A GENERAL FORM OF THE BEPPO LEVI’S LEMMA

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ABSTRACT. In this article we will prove a version of the Beppo Levi’s lemma for the complex valued functions. This achieved by making a more stronger assumption that is assumed in Beppo Levi’s lemma. We will assume that the sum of measurable functions that is absolutely convergent almost everywhere is integrable. We will prove that it implies the assumptions of the Beppo Levi lemma, if we consider functions that are non-negative. It can be argued that our version is more suitable to applications, and we will prove a new probability law. We will show that with our assumptions in probability theory it follows that the expected value is countable additive. Moreover, it follows that in strong law of large numbers we don’t need to make any assumptions on distributions and the mean of the sample will convergence almost surely to the mean of the expected values.

1. Introduction

The Beppo Levi lemma is the following.

Lemma 1. Let \((X, A, \mu)\) be a measure space and let be a pointwise increasing sequence of nonnegative measurable functions defined on a measurable set \(E\). If the sequence \(\int \sum_{i=1}^{n} f_i(x)\) is bounded, then the sum converges pointwise to a measurable function \(f\) on \(E\) and it holds that

\[ \int \sum_{i=1}^{\infty} f_i(x) = \sum_{i=1}^{\infty} \int f_i(x). \]

The lemma 1 works even if the integral is infinite and we let \(\int \sum_{i=1}^{n} f_i(x)\) be unbounded. However, there seems to be very little payoff allowing the integral of the series to be infinite. Basically all we got is that it diverges on a set of a finite measure, but we don’t see right away how to generalize to, say \(\mathbb{R}\) valued functions, because we run into a infinity minus infinity situation. We will show that if we make a stronger assumption that

\[ \int \sum_{i=1}^{\infty} f_i(x) < \infty, \]

we can extend the theory to complex valued functions and find new important applications of the theorem. In probability theory it follows from the assumption (1) that the expected value is countable additive. It follows that in strong law of large numbers we don’t need to make any assumptions on

1991 Mathematics Subject Classification. 97K50.

Key words and phrases. Beppo Levi lemma, probability theory, strong law of large numbers.
distributions and the mean of the sample will convergence in probability to the mean of the expected values.

**Definition 1.** We say that \( \sum_{j=1}^{\infty} f_j(z) \) is a formal series of measurable complex valued functions on \((X, \mathcal{A}, \mu)\), if the formal series \( \sum_{i=1}^{\infty} |f_j(z)| \) converges almost everywhere.

For any \( f_j(z) \) we define the non-negative real part \( v_{j1}(z) \geq 0 \), negative part \( v_{j2}(z) \geq 0 \), non-negative imaginary part \( v_{j3}(z) \geq 0 \) and negative imaginary part \( v_{j4}(z) \geq 0 \) via

\[
f_j(z) = v_{j1}(z) - v_{j2}(z) + iv_{j3}(z) - iv_{j4}(z).
\]

Now, we have almost everywhere that

\[
\sum_{j=1}^{\infty} f_j(z) = \sum_{j=1}^{\infty} v_{j1}(z) - \sum_{j=1}^{\infty} v_{j2}(z) + \sum_{j=1}^{\infty} iv_{j3}(z) - \sum_{j=1}^{\infty} iv_{j4}(z).
\]

because the series is absolutely summable almost everywhere. That the limit of increasing functions is measurable shows together with the composition (2) that the function \( \sum_{j=1}^{\infty} f_j(z) \) is measurable. The above composition (2) together with (1) implies that the functions \( f_j \) are integrable because \( v_{j1}, v_{j2}, v_{j3} \) and \( v_{j4} \) get dominated pointwise by the integrable sums in (2).

**Theorem 1** (General Beppo Levi’s lemma). Let \( \sum_{i=1}^{\infty} f_i(x) \) be a formal series of measurable complex valued functions on \( X \). If it holds that

\[
\int \left| \sum_{i=1}^{\infty} f_i(x) \right| < \infty,
\]

then

\[
\int \sum_{i=1}^{\infty} f_i(x) = \sum_{i=1}^{\infty} \int f_i(x).
\]

We see from the theorem 1 that the assumption (1) implies the assumptions of the Beppo Levi lemma 1. The expectation value of an random variable is defined to be the integral of the random variable.

\[
Ef(x) := \int f(x) \mu
\]

Our main corollaries to the theorem 1 are the following. The first can be thought as an new kind of more general probability law.

**Corollary 1** (The new probability law). Let \( \sum_{i=1}^{\infty} g_i(x) \) be a formal series of random variables on a probability space. If it holds that

\[
\int \left| \lim_{n \to \infty} \sum_{i=1}^{n} g_i(x) \right| < \infty,
\]

then almost surely

\[
\sum_{i=1}^{\infty} g_i(x) = \sum_{i=1}^{\infty} Eg_i(x),
\]

where \( E \) is the expectation value operator.
The second one is our more general formulation of the strong law of large numbers and it is clearly an instance of the first corollary. However, in our version the averages can be seen as an summation method.

**Corollary 2** (General strong law of large numbers). Let \( \sum_{i=1}^{\infty} g_i(x) \) be a formal series of random variables. If it holds that

\[
\int | \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g_i(x) | < \infty,
\]

then almost surely

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g_i(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E g_i(x),
\]

where \( E \) is the expectation value operator.

The usual form of the strong law of large numbers is achieved if we let all the expectation values be equal. The next corollary is the countable additivity of the expectation values.

