Refutation of extended truth definitions to Peano arithmetic

Abstract: We evaluate proof-theoretic analysis by iterated reflection and ordinal analysis of iterated arithmetical comprehension. The former is *not* tautologous, and the latter is a contradiction. This refutes the extended truth definitions as proffered on Peano arithmetic and forms a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, · , ⊗ ; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠, ⊕;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1; (%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Beklemishev, L.D.; Pakhomov, F.N. (2019). Reflection algebras and conservation results for theories of iterated truth. arxiv.org/pdf/1908.10302.pdf lbekl@yandex.ru

Abstract We consider extensions of the language of Peano arithmetic by transfinitely iterated truth definitions satisfying uniform Tarskian biconditionals. Without further axioms, such theories are known to be conservative extensions of the original system of arithmetic. Much stronger systems, however, are obtained by adding either induction axioms or reflection axioms on top of them. Theories of this kind can interpret some well-known predicatively reducible fragments of second order arithmetic such as iterated arithmetical comprehension.

8 Proof-theoretic analysis by iterated reflection

8.3 A case study: analysis of ACA This method of analysis, in the simplest situation going beyond Peano arithmetic, can be illustrated by the well-known example of the second order theory ACA. This system extends PA by the schemata of induction, for all second order formulas, and by the comprehension schema: [for each arithmetical formula ϕ (possibly with first- and second-order parameters but not containing Y as a parameter).]

$$\exists Y \forall x (x \in Y \leftrightarrow \phi(x)) \qquad (8.3.9.1)$$

LET p, q, r, s: φ or X, x, Y, S.

(#q<%r)=(p&#q) ; `TTCT TTTC TTCT TTTC` (8.3.9.2)

Remark 8.3.9.2: Eq. 8.3.9.2 as rendered is *not* tautologous, hence refuting the comprehension schema.

9 Analysis of second order systems In this section we show how Theorem 8 can be used to obtain ordinal analysis of some systems of second order arithmetic of ‘predicative’ strength.

9.1 Ordinal analysis of iterated arithmetical comprehension … The base theory of second-order arithmetic we consider is the well-known theory ACA_0, that is, the extension of EA by the scheme of arithmetic comprehension (9) and the axiom of set-induction

$$0 \in X \wedge \forall x\,(x \in X \rightarrow S(x) \in X) \rightarrow \forall x\,(x \in X) \qquad (9.1.1)$$

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((s@s)<q)&( ((#q<p)>((s&#q)<p))>(#q<p)) ;
                                        FFFF FFFF FFFF FFFF        (9.1.2)
```

Remark 9.1.2: Eq. 9.1.2 is *not* tautologous, and in fact is a contradiction. This refutes the base theory of second-order arithmetic, chosen as ACA_0, that is, the extension of EA by the scheme of arithmetic comprehension (8.3.9.2) and the axiom of set-induction [not given in the text].