Explicit Upper Bound for all Prime Gaps

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Declarations of interest: none

Abstract: Let p_s denote the greatest prime with squared value less than a given number. We call the interval from one prime's square to the next, a prime's *season*. By improving on the well known proof of arbitrarily large prime gaps, here we show that for all seasons, the upper bound of prime gap length is $2p_s$.

Introduction: Prime gaps are still of interest (1) but the well known proof of arbitrarily long prime gaps is suboptimal. The standard proof, (e.g. (1)) exploits the corraling of prime factor orbits by factorial numbers. It levereges the coincident appearance of,all factors in a product and sees that it imitates the prime number positions at the origin. The resulting gaps found from factorials will necessarily be longer than the *n* of the factorial, extending from *n! + 2* to *n! + p*, where *p* is the least prime greater than *n*. We reduce the cardinality of identified gaps by targeting *ex-primorial* numbers, i.e. integer multiples of primorial numbers *p#*, where primorials are identical to factorials except excluding composite factors. Additionally, by paying attention to the prime factors in orbit, and their well ordering, we recognize *rogue orbits*, occasions with the *np#* ±1 positions occupied can uniquely boost the gap length

Calculations: We call prime factor orbits not included as a factor with the others *rogue*, and indicate their occupancy of positions ±1 an *ex-primorial* with post-superscript notation *r* {0, 1, 2}. For a set of primes, the computationally identified prime gaps optimize the composite density of the orbiting prime factors by either having them all together mimicking the origin, in an ex-primorial, or allowing the greatest two orbits go rogue. If not the greatest two orbits, any rogue contribution is comparitavely suboptimal.

Factorially based prime gap: with unknown rogue orbits. $\boldsymbol{|g_n^{r0}| : n! \geq n-1}$

$$. (0).\ (p_?, \dots, -n, \dots, -2], -\frac{1}{r}.\frac{(n!)}{0}.+\frac{1}{r}, [2, \dots, n, \dots, p_?).$$

Exprimorally based prime gap: with no rogue orbits. $|g_n^{r0}| : \boldsymbol{kp_s\# = p_{s+1} - 1}$

$$❶.\ (-\boldsymbol{p_{s+1}}, \dots, \boldsymbol{p_s}, \dots, -\boldsymbol{2}], -1.\frac{(\boldsymbol{kp_s\#})}{\boldsymbol{0}}.+1.\,[\boldsymbol{2}, \dots, \boldsymbol{p_s}, \dots, \boldsymbol{p_{s+1}}).$$

Exprimorally based prime gap: with two rogue orbits. $\boldsymbol{g_n^{r2} : kp_{s-2}\# = 2p_{s-1}}$.

$$❷.\ (-\boldsymbol{p_{s-1}}, \dots, \boldsymbol{p_{s-2}}, \dots, -\boldsymbol{2}], r\,.\frac{(\boldsymbol{kp_{s-2}\#})}{\boldsymbol{0}}.\,r', [\boldsymbol{2}, \dots, \boldsymbol{p_{s-2}}, \dots, \boldsymbol{p_{s-1}})$$

If the next prime's square occurs during a prime gap in progress, this raises the longest possible gap to the next season's limit, hence the least expression valid for prime gaps in any season is ❸ $2p_s$.

Results:

Table 1 Comparison and placement of the first five seasons' theoretical upper bound without rogues (1), with contributing rogues (3), and their proximity to primorials and empirical maximal gaps.

Season	Range	Max Gap ❶ $p_{s+1} - 1$	Max Gap ❸ $2p_s$	Primorial	Maximal Gaps from n=2 (prime initiating)
I.2	4-8	4	4	2	2 (3)
II.3	9-24	6	6	6	4 (7)
III.5	25-48	10	10	30	6 (23)
IV.7	49-120	12	14	210	8 (89)
V.11	121-168	16	22	2310	14 (113)
VI.13	169-288	22	26	30,030	18 (523)

Conclusion:

Bonse inequality, observes that for primes seven and above, the next greater prime's square is less than the primorial, of that prime (2). The prime gap from 113 to 127, centered on *ex*-primorial 120, is the most efficient prime gap in the number line. Thereafter, the Bonse type inequality only gets stronger, forcing the maximal gaps to rely on suboptimal rogue orbits, not the greatest in the season. Empirically, all subsequent maximal prime gaps stay well below the theoretical supremum discovered here and cannot reverse the trend. Hence, $2p_s$ is the prime gap supremum in all seasons.

This proves the *prime-intersquare* (*Legendre's*) conjecture.

Proof:. We wish to show that ❸ $(2p_s)$, the prime gap supremum, is less than the difference between squares in all seasons.

1. $\forall n \in \mathbb{N}, \exists\, p \in \mathbb{P} : n^2 < p < (n+1)^2$. **Assertion**
2. $((n+1)^2 - n^2 = 2n + 1$. **By algebra.**
3. $n \geq p_s$. **By definition of a season**.
4. $2n + 1 > 2n$. **By definition of inequality. ■.**

References

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