**Corollary 3** (The countable additivity of the expectation value operator). Let \( \sum_{i=1}^{\infty} g_i(x) \) be a formal series of random variables. If it holds that

\[
\int | \sum_{i=1}^{\infty} g_i(x) | < \infty,
\]

then

\[
E \sum_{i=1}^{\infty} g_i(x) = \sum_{i=1}^{\infty} E g_i(x),
\]

where \( E \) is again the expectation value operator.

2. The proofs of the theorems and corollaries

We will use a very basic and well known theorem.

**Theorem 2.** If

\[
\sum_{i=1}^{\infty} z_i = z
\]

then

\[
\lim_{k \to \infty} \sum_{i=k}^{\infty} z_i = 0.
\]

**Proof.** For any \( k \) it holds that

\[
\sum_{i=1}^{\infty} z_i = \sum_{i=1}^{k} z_i + \lim_{k \to \infty} \sum_{i=k+1}^{\infty} z_i = 0.
\]

Thus, for any \( \epsilon > 0 \) there exist \( k \) such that

\[
\epsilon > \left| \sum_{i=1}^{\infty} z_i - \sum_{j=1}^{k} z_i \right| = \left| \lim_{k \to \infty} \sum_{i=k+1}^{\infty} z_i \right|,
\]

and we have the claim. \( \square \)
Next we define a zero function for each $k$

\begin{equation}
0 = 0_k = \int (\sum_{i=1}^{\infty} f_i(z) - \sum_{i=1}^{\infty} f_i(z)) = \int \sum_{i=1}^{k} f_i(z) - \sum_{i=1}^{k} \int f_i(z) - \int \sum_{i=k+1}^{\infty} f_i(z),
\end{equation}

where the sums are defined almost everywhere and we used the finite linearity of the integral and the fact that the functions $f_i(z)$ are integrable. We assumed in our definition of formal series 1 the absolute convergence almost everywhere. So that it holds almost everywhere that

\begin{equation}
\sum_{j=1}^{\infty} f_j(z) = \sum_{j=1}^{\infty} v_{j1}(z) - \sum_{j=1}^{\infty} v_{j2}(z) + i \sum_{j=1}^{\infty} v_{j3}(z) - i \sum_{j=1}^{\infty} v_{j4}(z),
\end{equation}

where each $v_{j\ell}$ is a non-negative function. So those integrable non-negative sum-functions $\sum_{j=1}^{\infty} v_{j1}(z)$ functions dominate the non-negative sum functions $\sum_{j=k+1}^{\infty} v_{j1}(z)$ pointwise. In other words

\begin{equation}
\sum_{j=k+1}^{\infty} v_{j1}(z) \leq \sum_{j=1}^{\infty} v_{j1}(z).
\end{equation}

It follows that we can use the dominated convergence theorem to $\int \sum_{i=k+1}^{\infty} f_i(z)$. So we let $k$ go to infinity in (3) and obtain

\begin{equation}
0 = \lim_{k \to \infty} 0_k = \int \lim_{k \to \infty} \sum_{i=1}^{k} f_i(z) - \lim_{k \to \infty} \sum_{i=1}^{k} \int f_i(z) - \lim_{k \to \infty} \sum_{i=k+1}^{\infty} f_i(z).
\end{equation}

We proved in theorem 2 that $\lim_{k \to \infty} \sum_{i=k+1}^{\infty} f_i(z)$ converges to zero. So we have

\begin{equation}
0 = \int \lim_{k \to \infty} \sum_{i=1}^{k} f_i(z) - \lim_{k \to \infty} \sum_{i=1}^{k} \int f_i(z),
\end{equation}

which is our main theorem 1. If we have a measure space of unit measure $\mu(X) = 1$, we have that

\begin{equation}
\int \lim_{k \to \infty} \sum_{i=1}^{k} \int f_i(z) = \lim_{k \to \infty} \sum_{i=1}^{k} \int f_i(z).
\end{equation}

Thus, from (5) and (6) we obtain

\begin{equation}
0 = \int \lim_{k \to \infty} \sum_{i=1}^{k} f_i(z) - \lim_{k \to \infty} \sum_{i=1}^{k} \int f_i(z) = \int (\lim_{k \to \infty} \sum_{i=1}^{k} f_i(z) - \lim_{k \to \infty} \sum_{i=1}^{k} \int f_i(z)),
\end{equation}

which implies that

\begin{equation}
\lim_{k \to \infty} \sum_{i=1}^{k} f_i(z) = \lim_{k \to \infty} \sum_{i=1}^{k} \int f_i(z),
\end{equation}

where the sums are defined almost everywhere and we used the finite linearity of the integral and the fact that the functions $f_i(z)$ are integrable. We assumed in our definition of formal series 1 the absolute convergence almost everywhere. So that it holds almost everywhere that
almost everywhere (1). If we take \( \mu \) to be a probability measure we have from (7) that

\[
\lim_{k \to \infty} \sum_{i=1}^{k} f_i(z) = \int \lim_{k \to \infty} \sum_{i=1}^{k} f_i(z) = \int \lim_{k \to \infty} \sum_{i=1}^{k} Ef_i(z),
\]

almost surely, which is our new probability law 1. The general strong law of large numbers 2 then easily follows from (8). The countable additivity of the expectation value operator 3 is clearly an instance of our main theorem 1.

References


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