Refutation of Lindström theorems for fragments of first-order logic

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Abstract: We evaluate three equations for the example property as not tautologous. This refutes virtually all conjectures, such as Lindström theorems for the six upper fragments of logic in Fig. 1, and the two proof approaches claimed as clever. These form a non tautologous fragment of the universal logic VL4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, F as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET  ¬ Not, ∼;  + Or, ∨, ∪;  - Not Or;  & And, ∧, ∩, ·, ⊓; \ Not And;
> Imply, greater than, →, ⇒, ⊃, ⊩;  < Not Imply, less than, ∈, ⊂, ⊆, ⊄, ⊆;
≡ Equivalent, ≡, =, ↔, ≡, ≡;  @ Not Equivalent, ≠, ⊖;
% possibility, for one or some, ∃, ◊, M;  # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3;  (z@z) F as contradiction, Ø, Null, ⊥;
(%z>≠z) N as non-contingency, ∆, ordinal 1;  (%z<≤z) C as contingency, ∇, ordinal 2;
( y < x) ( x ≤ y), ( x ∈ y), ( x ⊆ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.


Abstract. Lindström theorems characterize logics in terms of model-theoretic conditions such as Compactness and the Löwenheim-Skolem property. Most existing characterizations of this kind concern extensions of first-order logic. … We use two different proof techniques. One is a modification of the original Lindström proof. The other involves the modal concepts of bisimulation, tree unraveling, and finite depth.

1. Introduction The original Lindström theorem for first-order logic, in one of its most widely used formulations, says the following:

The first-order Lindströom Theorem [13] An extension of first-order logic satisfies Compactness and the Löwenheim-Skolem property if and only if it is no more expressive than first-order logic.

We are not aware of a similar characterization for modal logic involving the Löwenheim-Skolem property. Note that first-order logic itself is a compact proper extension of modal logic that has the latter property.
2. From first-order logic downwards

Claim: Each of the following [nine] properties of this model [Fig.2, not shown here] can be expressed by a sentence of L:

(3) Each such partial bijection preserves structure on the submodels defined by A and B, as far as the finitely many relations occurring in \( \varphi \) are concerned (recall that L has the Finite Occurrence Property). (2.3.0.0)

Proof of claim: The first eight properties can already be expressed in FO\(^3\) by a clever reuse of variables, and the ninth property can be expressed in L by closure under the Boolean connectives and relativisation by unary predicates. For instance, the third property can be expressed as the conjunction of all FO\(^3\)-formulas of the following forms, where S \( \in \text{REL}(\varphi) \) is a binary relation symbol, and Q \( \in \text{REL}(\varphi) \) is a unary relation symbol:

\[
\forall xy[ P x \land P y \land \exists z(F z \land R x z \land R y z) \rightarrow \exists z(R x z \land A z \land \exists x(R y x \land A x \land S z x)) \leftrightarrow \\
\exists z(R x z \land B z \land \exists x(R y x \land B x \land S z x)) ]
\] (2.3.1.1)

\[
\text{LET } p, q, r, s, t, u, v, x, y, z:
\]

\[
(((p \land x) \land (p \land y)) \land ((v \land z) \land ((r \land (z \land x)) \land (r \land (z \land y)))) \land ((r \land (x \land z)) \land (t \land z)) \land ((r \land (y \land x)) \land (t \land x)) \land (s \land (z \land x))) ;
\]

\[
\begin{array}{cccccccccc}
\text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} \\
\text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} \\
\end{array}
\]

\( \text{FFFF} \) \times 2

\[
\text{FFFF} \text{FFFF FFFF FFFF } \}
\]

\[
\text{FFFF} \text{FFFF FFFF FFFF } \}
\]

\[
\text{FFFF} \text{FFFF FFFF FFFF } \}
\]

(2.3.1.2)

and

\[
\forall x[ P x \rightarrow \exists z(R x z \land A z \land Q z) \leftrightarrow \exists z(R x z \land B z \land Q z)]
\] (2.3.2.1)

\[
(p \land x) > (((r \land (x \land z)) \land (t \land z)) \land (q \land z)) = (((r \land (x \land z)) \land (u \land z)) \land (q \land z))) ;
\]

\[
\begin{array}{cccccccccc}
\text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} \\
\text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} & \text{TTTT} \\
\end{array}
\]

\( \text{FFFF} \) \times 2

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\text{FFFF} \text{FFFF FFFF FFFF } \}
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\]

(2.3.2.2)

Remark 2.3: It is unclear if the authors mean to calculate Eqs. 2.3.1.1 and 2.3.2.1, so we perform that anyway.

\[
(((p \land x) \land (p \land y)) \land ((v \land z) \land ((r \land (z \land x)) \land (r \land (z \land y)))) \land ((r \land (x \land z)) \land (t \land z)) \land ((r \land (y \land x)) \land (t \land x)) \land (s \land (z \land x))) ;
\]

\[
\begin{array}{cccccccccc}
\text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} \\
\text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} & \text{FFFF} \\
\end{array}
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\( \text{FFFF} \) \times 2

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\[
\text{FFFF} \text{FFFF FFFF FFFF } \}
\]

(2.3.3)

Remark 2.3.3.2: For Models 2 in the table above, all replace F with U; and 2.1 N with E, 2.3.1 N with I, and 2.3.2 N with P.
Eqs. 2.3.3.1, .3.2, and .33 as rendered for the example property are *not* tautologous. This refutes virtually all conjectures, such as Lindström theorems for the six upper fragments of logic in Fig. 1, and the two proof approaches claimed as clever.