Determination of a Triangle from Symmedian Point and Two Vertexes

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Abstract

Given three noncollinear points P, B and C, we investigate the construction of the triangle DBC with symmedian point P.

1 Introduction

The problem of constructing a triangle ABC given two vertices A, B and the symmedian point K is solved of Michel Bataille in [1].

In this paper we give another construction.

2 Preliminaries

We shall work with homogeneous barycentric coordinates. We consider a nondegenerate triangle ABC as the reference triangle, and set up a coordinate system for points in the plane of the triangle ($a = |BC|, b = |CA|, c = |AB|$).

$$A = (1 : 0 : 0)\ , \quad B = (0 : 1 : 0)\ , \quad C = (0 : 0 : 1)$$

We shall make use of John H. Conway's notations [5, §3.4.1]. Let S denote *twice* the area of triangle ABC. For a real number θ, denote $S_\theta = S\cot\theta$. In particular,

$$S_A = \frac{b^2 + c^2 - a^2}{2}\ , \quad S_B = \frac{c^2 + a^2 - b^2}{2}\ , \quad S_C = \frac{a^2 + b^2 - c^2}{2}$$

$$S_B + S_C = a^2\ , \quad S_C + S_A = b^2\ , \quad S_A + S_B = c^2$$

$$S_{AB} = S_A S_B\ , \quad S_{BC} = S_B S_C\ , \quad S_{CA} = S_C S_A$$

$$S^2 = S_{AB} + S_{BC} + S_{CA} = \frac{1}{4}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)$$

Definition 1. The symmedian point K of the triangle ABC (Lemoine point, Grebe point, point X(6) in ETC, [4]) is the isogonal conjugate of the centroid G. The symmedian point K has homogeneous barycentric coordinates $(a^2 : b^2 : c^2)$.

Theorem 1. *[5, §4.6.2], [2, §118]*

The symmedian point is the only point which is the centroid of its own pedal triangle.

Theorem 2. *[2, §125]*

The symmedian point is the point of intersection of lines joining the midpoints of the sides of ABC to the midpoints of corresponding perpendiculars.

Definition 2. [5, §7.1] **The distance formula in homogeneous barycentric coordinates**

If $P = (x : y : z)$ and $Q = (u : v : w)$, the **square distance** between P and Q is given by:

$$|PQ|^2 = \frac{1}{(u+v+w)^2(x+y+z)^2} \sum_{cyclic} S_A((v+w)x - u(y+z))^2$$

3 The point D

Given three noncollinear points P, B and C, to be construct a triangle DBC with symmedian point P.

Let the point $P = (u : v : w), u \neq 0$ with respect to triangle ABC. Let the point $D = (x : y : z)$.

The symmedian point K_D of the triangle DBC has homogeneous barycentric coordinates with respect to triangle DBC : $K_D = (|BC|^2 : |CD|^2 : |DB|^2)$.

The square distances between D, B and C are (see Definition 2):

$$\begin{aligned} |BC|^2 &= a^2 \\ |CD|^2 &= \frac{b^2x^2 + a^2y^2 + (a^2+b^2-c^2)xy}{(x+y+z)^2} \\ |DB|^2 &= \frac{c^2x^2 + a^2z^2 + (a^2-b^2+c^2)xz}{(x+y+z)^2} \end{aligned} \tag{1}$$

With respect to triangle ABC, the symmedian point of triangle DBC has homogeneous barycentric coordinates:

$$\begin{aligned} K_D &= |BC|^2 D + |CD|^2 B + |DB|^2 C \\ &= a^2(x : y : z) + |CD|^2(0 : 1 : 0) + |DB|^2(0 : 0 : 1) \\ &= (a^2x(x+y+z) : b^2x(x+y) + y(-c^2x + a^2(2x+2y+z)) : c^2x(x+z) + z(-b^2x + a^2(2x+y+2z))) \end{aligned} \tag{2}$$

