THE RIEMANN HYPOTHESIS

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0.1 Abstract

The Proof involves Functional Equation of the Riemann Zeta function defined on the whole of Complex plane except for a Pole at s=1.

Further we prove that the absolute value of the Riemann Zeta Function is monotonically decreasing and monotonically increasing on specific intervals respectively. By using the above monotonicity property of zeta , work on it's non trivial xeroes

THE RIEMANN HYPOTHESIS (1859): The real part of every non trivial zero of Riemann Zeta Function is 1/2.

0.2 Proof

The analytic continuation (Ref.-[1]) of the Zeta Function is

$\zeta^*(s) = s\int_0^\infty([x] - x)/x^{s+1}dx \;\; ; 0 < Re(s) < 1$

$let, \; s = \sigma + i\eta; \;\; 0 < \sigma < 1.$

$The Functional\ equation\ of\ the\ Riemann\ Zeta\ function\ is\ given\ as,$

$\zeta^*(s) = 2^s \; \pi^{s-1} \; sin(\pi s/2) \; \Gamma(1-s) \; \zeta(1-s)$

$\zeta^*(\sigma + it) = 2^{\sigma+i\eta} \; (\pi)^{\sigma-1+i\eta} \; sin(\pi(\sigma+i\eta)/2) \; \Gamma(1-\sigma-i\eta) \; \zeta(1-\sigma-i\eta)$

$\mid \zeta^*(\sigma + it) \mid / \mid \zeta^*(\rho + it) \mid =$

$(2\pi)^{\sigma-\rho} \mid \; sin(\pi(\sigma+i\eta)/2) \mid \mid \Gamma(1-\sigma-i\eta) \mid \mid \zeta(1-\sigma-i\eta) \mid \; /$
$\mid sin(\pi(\rho+i\eta)/2) \mid \mid \Gamma(1-\rho-i\eta) \mid \;\; \mid \zeta(1-\rho-i\eta) \mid$

0.3 Claim

$| \zeta^{*}(\sigma + i\eta) | \; is \;\; monotonically \;\; increasing.$

$To \; Prove : | \zeta^{*}(\sigma + i\eta) | \leq | \zeta^{*}(\rho + i\eta). |$

Idea is to prove each term in Eq.-A less than 1.

let, $\sigma < \rho$.

$Consider,$

$| sin(\pi(\sigma + i\eta)/2) |^2 - | sin(\pi(\rho + i\eta)/2) |^2$

$= | sin(\pi\sigma/2)cosh(\pi\eta/2) + icos(\pi\sigma/2)sinh(\pi\eta/2) |^2 -$

$| sin(\pi\rho/2)cosh(\pi\eta/2) + icos(\pi\rho/2)sinh(\pi\eta/2) |^2$

$\leq [sin^2(\pi\sigma/2)cosh^2(\pi\eta/2) + (cos^2(\pi\sigma/2)sinh^2(\pi\eta/2)) -$

$(sin^2(\pi\rho/2)cosh^2(\pi\eta/2) - (cos^2(\pi\rho/2)sinh^2(\pi\eta/2))$

$Since, sin\theta \; is \; increasing \; and \; cos\theta \; is \; decreasing \; on \; [1/2, 1) \; \subset (0, \pi)$

$\leq [sin^2(\pi\rho/2)cosh^2(\pi\eta/2) + (cos^2(\pi\sigma/2)sinh^2(\pi\eta/2)) -$

$(sin^2(\pi\rho/2)cosh^2(\pi\eta/2) - (cos^2(\pi\rho/2)sinh^2(\pi\eta/2))$

$(cos^2(\pi\sigma/2)sinh^2(\pi\eta/2))^2 -$

$(cos^2(\pi\rho/2)sinh^2(\pi\eta/2))$

$Since, 1/2 \leq \rho < 1.$

$\leq (cos^2(\pi/2)sinh^2(\pi\eta/2))^2 -$

$(cos^2(\pi\rho/2)sinh^2(\pi\eta/2))$

$\leq (cos^2(\pi/2)sinh^2(\pi\eta/2))^2 \leq 0.$

thus,

$\mid sin(\pi(\sigma + i\eta)/2) \mid^2 \leq \mid sin(\pi(\rho + i\eta)/2) \mid^2 \; hence,$

$\mid sin(\pi(\sigma + i\eta)/2) \mid \leq \mid sin(\pi(\rho + i\eta)/2) \mid . - [Eq. - 1].$

$\mid \Gamma(1 - \sigma - i\eta) \mid = \mid \int_0^\infty e^{-t} t^{-\sigma - i\eta} dt \mid$

