THE RIEMANN HYPOTHESIS

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0.1 Abstract

The Proof involves Functional Equation of the Riemann Zeta function defined on the whole of Complex plane except for a Pole at $s=1$.

Further we prove that the absolute value of the Riemann Zeta Function is monotonically decreasing and monotonically increasing on specific intervals respectively. By using the above monotonicity property of zeta , work on it’s non trivial xeroes

THE RIEMANN HYPOTHESIS (1859): The real part of every non trivial zero of Riemann Zeta Function is 1/2.

0.2 Proof

The analytic continuation (Ref.-[1]) of the Zeta Function is

$$
\zeta^*(s) = s \int_0^\infty ([x] - x)/x^{s+1}dx \ ; 0<\text{Re}(s)<1
$$

let, $s = \sigma + i\eta; \ 0<\sigma<1$.

The Functional equation of the Riemann Zeta function is given as,

$$
\zeta^*(s) = 2^\sigma \pi^{s-1} \sin(\pi s/2) \Gamma(1 - s) \zeta(1 - s)
$$

$$
\zeta^*(\sigma + it) = 2^{\sigma + i\eta} (\pi)^{\sigma-1+i\eta} \sin(\pi(\sigma + i\eta)/2) \Gamma(1 - \sigma - i\eta) \zeta(1 - \sigma - i\eta)
$$

$$
| \zeta^*(\sigma + it) | / | \zeta^*(\rho + it) |
$$

$$
(2\pi)^{\sigma-\rho} \ | \sin(\pi(\sigma + i\eta)/2) | \ | \Gamma(1 - \sigma - i\eta) | \ | \zeta(1 - \sigma - i\eta) | / \\
| \sin(\pi(\rho + i\eta)/2) | \ | \Gamma(1 - \rho - i\eta) | \ | \zeta(1 - \rho - i\eta) |
$$
0.3 Claim

$|\zeta^*(\sigma + i\eta)|$ is monotonically increasing.

To Prove: $|\zeta^*(\sigma + i\eta)| \leq |\zeta^*(\rho + i\eta)|$

Idea is to prove each term in Eq.-A less than 1.

let, $\sigma < \rho$.

Consider,

$|\sin(\pi(\sigma + i\eta)/2)|^2 - |\sin(\pi(\rho + i\eta)/2)|^2$

$= |\sin(\pi\sigma/2)cosh(\pi\eta/2) + icos(\pi\sigma/2)sinh(\pi\eta/2)|^2 -$

$|\sin(\pi\rho/2)cosh(\pi\eta/2) + icos(\pi\rho/2)sinh(\pi\eta/2)|^2$

$\leq [\sin^2(\pi\sigma/2)cosh^2(\pi\eta/2) + (\cos^2(\pi\sigma/2)sinh^2(\pi\eta/2)) -$

$(\sin^2(\pi\rho/2)cosh^2(\pi\eta/2) - (\cos^2(\pi\rho/2)sinh^2(\pi\eta/2))]

Since, $\sin\theta$ is increasing and $\cos\theta$ is decreasing on $[1/2, 1) \subset (0, \pi)$

$\leq [\sin^2(\pi\rho/2)cosh^2(\pi\eta/2) + (\cos^2(\pi\sigma/2)sinh^2(\pi\eta/2)) -$

$(\sin^2(\pi\rho/2)cosh^2(\pi\eta/2) - (\cos^2(\pi\rho/2)sinh^2(\pi\eta/2))$

$(\cos^2(\pi\sigma/2)sinh^2(\pi\eta/2))^2 -$

$(\cos^2(\pi\rho/2)sinh^2(\pi\eta/2))$
Since, $1/2 \leq \rho < 1.$

\[
\leq (\cos^2(\pi/2) \sinh^2(\pi \eta/2))^2 - (\cos^2(\pi \rho/2) \sinh^2(\pi \eta/2)) \\ \leq (\cos^2(\pi/2) \sinh^2(\pi \eta/2))^2 \leq 0.
\]

thus,

\[
| \sin(\pi(\sigma + i \eta)/2) |^2 \leq | \sin(\pi(\rho + i \eta)/2) |^2 \text{ hence.}
\]

\[
| \sin(\pi(\sigma + i \eta)/2) | \leq | \sin(\pi(\rho + i \eta)/2) |. - [Eq. - 1].
\]

\[
| \Gamma(1 - \sigma - i \eta) | = | \int_0^\infty e^{-t} t^{\sigma-i\eta} dt |
\]

\[
\leq \int_0^\infty e^{-t} t^{\sigma} dt
\]

Now, $1/2 \leq \sigma < 1,$

$1/t < 1/t^{\sigma} \leq 1/t^{1/2}.$

Using the above inequality,

\[
| \Gamma(1 - \sigma - i \eta) | \leq \int_0^\infty e^{-t} t^{-1/2} dt
\]

\[
| \Gamma(1 - \sigma - i \eta) | \leq \gamma(1/2)
\]

\[
| \Gamma(1 - \sigma - i \eta) | \leq \pi^{1/2}
\]