The point $P = (u : v : w)$ is the symmedian point of the triangle DBC if and only if

$$K_D = P = \frac{u}{u+v+w}A + \frac{v}{u+v+w}B + \frac{w}{u+v+w}C$$

Solving these equations we obtain two solutions (may be not real) for point D:

$$
\begin{aligned}
D_1 &= (-4u(a^2\sqrt{f}+2a^4u(u+v+w)) \\
&\quad : (b^2-c^2)u(\sqrt{f}+3(b^2-c^2)u^2)+a^4u(8u^2-3v^2+2vw+w^2+2u(v+3w)) \\
&\quad + a^2(\sqrt{f}(2u-v+w)-2u^2(b^2(3u-2w)+c^2(5u+2w))) \\
&\quad : (-b^2+c^2)\sqrt{f}u+3(b^2-c^2)^2u^3+a^2(-2u^2(c^2(3u-2v)+b^2(5u+2v)) \\
&\quad + \sqrt{f}(2u+v-w))+a^4u(8u^2+v^2+2vw-3w^2+2u(3v+w))) \\
D_2 &= (4u(a^2\sqrt{f}-2a^4u(u+v+w)) \\
&\quad : (-b^2+c^2)\sqrt{f}u+3(b^2-c^2)^2u^3+a^4u(8u^2-3v^2+2vw+w^2+2u(v+3w)) \\
&\quad - a^2(\sqrt{f}(2u-v+w)+2u^2(b^2(3u-2w)+c^2(5u+2w))) \\
&\quad : (b^2-c^2)u(\sqrt{f}+3(b^2-c^2)u^2)-a^2(2u^2(c^2(3u-2v)+b^2(5u+2v)) \\
&\quad + \sqrt{f}(2u+v-w))+a^4u(8u^2+v^2+2vw-3w^2+2u(3v+w))) \\
f &= u^2((16a^4-24a^2b^2+9b^4-24a^2c^2-18b^2c^2+9c^4)u^2+a^4v^2+a^4w^2 \\
&\quad + 2a^2(4a^2-3b^2+3c^2)uv+14a^4vw+2a^2(4a^2+3b^2-3c^2)wu)
\end{aligned} \tag{3}
$$

The points $D_{1,2}$ are real if and only if $f \geqslant 0$. This is equal to be the points D lie inside or on the conic χ:

$$
\begin{aligned}
\chi :\ & (16a^4-24a^2b^2+9b^4-24a^2c^2-18b^2c^2+9c^4)x^2+a^4y^2+a^4z^2 \\
& + 2a^2(4a^2-3b^2+3c^2)xy+14a^4yz+2a^2(4a^2+3b^2-3c^2)zx = 0
\end{aligned} \tag{4}
$$

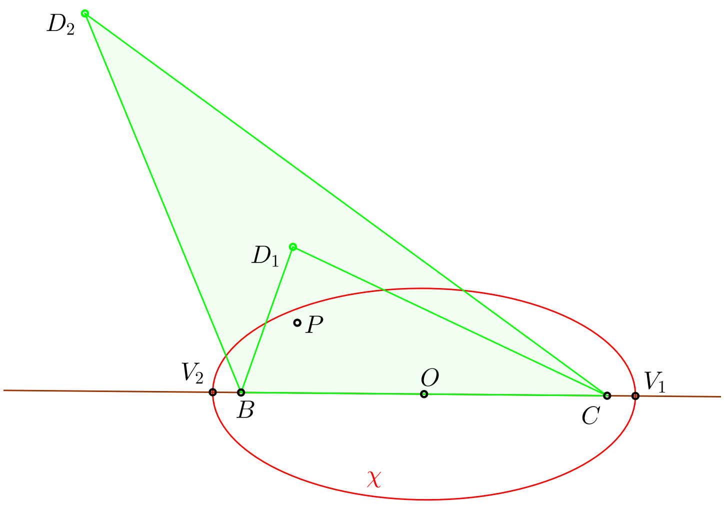

Figure 1: Triangle DBC and conic χ

The type of the conic is depending on its discriminant for the infinity point $(x : y : z), x + y + z = 0$, see [5, §10.7.1]. The conic χ has discriminant $-576a^2S^2 < 0$ and χ is a ellipse.

