_o = C - A_e - S + B_e = 0 \tag{53}$$

Trade in using 34 and 35 for the even sums.

$$C - K_c C + K_s S + S - K_s C - K_c S = 0 \tag{54}$$

$$C - K_c C + K_s S - S + K_s C + K_c S = 0 \tag{55}$$

Rearranging each for the ratio C/S gives 56 and 57.

$$\frac{C}{S} = \frac{K_c - K_s - 1}{-K_c - K_s + 1} \tag{56}$$

$$\frac{C}{S} = \frac{-K_c - K_s + 1}{-K_c + K_s + 1} \tag{57}$$

Setting 56 and 57 equal, cross multiplying, and simplifying, gives eq.58.

$$K_c^2 - 2K_c + K_s^2 + 1 = 0 \tag{58}$$

Even though there are now already 2 equations and 2 unknowns, the process can be duplicated on the even sums. This is included, as it helps in understanding the nature of the role of the sum constants K_c and K_s within the system.

$$A_e + B_e = 0 \quad \text{AND} \quad A_e - B_e = 0 \tag{59}$$

Again using 34 and 35 on 59 gives 60 and 61.

$$K_cC - K_sS + K_sC + K_cS = 0 \tag{60}$$

$$K_cC - K_sS - K_sC - K_cS = 0 \tag{61}$$

Rearranging each for the ratio C/S gives 62 and 63.

$$\frac{C}{S} = \frac{-K_c + K_s}{K_c + K_s} \tag{62}$$

$$\frac{C}{S} = \frac{K_c + K_s}{K_c - K_s} \tag{63}$$

Setting 62 and 63 equal, cross multiplying, and simplifying, gives 64.

$$K_c^2 + K_s^2 = 0 \tag{64}$$

As the last steps, it is now possible to gather the results from equations 52, 58, and 64, to do the substitutions back using K_c and K_s, and to show the requirements on a and b from the original Dirichlet Eta sum.

8 Solving the Conditions Generated by the Ratios of Sums

At this point, all 3 of those equations must make sense for the Dirichlet Eta sum to equal 0. The odd ball is equation 64, the condition from the even sums, so it is addressed first.

Substituting eq.26 and eq.27 back into eq.64, and simplifying, gives 65.

$$\frac{1}{2^{2a}} = 0 \tag{65}$$

However, there is clearly no solution for a that would make eq.65 possible. So what does it mean? Well, looking at eq.34 and eq.35, the even sums are the only ones directly composed from sums that include coefficients K_c and K_s in the fronts. When the sum and difference requirements are made on the even sums in eq.59, it's comparing the entire product of the coefficients with sine and cosine sums as a whole, rather than just the sum and difference of sums without coefficients, as in the other cases.

In very general terms, using temporary variables, the even sums' sum and difference has the form wx $\pm$ yz $=0$ instead of just x $\pm$ z $=0$. In this case, it means that the even portions are relying on the cosine and sine sums, which are dictated by the rest of the system, and the product, to balance them to zero, rather than on the constants K_c and K_s.

This leaves the other 2 equations and 2 unknowns, now a and b. Substituting eq.26 and eq.27 back into eq.52 gives equation 66, and back into eq.58 gives equation 67.

$$4 + 2^{2a} - 4 * 2^a \cos(b \ln 2) = 0 \tag{66}$$

$$1 + 2^{2a} - 2 * 2^a \cos(b \ln 2) = 0 \tag{67}$$

Both 66 and 67 can be studied separately for the relation of a and b dictating when the real and complex sums, or odd sums, would be 0 via the constants. Solving the system simultaneously is done by separating for the $\cos(b \ln 2)$ term in one equation, plugging it into the other, and then solving for a. Equation 68, is 67 in terms of the cosine.

$$\cos(b \ln 2) = \frac{1 + 2^{2a}}{2 * 2^a} \tag{68}$$

And finally, there it is. Plugging 68 into 66, gives $2 = 2^{2a}$, leading to $2a = 1$, and finally a=1/2. This shows that there is indeed only one possible choice for a that allows the correct ratio of sums and coefficients, such that all 8 sums, and thus the original Dirichlet Eta function, equals 0.

$$a = \frac{1}{2} \tag{69}$$

Therefore, a must = 1/2, and Riemann's suspicions were correct!

Q.E.D.