Using Ratios to Show the Real Numbers are a Denumerable Set
by Jim Rock

Abstract. We show that Cantor’s diagonal argument starts with an invalid premise. We represent real numbers as the limit of their partial decimal sums. Using the binary system, we calculate the ratio of the number of ones to the total number of digits for decimals in the closed interval \([0, 1]\). We use these ratios to show that the real numbers are a denumerable set. Since the reals numbers have the same cardinality as the power set of the natural numbers, \(\mathcal{P}(\mathbb{N}) = \mathcal{P}(\omega)\). There is no hierarchy of infinities. We have created Level Set Theory.

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The natural numbers can be put in one to one correspondence with the terminating decimal fractions in the closed interval \([0, 1]\).

\[
0 \rightarrow .0, 1 \rightarrow .1, 2 \rightarrow .2, \ldots, 10 \rightarrow .01, \ldots
\]

Each terminating decimal is the mirror image reflection through the decimal point of a natural number. The mapping does not include any repeating decimals. From this mapping the set of all rational numbers would appear to be uncountable.

This shows that attempting to map the real numbers in the closed interval \([0, 1]\) to the natural numbers, by listing them as infinite decimal fractions is an invalid process. Cantor’s diagonal proof that the real numbers are an uncountable set can never even get started.

The problem is in defining the real numbers as the set of all infinite decimal expansions. That’s vague. Most non-algebraic real numbers cannot be explicitly referenced. We need a new definition. Let \(p\) be an integer.

Each real number \(S_n\) is the limit of its partial decimal sums:

\[
\text{the limit } m \rightarrow \infty \quad j = 1 \text{ to } m, \quad 0 \leq a_j \leq 9, \quad \sum p + a_j / 10^j = S_n
\]

Whenever a ratio \(r\) is referenced in this paper, it is the number of ones divided by the total number of digits of a binary decimal in the closed interval \([0, 1]\).

The canonical form of a terminating decimal with \(r = 0(1)\) begins with a finite number of (ones) zeros and ends with an infinite number of zeros (ones). By rearranging its digits, each terminating decimal maps to a unique canonical terminating decimal. All canonical terminating decimals’ digits can be rearranged in a countably infinite number of ways to generate all terminating decimals.

A canonical repeating decimal begins with \(O \geq 0\) ones and \(Z \geq 0\) zeros followed by an infinite pattern of alternating \((a)\) ones and \((b)\) zeros. We can create \(a\) different sets of ones of size \(\aleph_0\) null by taking from the infinite pattern of alternating \(a\) ones and \(b\) zeros the ones in position \(1, 1+a+b, 1+(2)(a+b), 1+(3)(a+b)\ldots \), \(2, 2+a+b, 2+(2)(a+b), 2+(3)(a+b)\ldots \) up to \(a, a+(a+b), a+(2)(a+b), a+(3)(a+b)\ldots \) \(b\) sets of zeros can be made in the same fashion, replacing \(1\) with \(a+1\), \(2\) with \(a+2\), \(3\) with \(a+3\ldots \) up to \(a\) with \(a+b\), giving \(a+b+(a+b), a+b+(2)(a+b), a+b+(3)(a+b)\ldots \), thus creating a ratio of ones to all digits of \(a / (a + b)\).

Conjecture: all pure (no rational component) algebraic irrationals \(0 < r < 1\) are generated by rearranging the digits of canonical repeating decimals (see appendix for details.)

If the ratio created by the digits of the partial decimal sums gets progressively smaller (larger) as the partial decimal sums get larger, \(r = 0(1)\) for the entire (conjecture transcendental) decimal. Conjecture: \(r = 0(1)\) transcendentals combined with themselves and algebraic numbers generate all transcendental numbers.

Since there are no more unique rearrangements of digits for \(r = 0, 1\) than there are for canonical repeating decimals, all possible rearranging of \(r = 0(1)\) digits generate a countably infinite number of \(r = 0(1)\) decimals. Since all rearranging of the digits of canonical repeating decimals and \(r = 0(1)\) generate countably infinite sets, the real numbers are denumerable.
Since the real numbers have the same cardinality as the power set of the natural numbers, there is no hierarchy of infinities. We have created Level Set Theory. $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|.$

Appendix. Rearranging the digits in an irrational decimal

When rearranging the digits of algebraic irrationals to form canonical repeating decimals, there is only one correct alternating pattern of $a$ ones and $b$ zeros. Use $c$ for $a$, and $d$ for $b$. If $(a / c) / (b / d) > 1$ there will be an infinite number of zeros left over. If $(a / c) / (b / d) < 1$ there will be an infinite number of ones left over. If the correct pattern is used, there can be only a finite number of zeros and ones (not multiples of both $a$ and $b$) left over.

Explore the detailed proofs and fascinating consequences of the real numbers as a denumerable set in

https://arxiv.org/abs/1002.4433

Addressing mathematical inconsistency: Cantor and Gödel refuted by J. A. Perez.

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