4 Ellipse χ

The center of the ellipse χ is the midpoint O of the segment BC, and focuses are the points B and C, see Figure 1, conf. [1, §3]. The vertices are the points

$$V_1 = (0 : 1 : -7 - 4\sqrt{3}),\ \ V_2 = (0 : 1 : -7 + 4\sqrt{3})$$

The ellipse is independent of the position of point A.

If we choose $A = P$, i.e. $u = 1, v = w = 0$, the condition $P = A = (1 : 0 : 0) \in \chi$ (see (4)) is $\delta = 0$ where:

$$\delta = 16a^4 - 24a^2b^2 + 9b^4 - 24a^2c^2 - 18b^2c^2 + 9c^4 \tag{5}$$

$$\begin{aligned}
\text{really } \delta =&(4a^2 - 3b^2 + 6bc - 3c^2)(4a^2 - 3b^2 - 6bc - 3c^2)\\
=&(2a + \sqrt{3}(b - c))(2a - \sqrt{3}(b - c))(2a + \sqrt{3}(b + c))(2a - \sqrt{3}(b + c))\ ;\\
&2a + \sqrt{3}(b + c) > 0\ ;\\
&2a + \sqrt{3}(b - c) = \sqrt{3}(a + b - c) + (2 - \sqrt{3})\, a > 0\ ;\\
&2a - \sqrt{3}(b - c) = \sqrt{3}(a - b + c) + (2 - \sqrt{3})\, a > 0
\end{aligned}$$

Condition point A be on the ellipse with focuses B, C and vertexes V_1, V_2 is

$$c + b = |AB| + |AC| = |V_1B| + |V_1C| = |V_1V_2| = \frac{2a}{\sqrt{3}}$$

i. e. $2a - \sqrt{3}(b + c) = 0$, or

$$\delta \begin{cases} > 0, & 2a - \sqrt{3}(b + c) > 0,\ A \text{ inside } \chi;\\ = 0, & 2a - \sqrt{3}(b + c) = 0,\ A \text{ on } \chi;\\ < 0, & 2a - \sqrt{3}(b + c) < 0,\ A \text{ outside } \chi. \end{cases} \tag{6}$$

Theorem 3. *Let P lies on the ellipse χ (4). Let $\mathcal{C}$ is a circle with center O and diameter V_1V_2. Construct pedal $P_\perp$ of the perpendicular of point P to line BC, and point P' — the reflection of $P_\perp$ in P. When P traverses the ellipse χ, the locus of P' is the circle $\mathcal{C}$ (Figure 2).*

Proof. The circle $\mathcal{C}$ has center $O = (0 : 1 : 1)$ — midpoint of the segment BC and radius $\frac{1}{2}|V_1V_2| = \frac{a}{\sqrt{3}}$. The equation of the circle (see [5, page 91]) is:

$$c^2xy+b^2xz+a^2yz-(x+y+z)\left(\left(-\frac{a^2}{3} + \frac{1}{4}(-a^2 + 2b^2 + 2c^2)\right) x - \frac{a^2y}{12} - \frac{a^2z}{12}\right) = 0$$

This can be rewritten as

$$\mathcal{C}:\ \ c^2xy + b^2xz + a^2yz - (x + y + z)\left(S_Ax - \frac{a^2(x + y + z)}{12}\right) = 0 \tag{7}$$

Let $P = (u, v, w)$. The infinity point of line BC is $(0 : -1 : 1)$. The infinity point of lines perpendicular to it is $(-S_B - S_C, S_C, S_B)$. The perpendicular from P