$\leq \int_0^\infty e^{-t} t^{-\sigma} dt$

$Now, 1/2 \leq \sigma < 1,$

$1/t < 1/t^\sigma \leq 1/t^{1/2}.$

$Using the above ineqality,$

$\mid \Gamma(1 - \sigma - i\eta) \mid \leq \int_0^\infty e^{-t} t^{-1/2} dt$

$\mid \Gamma(1 - \sigma - i\eta) \mid \leq \gamma(1/2)$

$\mid \Gamma(1 - \sigma - i\eta) \mid \leq \pi^{1/2}$

Similarly, since ρ *also belongs to the same domain* $[1/2, 1)$

$\mid \Gamma(1 - \rho - i\eta) \mid \leq \pi^{1/2}$

$| \Gamma(1-\sigma-i\eta) | - | \Gamma(1-\rho-i\eta) | \leq 0.$

$| \Gamma(1-\sigma-i\eta) | \leq | \Gamma(1-\rho-i\eta) | .. - Eq. - 2.$

Thus,

$1/2 \leq \sigma < \rho < 1,$

$Implies,$

$| \zeta^*(\sigma+i\eta) | \leq | \zeta^*(\rho+i\eta). |$

$So, \zeta^*(\sigma+it) is increasing on [1/2, 1).$

$| \zeta(1-\sigma-i\eta) | = | \sum_{n=1}^{\infty} 1/n^{1-\sigma-i\eta} |$

$; where] \;\; n^{-in} = e^{-i\eta lnn}$

$\leq \sum_{n=1}^{\infty} 1/n^{1-\sigma}$

$\sigma < \rho \;\; implies 1/n^{1-\sigma} < 1/n^{1-\rho}$

$\leq \sum_{n=1}^{\infty} 1/n^{1-\rho}$

$| \zeta(1-\sigma-i\eta) | < \sum_{n=1}^{\infty} 1/n^{1-\rho}$

$| \zeta(1-\rho-i\eta) | < \sum_{n=1}^{\infty} 1/n^{1-\rho} So,$

$| \zeta(1-\sigma-i\eta) | - | \zeta(1-\rho-i\eta) | \leq 0$

$| \zeta(1-\sigma-i\eta) |<| \zeta(1-\rho-i\eta) | -Eq-3$

$Also, \sigma < \rho \;\; implies$

$2\pi^{\sigma-\rho} < 1 \;\; .Eq-4$

$So, \; using \; Eq. \; 1,2,3 \; and \; 4 \; in \; Eq \; A,$

$When, \sigma<\rho; \;\; \sigma, \rho \in [1/2, 1)$

$| \zeta^*(\sigma+i\eta) |\leq| \zeta^*(\rho+i\eta). |$

Hence, $| \zeta^*(\sigma+i\eta) | \;\; is \;\; Monotonically \; Increasing \; on[1/2, 1),$

0.4 Claim

$\zeta^*(\sigma+i\eta) \; is \; Monotonically \; Decreasing \; on(0, 1/2]$

$Let, \sigma^1 = 1-\sigma$

$\rho^1 = 1-\rho$

$Since, \;\; 1/2 \leq \sigma<\rho<1$

$Thus, 0<\rho^1<\sigma^1 \leq 1/2.$

$| \zeta^*(1-\sigma^1+i\eta) |\leq| \zeta^*(1-\rho^1+in).$

$weknow \; | \zeta(s) |=| \zeta(1-s) |$

Thus,

when $\rho^1 < \sigma^1$,

$\mid \zeta^*(\sigma^1 - i\eta) \mid \leq \mid \zeta^*(\rho^1 - in)$.

So, $\mid \zeta^*(\sigma + i\eta) \mid isMonotonically\ \ Increasing\ on\ [1/2, 1), and$

$\mid \zeta^*(\sigma + i\eta) \mid isMonotonically\ \ Decreasing\ on\ (0, 1/2]$

$For, 1/2 \leq \sigma < 1,$

$\mid \zeta^*(1/2 + it) \mid \leq \mid \zeta^*(\sigma + it) \mid < \mid \zeta^*(1 + it) \mid$
.

For, $0 \leq \sigma \leq 1/2$,

$\mid \zeta^*(1/2 + it) \mid < \mid \zeta^*(\sigma + it) \mid \leq \mid \zeta^*(0 + it) \mid$.

Combining above two ineqalities, for all $\sigma \in (0, 1/2] \cup [1/2, 1)$,

$\mid \zeta^*(1/2 + it) \mid \leq \mid \zeta^*(\sigma + it) \mid$.

Now $\quad \zeta^*(\sigma + it) = 0 for, 0 < \sigma < 1.$

$\mid \zeta^*(1/2 + it) \mid \leq 0$

$\zeta^*(1/2 + it) = 0$

0.5 Case

let,

$\zeta^{*}(\sigma + i\eta) = 0$ *and* $\sigma \neq 1/2$

then,

$$0 \leq \mid \zeta^{*}(1/2 + it) \mid \leq \mid \zeta^{*}(\sigma + it) \leq \zeta_{*}(1 + it)$$

but,

$$\zeta_{*}(\sigma + it) = 0, for\ 0 < \sigma < 1 implies$$

$\mid \zeta_{*}(1/2 + it) \mid \leq 0$

this implies ,

$\mid \zeta^{*}(1/2 + it) \mid = 0$

which is a contradiction to our assumption that

$$\mid \zeta^{*}(\sigma + i\eta) \mid = 0$$

$if \sigma \neq 1/2,$

So we have non trivial zeroes of $\zeta^{*}(s)$ *implies* $\sigma = 1/2$, *on the line,*

$\sigma = 1/2$

0.6 References:-

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