Similarly, since $\rho$ also belongs to the same domain $[1/2, 1)$

\[
| \Gamma(1 - \rho - i \eta) | \leq \pi^{1/2}
\]
\[ |\Gamma(1 - \sigma - i\eta)| - |\Gamma(1 - \rho - i\eta)| \leq 0. \]

\[ |\Gamma(1 - \sigma - i\eta)| \leq |\Gamma(1 - \rho - i\eta)| \quad - \text{Eq. - 2.} \]

Thus,

\[ \frac{1}{2} \leq \sigma < \rho < 1, \]

Implies,

\[ |\zeta^*(\sigma + i\eta)| \leq |\zeta^*(\rho + i\eta)|. \]

So, \(\zeta^*(\sigma + it)\) is increasing on \([1/2, 1)\).

\[ |\zeta(1 - \sigma - i\eta)| = |\sum_{n=1}^{\infty} 1/n^{1-\sigma-i\eta}| \]

; where \(n^{-in} = e^{-in\ln n}\)

\[ \leq \sum_{n=1}^{\infty} 1/n^{1-\sigma} \]

\(\sigma < \rho\) implies \(1/n^{1-\sigma} < 1/n^{1-\rho}\)

\[ \leq \sum_{n=1}^{\infty} 1/n^{1-\rho} \]

\[ |\zeta(1 - \sigma - i\eta)| < \sum_{n=1}^{\infty} 1/n^{1-\rho} \]

\[ |\zeta(1 - \rho - i\eta)| < \sum_{n=1}^{\infty} 1/n^{1-\rho} \quad \text{So,} \]

\[ |\zeta(1 - \sigma - i\eta)| - |\zeta(1 - \rho - i\eta)| \leq 0 \]
\[
| \zeta(1 - \sigma - i\eta) | < | \zeta(1 - \rho - i\eta) | \quad -Eq - 3
\]

Also, \( \sigma < \rho \) implies 

\[2\pi^{\sigma - \rho} < 1 \quad Eq - 4\]

So, using Eq. 1, 2, 3 and 4 in Eq A,

When, \( \sigma < \rho \); \( \sigma, \rho \in [1/2, 1) \)

\[| \zeta^*(\sigma + i\eta) | \leq | \zeta^*(\rho + i\eta) | \]

Hence, \( | \zeta^*(\sigma + i\eta) | \) is Monotonically Increasing on\([1/2, 1), \]

### 0.4 Claim

\( \zeta^*(\sigma + i\eta) \) is Monotonically Decreasing on\((0, 1/2] \)

Let, \( \sigma^1 = 1 - \sigma \)

\( \rho^1 = 1 - \rho \)

Since, \( 1/2 \leq \sigma < \rho < 1 \)

Thus, \( 0 < \rho^1 < \sigma^1 \leq 1/2. \)

\[| \zeta^*(1 - \sigma^1 + i\eta) | \leq | \zeta^*(1 - \rho^1 + i\eta) | . \]

we know \( | \zeta(s) | = | \zeta(1 - s) | \)
Thus,

when $\rho^1 < \sigma^1$,

$$|\zeta^*(\sigma^1 - i\eta)| \leq |\zeta^*(\rho^1 - i\eta)|.$$

So, $|\zeta^*(\sigma + i\eta)|$ is Monotonically Increasing on $[1/2, 1]$, and $|\zeta^*(\sigma + i\eta)|$ is Monotonically Decreasing on $(0, 1/2]$

For, $1/2 \leq \sigma < 1$,

$$|\zeta^*(1/2 + it)| \leq |\zeta^*(\sigma + it)| < |\zeta^*(1 + it)|.$$

For, $0 \leq \sigma \leq 1/2$,

$$|\zeta^*(1/2 + it)| < |\zeta^*(\sigma + it)| \leq |\zeta^*(0 + it)|.$$

Combining above two inequalities, for all $\sigma \in (0, 1/2] \cup [1/2, 1)$,

$$|\zeta^*(1/2 + it)| \leq |\zeta^*(\sigma + it)|.$$

Now $\zeta^*(\sigma + it) = 0$ for, $0 < \sigma < 1$.

$$|\zeta^*(1/2 + it)| \leq 0$$

$\zeta^*(1/2 + it) = 0$
0.5 Case

let, 
\[ \zeta^*(\sigma + i\eta) = 0 \text{ and } \sigma \neq 1/2 \]

then,

\[ 0 \leq |\zeta^*(1/2 + it)| \leq |\zeta^*(\sigma + it)| \leq \zeta^*(1 + it) \]

but,

\[ \zeta^*(\sigma + it) = 0, \text{ for } 0 < \sigma < 1\]

implies

\[ |\zeta^*(1/2 + it)| \leq 0 \]

this implies ,

\[ |\zeta^*(1/2 + it)| = 0 \]

which is a contradiction to our assumption that

\[ |\zeta^*(\sigma + i\eta)| \neq 0 \]

if \( \sigma \neq 1/2 \),

So we have non trivial zeroes of \( \zeta^*(s) \) implies \( \sigma = 1/2 \), on the line, \( \sigma = 1/2 \)
0.6 References:-


Ariel Jacobs - The Proof of the Riemann Hypothesis.


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