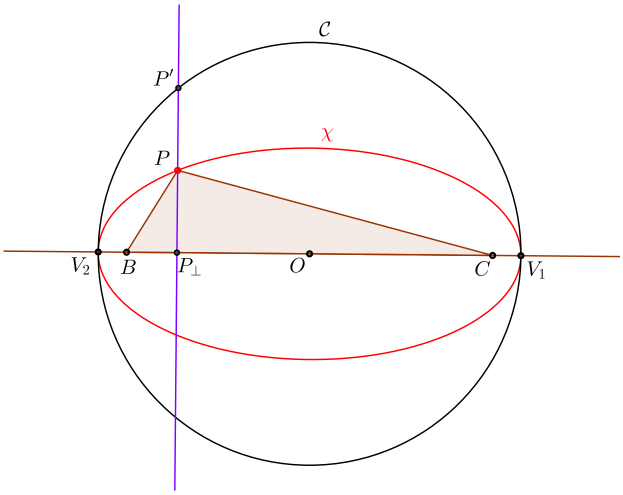

Figure 2: The circle $\mathcal{C}$

to BC is the line $(S_Bv - S_Cw)x - (S_Cw + S_B(u+w))y + (S_Bv + S_C(u+v))z = 0$ and the intersection with BC is the point $P_\perp = (0 : S_Bv + S_C(u+v) : S_Cw + S_B(u+w))$.

$$\overrightarrow{PP'} = \overrightarrow{P_\perp P}$$
$$P' - P = P - P_\perp$$
$$P' = 2P - P_\perp = (2(S_B + S_C)u : S_Bv + S_C(-u+v) : S_Cw + S_B(-u+w)) \tag{8}$$

The point P', (8) lies on $\mathcal{C}$, (7) if

$$\frac{1}{3}a^2(16a^4u^2 - 24a^2b^2u^2 + 9b^4u^2 - 24a^2c^2u^2 - 18b^2c^2u^2 + 9c^4u^2 + 8a^4uv - 6a^2b^2uv + 6a^2c^2uv + a^4v^2 + 8a^4uw + 6a^2b^2uw - 6a^2c^2uw + 14a^4vw + a^4w^2) = 0$$

But this is satisfied because condition to be point P lies on ellipse χ, (4) is

$$(16a^4u^2 - 24a^2b^2u^2 + 9b^4u^2 - 24a^2c^2u^2 - 18b^2c^2u^2 + 9c^4u^2 + 8a^4uv - 6a^2b^2uv + 6a^2c^2uv + a^4v^2 + 8a^4uw + 6a^2b^2uw - 6a^2c^2uw + 14a^4vw + a^4w^2) = 0$$

□

Theorem 4. *Let P be point of the ellipse χ. Let $(BP), (CP)$ are circles with diameters BP and CP respectively. The circles $(BP), (CP)$ tangent to the circle $\mathcal{C}$ with diameter V_1V_2 inwardly in points T_b, T_c respectively. The points P, T_b, T_c are collinear (Figure 3).*

Proof. The ellipse χ is independent of the position of point A. We choose $A = P$, i.e. $u = 1, v = w = 0$.

The circles $(BP) = (BA)$ with center midpoint of BA and radius $c/2$ (see [6, §9.6.1]) and $(CP) = (CA)$ with center midpoint of CA and radius $b/2$, have

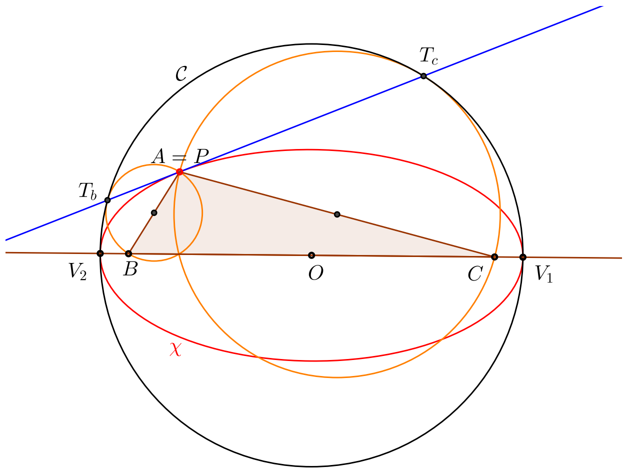

Figure 3: T_b, T_c

equations:

$$\begin{aligned} (BA): &\quad a^2yz + b^2zx + c^2xy - S_cz(x+y+z) = 0 \\ (CA): &\quad a^2yz + b^2zx + c^2xy - S_by(x+y+z) = 0 \end{aligned} \tag{9}$$

