1 Abstract

The Proof involves Analytic Continuation of the Riemann Zeta function expressed as a Hadamard Product

2 Proof

The Analytic Continuity of Riemann Zeta -function over

\[ 0 < \text{Re}(s) < 1 \]

defined as a Hadamard Product \[2\] is,

\[ \zeta(s) = \frac{1}{2} \Pi \rho(1 - s/\rho) \]

Let, \( s = \sigma + it \)

and \( \rho = a + ib. \)

let;

\[ 1/2 < \sigma < \eta < 1 \]

\[ \zeta(\sigma + it) \]
\[= \prod \rho \left[1 - (\sigma + it)/(a + ib)\right]\]

\[|\zeta(\sigma + it)| = \frac{1}{2} \prod \rho (\sigma - a)^2 + (t - b)^2)^{\frac{1}{2}}/(a^2 + b^2)^{\frac{1}{2}}\]

\[|\zeta(\eta + it)| = \frac{1}{2} \prod (\eta - a)^2 + (t - b)^2)^{\frac{1}{2}}/[a^2 + b^2)^{\frac{1}{2}}]\]

**CASE 1**: \(1/2 < a < \sigma < \eta < 1\) and \(t\) is fixed

\((\eta - a) > (\sigma - a) > 0\)

\((\eta - a)^2 > (\sigma - a)^2\)

\(|\zeta(\eta + it)| > \frac{1}{2} \prod (\sigma - a)^2 + (t - b)^2)^{\frac{1}{2}}.\]

\(|\zeta(\eta + it)| > |\zeta(\sigma + it)|\]

So, for fixed \(t\),

\(|\zeta(\sigma + it)|\) is Strictly Monotonically Increasing for \(1/2 < \sigma < 1\).

**CASE 2**:

\(1/2 < \sigma < \eta < 1\)

\((a - \eta) > (a - \sigma).\)

\((a - \eta)^2 > (a - \sigma)^2.\)

\(|\zeta(\sigma + it)| < |\zeta(\eta + it)|\)

\(|\zeta(\sigma + it)|\) is Strictly Monotonically Increasing.
CASE 3: \(0 < \sigma < a < \eta < 1/2\).

\((\eta - a) > (a - \sigma)\)

\((\eta - a)^2 > (a - \sigma)^2\)

\(|\zeta(\eta + it)| > |\zeta(\sigma + it)|\).

So, \(|\zeta(\sigma + it)|\) is Strictly Monotonically Increasing in this case.

This gives that \(\zeta(\sigma + it) \neq 0 \forall \sigma \in (0, 1) - [1/2]\).

but, by Hypothesis, \(\zeta(\sigma + it) = 0, \sigma \in (0, 1)\)

Hence,

\(\sigma = 1/2\).

So, the real part of all the non trivial zeroes is 1/2.
3 References:-


2. Titchmarsh- The theory of Functions .


8. A note on $S(t)$ and the zeros of the Riemann zeta-function - DA Goldston, SM Gonek.


Contact:

Shekhar Suman.


Email - shekharsuman068@gmail.com