Showing the Real Numbers are Denumerable Using Binary Digit Ratios

by Jim Rock

Abstract. Cantor’s diagonal argument starts with an invalid premise. Using binary decimals, we calculate the ratio of ones to the total number of digits for decimals in the closed interval \([0, 1]\). Assuming \(T_\theta\) the set of all transcendentals with \(r = \theta\) is the basis for all transcendental numbers, we use these ratios to show that the real numbers are denumerable. The reals numbers have the same cardinality as the power set of the natural numbers. There is no hierarchy of infinities in Level Set Theory. \(|\mathbb{I}| = |2^\mathbb{I}|\).

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The natural numbers can be put in one to one correspondence with the terminating decimal fractions in the closed interval \([0, 1]\), \(0 \to 0, 1 \to 1, 2 \to 2, \ldots, 10 \to 01, \ldots\). Each terminating decimal is the mirror image reflection through the decimal point of a natural number. The mapping does not include any repeating decimals. From this mapping the set of all rational numbers would appear to be uncountable.

Attempting to map the real numbers in the closed interval \([0, 1]\) to the natural numbers, by listing them sequentially as infinite decimal fractions is an invalid process. Cantor’s diagonal proof that the real numbers are an uncountable set starts with an invalid premise.

The problem is in defining the real numbers as the set of all infinite decimal expansions. That’s vague. Most non-algebraic real numbers cannot be explicitly referenced. We need a new definition. Each real number \(S_n\) is the limit of its partial decimal sums: the limit \(n \to \infty \ j = 1\) to \(n, 0 \leq a_j \leq 9 \sum a_j / 10^j = S_n\)

Whenever a ratio \(r\) is referenced in this paper it is the number of ones divided by the total number of digits of a binary decimal in the closed interval \([0, 1]\).

A canonical repeating decimal begins with \(O \geq 0\) ones and \(Z \geq 0\) zeros followed by an infinite pattern of alternating \((a)\) ones and \((b)\) zeros. We can create \(a\) different sets of ones of size aleph_null by taking from the infinite pattern of alternating \(a\) ones and \(b\) zeros the ones in position \(1, 1 + a + b, 1 + (2)(a + b), 1 + (3)(a + b)\ldots 2, 2 + a + b, 2 + (2)(a + b), 2 + (3)(a + b)\ldots\) up to \(a, a + (a + b), a + (2)(a + b), a + (3)(a + b)\ldots\) \(b\) sets of zeros can be made in the same fashion, replacing \(1\) with \(a + 1\), \(2\) with \(a + 2\), \(3\) with \(a + 3\ldots\) up to \(a\) with \(a + b\), giving \(a + b + (a + b), a + b + (2)(a + b), a + b + (3)(a + b)\ldots\) thus creating a ratio of ones to all digits of \(a / (a + b)\).

If the ratio created by the digits of irrational partial decimal sums gets progressively smaller as the partial decimal sums get larger, \(r = 0\). Liouville numbers \(\text{https://en.wikipedia.org/wiki/Liouville\_number}\) are transcendental numbers with \(r = 0\). The ever increasing gaps between ones in their decimal expansion allows Liouville numbers to approach rational numbers more quickly than is possible for an algebraic irrational.

Any numbers with similar gaps between ones are transcendental.

Any algebraic irrational \(a_r\) \((0 < r < 1)\) can be generated using the correct canonical repeating decimal \(c_r\) \((0 < r < 1)\) and two algebraic irrationals \(a_0, a_1\) with ratio \(r_1, 0 \leq r_1 < r < 1\) and \(a \ I \ I\) map between their ones. \(a_0\) contains ones in the same decimal places of \(c_r\) changing from ones to zeros. \(a_1\) contains ones in the same decimal places of \(c_r\) changing from zeros to ones. \(a_r = c_r + a_1 - a_0\).

By adding to (or replacing) \(a_0, a_1\) with transcendentalss \(t_0, t_1\) \((r=0\) and a \(I \ I\) map between their ones), multiple \(t_r\) \((0 < r < 1)\) can be generated by using the correct canonical repeating decimal \(c_r\) \((0 < r < 1)\).

For each pair of \(t_0, t_1\) \(t_0\) contains ones in the same decimal places of \(a_r\) \((c_r)\) changing from ones to zeros. \(t_1\) contains ones in the same decimal places of \(a_r\) \((c_r)\) changing from zeros to ones. \(t_r = a_r + t_1 - t_0\) \((t_r = c_r + t_1 - t_0)\).

Multiple \(t_{r2}\) can be generated using \(t_r\), and \(t_{r10}, t_{r11}\) with ratio \(r_1, 0 < r_1 < r < 1\) and a \(I \ I\) map between their ones. (For each pair \(t_{r10}, t_{r11}\) \(t_{r10}\) contains ones in the same decimal places of \(t_r\) changing from ones to zeros. \(t_{r11}\) contains ones in the same decimal places of \(t_r\) changing from zeros to ones. \(t_{r2} = t_r + t_{r11} - t_{r10}\).
Since the ones in any decimal in $T_0$ the set of all transcendental numbers with $r = 0$ maps to a subset of the ones in any decimal in $A$, the set algebraic numbers with $0 < r < 1$, $T_0$ maps to a subset of $A$. $T_0$ is countably infinite. Since $T_r$ the set of all transcendental numbers for all $r$ with $0 < r < 1$ is generated from $T_0$ and all algebraic numbers in $[0, 1]$, $T_r$ is countably infinite. $T_r$ combined with itself, $T_0$, and all algebraic numbers generates $T$ the set of all transcendental numbers; a countably infinite set.

Assuming $T_0$ the set of all transcendental numbers with $r = 0$ is the basis for all transcendental numbers, the real numbers are denumerable. Since the real numbers have the same cardinality as the power set of the natural numbers, there is no hierarchy of infinities. We have created Level Set Theory. $|\mathbb{Z}| = |2^\mathbb{N}|$.

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