The intersect points of circles $\mathcal{C}$ and (BA) are

$$\begin{aligned} &(4a^8 - 3(b^2-c^2)^3(b^2+c^2) - 3a^6(3b^2+5c^2) + a^4(3b^4+8b^2c^2+21c^4) \\ &+ a^2(5b^6+b^4c^2+7b^2c^4-13c^6) \pm 8S_{AC}S\sqrt{-\delta} \\ &: -2b^2(-3(a^6-3a^4(b^2+c^2)-(b^2-c^2)^2(b^2+c^2)+a^2(3b^4+2b^2c^2+3c^4)) \pm 4S_AS\sqrt{-\delta}) \\ &: 2S_A(4a^6-11a^4(b^2+c^2)-3(b^2-c^2)^2(b^2+c^2)+2a^2(5b^4+2b^2c^2+5c^4) \pm 4S_AS\sqrt{-\delta})) \end{aligned}$$

If $\delta < 0$ there are two intersect points. If $\delta = 0$ there is only one tangent point:

$$T_b = (4a^2 + 3b^2 - 3c^2 : 6b^2 : -4a^2 + 3(b^2+c^2)) = (a^2 + 6S_C : 6b^2 : -a^2 + 6S_A) \tag{10}$$

similarly:

$$T_c = (4a^2 - 3b^2 + 3c^2 : -4a^2 + 3(b^2+c^2) : 6c^2) = (a^2 + 6S_B : -a^2 + 6S_A : 6c^2)$$

$$\begin{vmatrix} T_b \\ P \\ T_c \end{vmatrix} = \begin{vmatrix} T_b \\ A \\ T_c \end{vmatrix} = \begin{vmatrix} a^2+6S_C & 6b^2 & -a^2+6S_A \\ 1 & 0 & 0 \\ a^2+6S_B & -a^2+6S_A & 6c^2 \end{vmatrix} = \delta = 0$$

It follows that the points T_b, P, T_c are collinear. □

Theorem 5. *Let P be point of the ellipse χ. Let $(BP), (CP)$ are circles with diameters BP and CP respectively. Construct pedal $P_\perp$ of the perpendicular of point P to line BC, and point P' — the reflection of $P_\perp$ in P. Let P'' be midpoint of segment PP'. Let τ is perpendicular of point P'' to line OP', O is midpoint of BC (Figure 4). Then τ is the common tangent of the circles (BP) and (CP).*

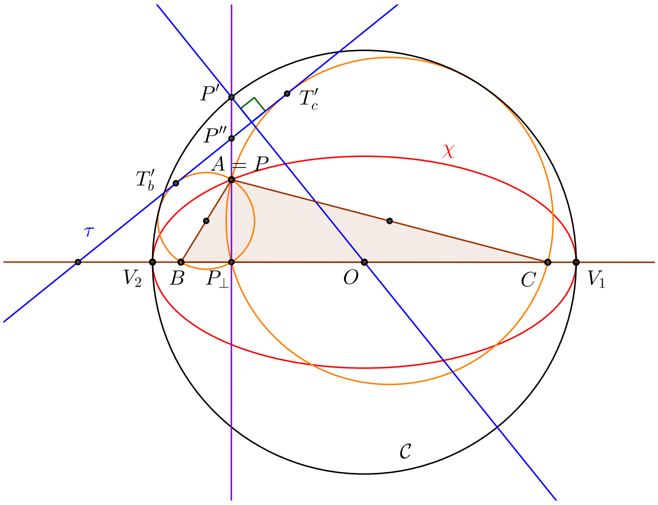

Figure 4: The line τ

Proof. The ellipse χ is independent of the position of point A. We choose $A = P = (1 : 0 : 0)$. Then $P_\perp = (0 : S_C : S_B)$ and $P' = (2a^2 : -S_C : -S_B)$, see Theorem 3 and (8). The equation of the line joining points $O = (0 : 1 : 1)$ and P', is:

$$OP' : \quad (b^2 - c^2)x + 2a^2y - 2a^2z = 0$$

The infinite point of line OP' has homogeneous coordinates $(4a^2 : -2a^2-b^2+c^2 : -2a^2+b^2-c^2)$. The infinite point of the lines perpendicular to OP', see [5, §4.5], has homogeneous coordinates $(2a^2(b^2 - c^2) : 2a^4 + (b^2 - c^2)^2 - a^2(5b^2 + 3c^2) : -2a^4 - (b^2 - c^2)^2 + a^2(3b^2 + 5c^2))$. The midpoint of segment PP' is point $P'' = (6a^2 : -a^2 - b^2 + c^2 : -a^2 + b^2 - c^2)$.

