Refutation of resolution-based decision procedure for two variables with equality

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Abstract: We evaluate six equations and three conjectures for the decision procedure. None is tautologous. This refutes the procedure for two variables with equality, and forms a non tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, F as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨ , ∪ ; - Not Or; & And, ∧ , ∩ ; ⊔ ; \\ Not And;
> Imply, greater than, → , ⇒ , ⊳ , ⊳ ; < Not Imply, less than, ∈ , ⊀ , ∈ , ≠ , ≈ ;
= Equivalent, ≡ ; ⊤ , ≡ , ⊤ , ⊤ ; ⊥ ;
% possibility, for one or some, ∃ , ⊃ , M ; # necessity, for every or all, ∀ , □ , L;
(z=z) T as tautology, ⊤ , ordinal 3; (z@z) F as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~( y < x) ( x ≤ y), ( x ∈ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: de Nivelle, H.; Pratt-Hartmann, I. (2001). A resolution-based decision procedure for the two variable fragment with equality. nivelle@mpi.mpg-sb.de, ipratt@cs.man.ac.uk
www.cs.man.ac.uk/~ipratt/papers/logic/ijcar01.pdf

1 Introduction
The two-variable-fragment L2 ≈ is the set of formulas that do not contain function symbols, that possibly contain equality (∼), and that use only two variables. The two-variable fragment without equality L2 is the subset of L2 ≈ not involving the predicate ∼. For example, the formula

∀x∃y[r(x, y) ∧ ∀x(r(y, x) → x∼y)], (1.1.1)

LET p, q, r, s: a, x, r, y.
(r&(q&s))&((r&(s&q))>(q=s)) ;
FFFF FFFF FFFF FFNN (1.1.2)

stating that every element is r-related to some element whose only r-successor is itself, is in L2 ≈ (but not in L2). Note in particular the ‘re-use’ of the variable x by nested quantifiers in this example. In the same way, it is possible to translate modal formulas into L2, (without equality) by reusing variables. For example, the modal formula

□◊a (1.2.1)

#%#p = (s=s) ;
FNFN FNFN FFFF FFFF FFFF FFFF (1.2.2)

can be translated into

∀y(r(x, y) → ∃x(r(y, x) ∧ ∀y(r(x, y) → a(y)))). (1.3.1)
No equality is needed for translating modal formulas.

**Remark 1.3.2:** Eqs. 1.2.2 is not equivalent to 1.3.2 as claimed.

### 2 Motivation

A logic is said to have the *finite-model property* if any satisfiable formula in that logic is satisfiable in a finite structure. It is easy to see that any fragment of first order logic having the finite model property is decidable; and indeed, most of the known decidable fragments of first-order logic have the finite model property. … One such fragment of particular interest here is the so-called Gödel class: the set of first-order formulas *without equality* which, when put in prenex form, have quantifier prefixes matching the pattern $\exists^* \forall \forall \exists^*$. Gödel . . showed that the Gödel class has the finite model property, and is thus decidable. In the same paper, Gödel claimed that allowing $\approx$ in formulas of the Gödel class would not affect the finite model property, a claim which was later shown to be false by Goldfarb . . Between these two discoveries, Scott . . showed that any formula of the two-variable fragment can be transformed into a formula in the Gödel fragment which is equisatisfiable. Relying on Gödel’s incorrect claim, Scott concluded decidability for $L_2 \approx$: Of course, what Scott actually showed was the decidability for $L_2$ only. That the full two-variable fragment does indeed have the finite model property was eventually established by Mortimer . . The fragment $L_2 \approx$ is of particular interest when dealing with natural language input, because many simple natural language sentences translate into $L_2 \approx$. To give a somewhat fanciful example, the sentence

Every meta-barber shaves every man who shaves no man who shaves himself.

translates to the two-variable formula

$$\forall x (\text{meta-barber}(x) \rightarrow \forall y ((\text{man}(y) \land \forall x ((\text{man}(x) \land \text{shave}(x,x)) \rightarrow \text{shave}(y,x)) \rightarrow \text{shave}(x,y))).$$

(2.1.1)

**Remark 2.1.2:** Eq. 2.1.2 as stated is not tautologous, meaning the example is not a theorem as presumably it should be.

### 3 Making equality disappear

In this section, we give a method for removing equality from a formula in $L_2 \approx$, based on resolution. … Occurrences of the $\approx$-symbol fall into two groups. Negative occurrences can be ’simulated’ without recourse to equality. Positive occurrences can be restricted to those belonging to a $\exists^!$ quantifier.

**Remark 3:** We interpret the quantifier $\exists^!$ as $\exists$, due to the non tautologous performances of Eqs. 1… and 2… above.
Lemma 5. Let $\gamma(x)$ be a formula not involving the variable $y$ and let $\delta(y)$ a formula not involving the variable $x$. Then the formulas

$$\forall x \forall y (\gamma(x) \lor \delta(y) \lor x \approx y) \quad (5.1.1)$$

\[\text{LET p, q, r, s: } \gamma, \delta, x, y.\]

\[((p\land r)+(q\land s))+(r\approx s); \quad \text{TTTT CTCT CCTT TTTT} \quad (5.1.2)\]

and

$$\forall x \gamma(x) \lor \forall x \delta(x) \lor (\exists !x \neg \gamma(x) \land \forall x (\gamma(x) \leftrightarrow \delta(x))) \quad (5.2.1)$$

\[((p\land r)+(q\land r))+((r\land \neg (p\land r))\land ((p\land r)\approx (q\land r))); \quad \text{CCCC TNTN CCCC TNTN} \quad (5.2.2)\]

are logically equivalent. Using this, we can use the splitting rule to decompose the disjunctions of the Type 3 formula. The result is a formula, in all positive occurrences of $\approx$ belong to a $\exists !$ quantifier. These can be eliminated by introducing new individual constants.

Remark 5.2.2: Eqs. 5.1.2 and 5.2.2 not logically equivalent, thereby refuting Lemma 5.