ENVELOPES IN FUNCTION SPACES WITH RESPECT TO CONVEX SETS

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We discuss the existence of an envelope of a function from a certain subclass of function space. Here we restrict ourselves to considering the model space $L^1_{\mathrm{loc}}(D)$ of functions locally integrable with respect to the *Lebesgue measure* λ in an open connected subset, i. e., *domain* D from the *d-dimensional Euclidean space* $\mathbb{R}^d$, where $d \in \mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{R}$ is the *real line.* Let H be a subset in $L^1_{\mathrm{loc}}(D)$, and $F\colon D \to \mathbb{R}_{\pm\infty} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ be a extended numerical function belonging to $L^1_{\mathrm{loc}}(D)$. We say that there exists a *lower envelope for F with respect to H* if there is a function $h \in H$ such that $h \leq F$ on D. Denote by $\operatorname{sbh}(D)$, $C(D)$, and $C^k(D)$ with $k \in \mathbb{N} \cup \{\infty\}$ the classes of *subharmonic, continuous,* and *k times continuously differentiable functions* on D, respectively [1]. The class $\operatorname{sbh}(D)$ contains the minus-infinity function $\boldsymbol{-\infty}\colon x \mapsto -\infty$ identically equal to $-\infty$; $\operatorname{sbh}_*(D) := \operatorname{sbh}(D) \setminus \{\boldsymbol{-\infty}\}$.

The *Alexandroff compactification* of $\mathbb{R}^d$ is denoted by $\mathbb{R}^d_\infty := \mathbb{R}^d \cup \{\infty\}$. Given a subset S of $\mathbb{R}^d_\infty$, the *closure* $\operatorname{clos} S$ and the *interior* $\operatorname{int} S$ will always be taken relative $\mathbb{R}^d_\infty$. For $S' \subset S \subset \mathbb{R}^d_\infty$ we write $S' \Subset S$ if $\operatorname{clos} S' \subset \operatorname{int} S$.

Let $\operatorname{Borel}(S)$ be the *class of all Borel subsets in* $S \in \operatorname{Borel}(\mathbb{R}^d_\infty)$. We denote by $\operatorname{Meas}(S)$ the *class of all Borel signed measures on* $\operatorname{Borel}(S)$; $\operatorname{Meas}_{\mathrm{cmp}}(S)$ is the class of measures $\mu \in \operatorname{Meas}(S)$ with a compact *support* $\operatorname{supp} \mu \Subset S$; $\operatorname{Meas}^+(S) := \{\mu \in \operatorname{Meas}(S)\colon \mu \geq 0\}$, $\operatorname{Meas}^+_{\mathrm{cmp}}(S) := \operatorname{Meas}_{\mathrm{cmp}}(S) \cap \operatorname{Meas}^+(S)$.

Definition (of linear and affine balayage of measures [2, Definition 7.1], [3], [4]). Let $H \subset \operatorname{sbh}(D)$, $\nu \in \operatorname{Meas}^+(D)$, and $\mathcal{M} \subset \operatorname{Meas}^+(D)$. We say that a measure $\mu \in \mathcal{M}$ is *a linear balayage of the measure ν with respect to H in $\mathcal{M}$* and write $\nu \preceq_{H,\mathcal{M}} \mu$ if

$$\int_D h \, \mathrm{d}\nu \leqslant \int_D h \, \mathrm{d}\mu \quad \textit{for all functions } h \in H.$$

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Let $c \in \mathbb{R}$, and $\mathbf{1}\colon x \mapsto 1$ be the function identically equal to 1 on $D \ni x$. We say that an affine function $\mu + c := \mu + c \cdot \mathbf{1} \in \mathcal{M} + \mathbb{R}\mathbf{1}$ is *an affine balayage of the measure ν with respect to H in $\mathcal{M} + \mathbb{R}\mathbf{1}$*, and write $\nu \curlyeqprec_{H,\mathcal{M}} \mu + c$, if

$$\int h \,\mathrm{d}\nu \leqslant \int h \,\mathrm{d}\mu + c \quad \text{for all functions } h \in H.$$

Proposition ([2, Proposition 8.1], [4, Proposition 7.1]). *Let $\varnothing \neq H \subset \operatorname{sbh}_*(D)$, $\mathcal{M} \subset \operatorname{Meas}^+_{\mathrm{cmp}}(D)$, and $0 \neq \nu \in \operatorname{Meas}^+_{\mathrm{cmp}}(D)$ be such that $-\infty < \int_D h \,\mathrm{d}\nu$ for all $h \in H$. Let the restriction of F to each compact subset $K \Subset D$ is $\mu\bigm|_K$-measurable for every measure $\mu \in \mathcal{M}$, where $\mu\bigm|_K$ is the restriction of μ to K. If there exists a lower envelope $h \leq F$ on D for F with respect to $H \ni h$, then*

$$-\infty < \inf\left\{\int_D F \,\mathrm{d}\mu \colon \nu \preceq_{H,\mathcal{M}} \mu\right\}, \tag{1lin}$$

$$-\infty < \inf\left\{\int_D F \,\mathrm{d}\mu + c \colon \nu \curlyeqprec_{H,\mathcal{M}} \mu + c\right\}. \tag{1aff}$$

This Proposition is somewhat reversible if H is convex. For simplicity and brevity, we formulate such an almost inverse statement only for $F \in C(D)$.

Theorem (general case in [2, Theorem 6], special case in [4, Theorem 7.1]). *Let $F \in C(D)$, $\mathbb{R}\mathbf{1} \subset H \subset \operatorname{sbh}_*(D)$, $0 \neq \nu \in \operatorname{Meas}^+_{\mathrm{cmp}}(D)$, $\operatorname{supp}\nu \subset U_0 \Subset D$, where U_0 is a domain, $\mathcal{M} := \bigl\{\mu \in \operatorname{Meas}^+_{\mathrm{cmp}}(D \setminus U_0) \colon \mathrm{d}\mu = m \,\mathrm{d}\lambda$, where $m \in C^\infty(D)\bigr\}$. Suppose that one of the following two conditions is fulfilled:*

[H1] *for any locally bounded from above sequence of functions $(h_k)_{k\in\mathbb{N}} \subset H$, the upper semi-continuous regularization of the upper limit $\limsup_{k\to\infty} h_k$ belong to H provided that $\limsup_{k\to\infty} h_k \not\equiv -\infty$ on D;*

[H2] *H is sequentially closed in $L^1_{\mathrm{loc}}(D)$.*

[L] *If H is convex cone, and the condition* (1lin) *is fulfilled, then there exists a lower envelope $h \leq F$ on D for F with respect to $H \ni h$.*

[A] *If H is convex set, and the condition* (1aff) *is fulfilled, then there exists a lower envelope $h \leq F$ on D for F with respect to $H \ni h$.*

In our review [2], this Theorem is proved in a much more general form for arbitrary functions $F \in L^1_{\mathrm{loc}}(D)$ without condition $\mathbb{R}\mathbf{1} \in H$. Special cases of this Theorems and corollaries from it have been successfully applied in our articles [5]–[10] to study the distribution of zero sets of holomorphic functions under restrictions on their growth. Research on the application of this Theorem in complex analysis will be continued. More general abstract forms of our Theorem from [2] and [3] can find applications in other functional spaces far from the space $L^1_{\mathrm{loc}}(D)$ since they are given for projective limits of vector lattices or topological projective limits of Frechet lattices (see [2, Ch. 1], [3] and bibliography in them).

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