The perpendicular from P'' to OP' is the line τ , which has equation

$$\begin{vmatrix} 6a^2 & -a^2 - b^2 + c^2 & -a^2 + b^2 - c^2 \\ 2a^2(b^2 - c^2) & 2a^4 + (b^2 - c^2)^2 - a^2(5b^2 + 3c^2) & -2a^4 - (b^2 - c^2)^2 + a^2(3b^2 + 5c^2) \\ x & y & z \end{vmatrix} = 0$$

this is

$$-4S^2x + (3a^4 + 2(b^2 - c^2)^2 - a^2(5b^2 + 7c^2))y + (3a^4 + 2(b^2 - c^2)^2 - a^2(7b^2 + 5c^2))z = 0 \quad (11)$$

The intersect points of line τ (11) and circle (BA) (9) are T'_{b1}, T'_{b2}:

$$\begin{aligned} &2a^2(12a^4 + 9(b^2 - c^2)^2 - 2a^2(11b^2 + 13c^2)) \pm 2a^2(b^2 - c^2)\sqrt{\delta} \\ &: -(b^2 - c^2)(6a^4 + 3(b^2 - c^2)^2 - a^2(7b^2 + 9c^2)) \pm (2a^4 + (b^2 - c^2)^2 - a^2(5b^2 + 3c^2))\sqrt{\delta} \\ &: -8a^6 + 3(b^2 - c^2)^3 + 2a^4(9b^2 + 7c^2) + a^2(-13b^4 + 10b^2c^2 + 3c^4) \\ &\mp (2a^4 + (b^2 - c^2)^2 - a^2(3b^2 + 5c^2))\sqrt{\delta}) \end{aligned}$$

where δ is given in (5) above.

If A inside χ (6), $\delta > 0$, there are two intersect points T'_{b1}, T'_{b2}. If A lies on χ, $\delta = 0$, there is only one tangent point:

$$\begin{aligned} T'_b = &(2a^2(12a^4 + 9(b^2 - c^2)^2 - 2a^2(11b^2 + 13c^2)) : -(b^2 - c^2)(6a^4 + 3(b^2 - c^2)^2 - a^2(7b^2 + 9c^2)) \\ &: -8a^6 + 3(b^2 - c^2)^3 + 2a^4(9b^2 + 7c^2) + a^2(-13b^4 + 10b^2c^2 + 3c^4)) \end{aligned} \quad (12)$$

Similarly the intersect points of line τ (11) and circle (CA) (9) are T'_{c1}, T'_{c2}:

$$\begin{aligned}&(2a^2(12a^4+9(b^2-c^2)^2-2a^2(13b^2+11c^2)) \pm 2a^2(-b^2+c^2)\sqrt{\delta}\\&:-8a^6-3(b^2-c^2)^3+2a^4(7b^2+9c^2)+a^2(3b^4+10b^2c^2-13c^4)\\&\mp(2a^4+(b^2-c^2)^2-a^2(5b^2+3c^2))\sqrt{\delta}\\&:(b^2-c^2)(6a^4+3(b^2-c^2)^2-a^2(9b^2+7c^2)) \pm (2a^4+(b^2-c^2)^2-a^2(3b^2+5c^2))\sqrt{\delta})\end{aligned}$$

The tangent point of τ (11) and (CA) (9), $\delta = 0$, is:

$$\begin{aligned}T'_c =\ &(2a^2(12a^4+9(b^2-c^2)^2-2a^2(13b^2+11c^2)) : -8a^6-3(b^2-c^2)^3+2a^4(7b^2+9c^2)\\&+a^2(3b^4+10b^2c^2-13c^4) : (b^2-c^2)(6a^4+3(b^2-c^2)^2-a^2(9b^2+7c^2)))\end{aligned}$$

