On the Emergence of Spacetime Dimensions from the Kolmogorov Entropy

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Abstract

This short pedagogical report is based on a couple of premises. First, it was recently shown that the long run of non-equilibrium Renormalization Group flows is prone to end up on strange attractors. As a result, multifractals are likely to provide the proper framework for the characterization of effective field theories. Secondly, it is known that multifractal analysis uses the Kolmogorov entropy (K-entropy) to quantify the degree of disorder in chaotic systems and turbulent flows. Building on these premises, the report details the remarkable connection between K-entropy, multifractal sets and spacetime dimensions. It also supports the proposal that near and above the Fermi scale, spacetime is defined by continuous and arbitrarily small deviations from four-dimensions.

Key words: Kolmogorov entropy, multifractals, minimal fractal manifold, effective field theory.

We have conjectured in [1, 2] that the flow from the ultraviolet (UV) to the infrared (IR) sector of any multidimensional nonlinear field theory approaches chaotic dynamics in a universal way. This result stems from several independent routes to aperiodic behavior and implies that the IR attractor of effective field theories is likely to replicate the properties of *strange attractors* and *multifractal sets*. In particular, the chaotic behavior of the Renormalization Group flow near or above the Fermi scale suggests that phenomena on or above this scale mimic the dynamics on a strange attractor [3-5, 9]. It is relevant to recall that the evolution of trajectories within the strange attractor may be *locally* unstable, but robust and regular at the *global* scale. In this sense, chaos can be viewed as the foundation of classical statistical mechanics, which is based on invariant measures and the ergodic hypothesis [10].

Let a generic UV to IR trajectory be described by the n- dimensional phase-space flow $x(\tau)$. Here, τ denotes the evolution parameter ("time") corresponding to the Renormalization Group scale μ

$$\tau = \log(\mu / \mu_0) \tag{1}$$

The random behavior of the flow near the strange attractor can be characterized by dividing the phase-space into n- dimensional hypercubes of side r, which are sampled at discrete time intervals $\Delta\tau$. The generalized *K-entropy* of order $q \neq 1$ is given by the equation [6]

$$K_q(X) = -\lim_{r\to 0}\lim_{\Delta\tau\to 0}\lim_{N\to\infty}\frac{1}{N\Delta\tau}\frac{1}{q-1}\ln\sum_{i_1,i_2,\dots i_N}^{M(r)} p^q_{i_1,i_2,\dots i_N} \tag{2}$$

where $X = x_i$ is the discrete random variable, that is, $x_i = x(\tau = i\Delta\tau)$, and $p_{i_1,i_2,\dots,i_M}$ stands for the joint probability that the trajectory $x(\tau = \Delta\tau)$ is in i_1, $x(\tau = 2\Delta\tau)$ is in i_2 and $x(\tau = M\Delta\tau)$ is in i_M. The K-entropy defines the asymptotic scenario where $r \to 0$ and the phase-space is sampled with an infinite number of steps ($N \to \infty$) at vanishing time intervals $(\Delta\tau \to 0)$. In the special case $N\Delta\tau = 1$ and when the joint probability is constant across all hypercubes ($M(r) = const.$, $p_{i_1,i_2,\dots,i_M} = p_i$), (2) turns into the *Rényi entropy* in the logarithm base b, which assumes the form

$$S_q(X) = \frac{1}{1-q} \log_b(\sum_{i=1}^{M} p_i^q) \tag{3}$$

Furthermore, (3) reduces to the familiar *thermodynamic entropy* when $q \to 1$ and Boltzmann constant is set to $k_B = 1$ [7]

$$S(X) = -\sum_{i=1}^{M} p_i \ln p_i \tag{4}$$

Finally, the concept of *generalized dimension* of order q is introduced in conjunction with (3) according to

$$D_q = \lim_{r \to 0} \frac{1}{1-q} \frac{\log_b(\sum_{i=1}^{M} p_i^q)}{\log r} \tag{5}$$

A particularly straightforward expression of (3) is obtained for the null order $q = 0$ and the natural logarithm base. It is referred to as *topological entropy* and is given by

$$S_0(r) = \ln \sum_{i=1}^{M} p_i^0 = \ln M \tag{6}$$

It is known that the *box-counting dimension* of a fractal object of normalized size r is defined as

$$D_0 \approx \frac{\ln M}{\ln r} \Rightarrow M \approx r^{D_0} = \varepsilon^{-D_0} \tag{7}$$

in which M denotes the number of covering boxes and $\varepsilon = r^{-1}$ is the normalized size of the box. The dimension of ordinary Euclidean space corresponds to integer and positive-definite values of the box-counting dimension, $D_0 = k$, $k = 0,1,2\ldots$.

Comparing (6) to (7) leads to the connection between the box-counting dimension and topological entropy via

$$\boxed{\varepsilon^{-D_0} = \exp[S_0(r)]} \tag{8}$$

Two straightforward conclusions may be drawn from (8):

- Maximal topological entropy ($S_0(r) \to \infty$) matches the limit $\varepsilon \to 0$ and corresponds to the four-dimensional continuum of both General Relativity and Quantum Field Theory.
- The steady increase of topological entropy along the flow implies that, near or above the Fermi scale, spacetime is endowed with a *continuous* spectrum of dimensions ($\varepsilon = 4 - D_0 << 1$), asymptotically reaching $D_0 = 4$ as $\varepsilon \to 0$ [8, 4].

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