Okumura's Disc Series Can Beyond the Crucial Point of Däumler-Puha's Horn Torus Models for the Riemann Sphere

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Abstract: In this paper, we will note a simple and pleasant new property that an Okumura's disc series can beyond the crucial point of Däumler-Puha's horn torus models for the Riemann sphere.

Key Words: Infinity, discontinuous, point at infinity, stereographic projection, Riemann sphere, horn torus, Däumler-Puha's horn torus, Okumura's disc series, conformal mapping, division by zero calculus.

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1 Introduction

In this paper, we will note a simple and pleasant new property that an Okumura's disc series can beyond the crucial point of Däumler-Puha's horn torus models for the Riemann sphere.

2 Backgrounds of the conclusion

For the statement of the conclusion, we will first recall the basic related backgrounds; division by zero calculus, examples and horn torus models.

2.1 Division by zero calculus

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z-a)^n,$$

we **define** the division by zero calculus by the identity

$$f(a) = C_0.$$

For many basic properties and applications of the division by zero calculus, see [5, 4].

2.2 Our life figure

As an interesting figure which shows an interesting relation between 0 and infinity, we will consider a sector Δ_α on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{ |\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2} \right\}.$$

We will consider the disc inscribed in the sector Δ_α whose center $(k, 0)$ with its radius r. Then, we have

$$r = k \sin \alpha.$$

Then, note that as k tends to zero, r tends to zero, meanwhile k tends to $+\infty$, r tends to $+\infty$. However, by our division by zero calculus, we see that immediately

$$[r]_{r=\infty} = 0.$$

$[r]_{r=\infty}$ means that it is the value at the point at infinity of the function r, that is not a limit to the point at infinity.

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to a point (the origin) suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

2.3 H. Okumura's example

The suprising example by H. Okumura will show a new phenomenon at the point at infinity.

On the sector Δ_α, we shall change the angle and we consider a fixed size circle $C_a, a > 0$ with its radius a inscribed in the sector. We see that when the circle tends to $+\infty$, the angles α tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with its center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}.$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive y axis is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a in a sense.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

These two subsections were taken from [2]. See also [3].

2.4 Horn torus model

We will consider the three circles represented by

$$\xi^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

$$\left(\xi - \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2,$$

and

$$\left(\xi + \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2.$$

By rotation on the space (ξ, η, ζ) on the (x, y) plane as in $\xi = x, \eta = y$ around ζ axis, we will consider the sphere with $1/2$ radius as the Riemann sphere and the horn torus made in the sphere.

The stereographic projection mapping from (x, y) plane to the Riemann sphere is given by

$$\xi = \frac{x}{x^2 + y^2 + 1},$$

$$\eta = \frac{y}{x^2 + y^2 + 1},$$

and

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}.$$

Of course,

$$\xi^2 + \eta^2 = \zeta(1 - \zeta),$$

and

$$x = \frac{\xi}{1 - \zeta}, y = \frac{\eta}{1 - \zeta}.$$

The mapping from (x, y) plane to the horn torus is given by

$$\xi = \frac{2x\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

$$\eta = \frac{2y\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

and

$$\zeta = \frac{(x^2 + y^2 - 1)\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2} + \frac{1}{2}.$$

This Puha mapping has a simple and beautiful geometrical correspondence. At first for the plane we consider the stereographic mapping to the Riemann sphere and next, we consider the common point of the line connecting the point and the center (0,0,1/2) and the horn torus. This is the desired point on the horn torus for the plane point.

W. W. Däumler discovered a surprising conformal mapping from the extended complex plane to the horn torus model:

https://www.horntorus.com/manifolds/conformal.html
and
https://www.horntorus.com/manifolds/solution.html

We obtain the complicated conformal mapping for the z plane to the horn torus, by a delicate arrangement of the Puha mapping.

We can represent the direct Däumler mapping from the z plane onto the horn torus as follows : With

$$\phi = 2\cot^{-1}(-\log|z|),$$

$$\xi = \frac{x \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2+y^2}},$$

$$\eta = \frac{y \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2+y^2}},$$

and

$$\zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}.$$

See the papers [1, 4] for the details.

3 Conclusion

On the horn torus models of Puha and Däumler, the example in Subsection 2.2 is clear.

Meanwhile, on the Okumura example, note that the series of discs tending to the point at infinity converges to the crucial point of the horm torus on the upper part. However, the disc with its center at the origin, of course, is mapped to the lower part of the horn torus. Therefore, we see the surprising result:

Conclusion: The Okumura's disc series can beyond the crucial point of Däumler-Puha's horn torus models for the Riemann sphere.

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