□

Theorem 6. *The point P'' is midpoint of $T'_bT'_c$. (See Theorem 5 and Figure 4).*

Proof. The condition the point P'' be midpoint of $T'_bT'_c$ is $T'_b + T'_c = 2P''$ (in absolute barycentric coordinates). The sum of the coordinates of the points are:

$$\begin{aligned}T'_b;\ T'_c &\to 4a^2(4a^4+3(b^2-c^2)^2-8a^2(b^2+c^2))\\P'' &\to 4a^2\end{aligned}$$

$$\begin{aligned}T'_b + T'_c =\ &\frac{12a^4+9(b^2-c^2)^2-2a^2(11b^2+13c^2)}{2(4a^4+3(b^2-c^2)^2-8a^2(b^2+c^2))}A - \frac{(b^2-c^2)(6a^4+3(b^2-c^2)^2-a^2(7b^2+9c^2))}{4a^2(4a^4+3(b^2-c^2)^2-8a^2(b^2+c^2))}B\\&+\frac{-8a^6+3(b^2-c^2)^3+2a^4(9b^2+7c^2)+a^2(-13b^4+10b^2c^2+3c^4)}{4a^2(4a^4+3(b^2-c^2)^2-8a^2(b^2+c^2))}C\\&+\frac{12a^4+9(b^2-c^2)^2-2a^2(13b^2+11c^2)}{2(4a^4+3(b^2-c^2)^2-8a^2(b^2+c^2))}A + \frac{(b^2-c^2)(6a^4+3(b^2-c^2)^2-a^2(9b^2+7c^2))}{4a^2(4a^4+3(b^2-c^2)^2-8a^2(b^2+c^2))}C\\&+\frac{-8a^6-3(b^2-c^2)^3+2a^4(7b^2+9c^2)+a^2(3b^4+10b^2c^2-13c^4)}{4a^2(4a^4+3(b^2-c^2)^2-8a^2(b^2+c^2))}B\\&= 3A - \frac{a^2+b^2-c^2}{2a^2}B - \frac{a^2-b^2+c^2}{2a^2}C = 2P''\end{aligned}$$

□

Remark. Similarly we prove that the point P'' is midpoint of $T'_{bi}T'_{ci},\ i = 1, 2$. (See Theorem 5).

Theorem 7. *The lines $OP', BT'_{bi}, CT'_{ci}, i = 1, 2$, see Theorem 5, are concurrent.*

Proof. The equations of the lines are:

$$\begin{aligned}OP' :\ &(b^2-c^2)x+2a^2y-2a^2z=0;\\BT'_{b1,2} :\ &(-8a^6-2a^4(-9b^2-7c^2\pm\sqrt{\delta})-(b^2-c^2)^2(-3b^2+3c^2\pm\sqrt{\delta})\\&+a^2(-13b^4+3c^4\pm5c^2\sqrt{\delta}+b^2(10c^2\pm3\sqrt{\delta})))x\\&+(-24a^6+a^4(44b^2+52c^2)-2a^2(b^2-c^2)(9b^2-9c^2\pm\sqrt{\delta}))z=0\\CT'_{c1,2} :\ &(8a^6+2a^4(-7b^2-9c^2\pm\sqrt{\delta})+(b^2-c^2)^2(3b^2-3c^2\pm\sqrt{\delta})\\&-a^2(3b^4-13c^4\pm3c^2\sqrt{\delta}+5b^2(2c^2\pm\sqrt{\delta})))x\\&+(24a^6-4a^4(13b^2+11c^2)-2a^2(b^2-c^2)(-9b^2+9c^2\pm\sqrt{\delta}))y=0\end{aligned}$$

Three lines $p_i x + q_i y + r_i z = 0, i = 1, 2, 3$, are concurrent if and only if (see [5, §4.3], [3])

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

For the lines $OP', BT'_{bi}, CT'_{ci}, i = 1, 2$ this is true. □

5 Construction

Construct a triangle DBC given the side BC and symmedian point P, not on the line BC.

Construction. (See Figure 5.)

1. The given noncollinear points P, B and C
2. Midpoint O of BC
3. Perpendicular of P to BC, which intersect BC in point $P_\perp$
4. Point P' — the reflection of $P_\perp$ in P.
5. The midpoint P'' of PP'
6. Circles $(BP), (CP)$ with diameters BP and CP
7. Perpendicular line τ from P'' to OP'
8. Intersections T'_{b1} and T'_{b2} of the line τ and circle (BP)
9. $D_1 = BT'_{b1} \cap OP'$ and $D_2 = BT'_{b2} \cap OP'$
10. Triangles D_1BC and D_2BC.

□

Proof. Let D_1 be a point constructed by method given above, see Figure 6. Likewise is proof for point D_2.

By construction, $D_1 = BT'_{b1} \cap OP'$. By Theorem 7, $D_1 \in CT'_{c1}$.

By construction, $PP_\perp \perp BC$. From triangle BPT'_{b1}, inscribed in circle (BP), $\sphericalangle PT'_{b1}B = 90°$. Similarly from triangle CPT'_{c1} follow $\sphericalangle PT'_{c1}C = 90°$. Hence the triangle $P_\perp T'_{c1}T'_{b1}$ is pedal triangle of point P with respect to the triangle D_1BC.

By the remark after Theorem 6, point P'' is midpoint of $T'_{b1}T'_{c1}$ i.e. $P_\perp P''$ is median in triangle $P_\perp T'_{c1}T'_{b1}$. By construction, point P divides the median $P_\perp P''$ in the ratio $P_\perp P : PP'' = 2 : 1$. Hence P is centroid of triangle $P_\perp T'_{c1}T'_{b1}$.

By Theorem 1, P is the symmedian point of triangle D_1BC. □

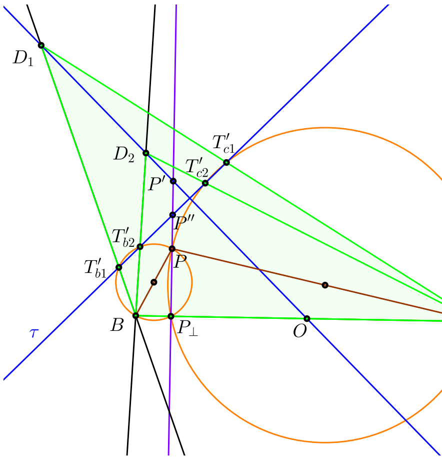

Figure 5: Construction

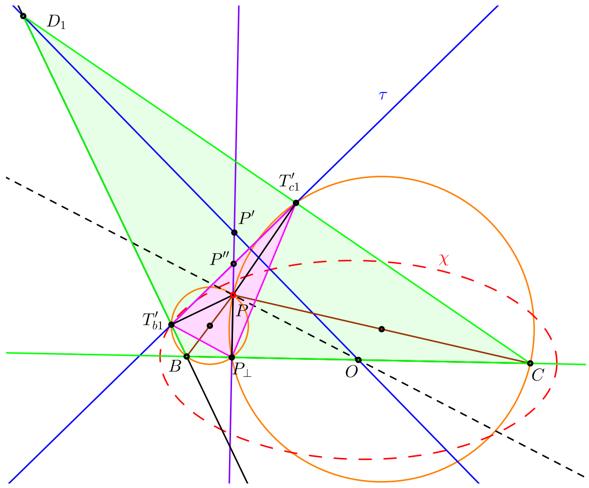

Figure 6: Existence of D

Existence of D**.** (See Figure 6.)

1. If the point P lies on the ellipse χ, (4), $\delta = 0$, see (6), the line τ tangent to circles $(BP), (CP)$ and exist a unique point D, such as P is symmedian point of triangle DBC;

2. If the point P is inside χ, not on the line BC , $\delta > 0$, the line τ intersect each of circles $(BP), (CP)$ in two points. Then exist two points D_1, D_2, such as P is symmedian point of triangles D_1BC and D_2BC;

3. If the point P is outside χ, $\delta < 0$, the line τ not intersect circles $(BP), (CP)$ (Theorem 5), and point D not exist.

□

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