Balayage of Measures and Their Potentials: Duality Theorems and Extended Poisson–Jensen Formula

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Abstract

We investigate some properties of balayage of measures and their potentials on domains or open sets in finite-dimensional Euclidean space. Main results are Duality Theorems for potentials of balayage of measures, for Arens–Singer and Jensen measures and potentials, and also a new extended and generalized variant of Poisson–Jensen formula for balayage of measure and their potentials.

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We have been considered in the survey [37] various general concepts of balayage. In this article we deal with a particular case of such balayage with respect to special classes of subharmonic functions. We use in this paper part of the results from the previous article [34]. But the main results on potentials from Sec. 2 in its main part are new, although studies on the of Jensen and Arens-Singer potentials and their special classes with applications were partially carried out in Gamelin’s monograph [10, 3.1, 3.3], in articles [1], [46], [43], as well as the first of the authors together with various co-authors previously in articles [18]-[36], [5], [38], [39], [44], and also in [41, III, C], [6] etc.

1 Definitions, notations and conventions

The reader can skip this Section 1 and return to it only if necessary. We use definitions, notations and conventions from [34] with some additions.

1.1 Sets, order, topology

As usual, $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all natural, real and complex numbers, respectively; $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ is French natural series, and $\mathbb{Z} := \mathbb{N}_0 \cup \mathbb{N}_0$.

For $d \in \mathbb{N}$ we denote by $\mathbb{R}^d$ the $d$-dimensional real Euclidean space with the standard Euclidean norm $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and the distance function $\text{dist}(\cdot, \cdot)$. For the real line $\mathbb{R} = \mathbb{R}^1$ with Euclidean norm-module $|\cdot|$, $\mathbb{R}_{\pm \infty} := \mathbb{R}_- \cup \mathbb{R}_{+ \infty}$ is extended real line in the end topology with two ends $\pm \infty$, with the order relation $\leq$ on $\mathbb{R}$ complemented by the inequalities $-\infty \leq x \leq \pm \infty$ for $x \in \mathbb{R}_{\pm \infty}$, with the positive real axis

\[ \mathbb{R}_+ := \{x \in \mathbb{R}: x \geq 0\}, \mathbb{R}_{+ \infty} := \mathbb{R}^+ \cup \{\pm \infty\}, \]

\[ x^+ := \max\{0, x\}, \quad x^- := (-x)^+, \quad \text{for } x \in \mathbb{R}_{\pm \infty}, \quad (1.1^+) \]

\[ S^+ := \{x \geq 0: x \in S\}, \quad S^* := S \setminus \{0\} \quad \text{for } S \subset \mathbb{R}_{\pm \infty}, \quad \mathbb{R}_+^* := (\mathbb{R}_+)^*, \quad (1.1^+) \]

\[ x \cdot (\pm \infty) := \pm \infty =: (-x) \cdot (\mp \infty) \quad \text{for } x \in \mathbb{R}_+^* \cup (+\infty), \quad (1.1_0) \]

\[ \frac{x}{\pm \infty} := 0 \quad \text{for } x \in \mathbb{R}, \quad \text{but } 0 \cdot (\pm \infty) := 0 \quad (1.1_0) \]

unless otherwise specified. An open connected (sub-)set of $\mathbb{R}_{\pm \infty}$ is a (sub-)interval of $\mathbb{R}_{\pm \infty}$. The Alexandroff one-point compactification of $\mathbb{R}^d$ is denoted by $\mathbb{R}_{\pm \infty} := \mathbb{R}^d \cup \{\infty\}$.

The same symbol $0$ is used, depending on the context, to denote the number zero, the origin, zero vector, zero function, zero measure, etc. The positiveness is everywhere
understood as $\geq 0$ according to the context. Given $x \in \mathbb{R}^d$ and $\mathbf{r} \in \mathbb{R}^d_+$, we set

\begin{align}
B(x, r) := \{ x' \in \mathbb{R}^d : |x' - x| < r \}, & \quad \overline{B}(x, r) := \{ x' \in \mathbb{R}^d : |x' - x| \leq r \}, \\
B(\infty, r) := \{ x \in \mathbb{R}^d_\infty : |x| > 1/r \}, & \quad \overline{B}(\infty, r) := \{ x \in \mathbb{R}^d_\infty : |x| \geq 1/r \}, \\
B(r) := B(0, r), & \quad \mathbb{B} := B(0, 1), \quad \overline{B}(r) := \overline{B}(0, r), \quad \mathbb{B} := \overline{B}(0, 1), \\
B_0(x, r) := B(x, r) \setminus \{ x \}, & \quad \overline{B}_0(x, r) := \overline{B}(x, r) \setminus \{ x \}.
\end{align}

Thus, the basis of open (respectively closed) neighborhood of the point $x \in \mathbb{R}^d_\infty$ is open (respectively closed) balls $B(x, r)$ (respectively $\overline{B}(x, r)$) centered at $x$ with radius $r > 0$.

Given a subset $S$ of $\mathbb{R}^d$, the closure $\text{clos} S$, the interior $\text{int} S$ and the boundary $\partial S$ will always be taken relative $\mathbb{R}^d_\infty$. For $S' \subset S \subset \mathbb{R}^d_\infty$ we write $S' \Subset S$ if $\text{clos} S' \subset \text{int} S$. An open connected (sub-)set of $\mathbb{R}^d_\infty$ is a (sub-)domain of $\mathbb{R}^d_\infty$.

### 1.2 Functions

Let $X, Y$ are sets. We denote by $Y^X$ the set of all functions $f : X \to Y$. The value $f(x) \in Y$ of an arbitrary function $f \in Y^X$ is not necessarily defined for all $x \in X$. The restriction of a function $f$ to $S \subset X$ is denoted by $f|_S$. If $F \subset Y^X$, then $F|_S := \{ f|_S : f \in F \}$. We set

\begin{align}
\mathbb{R}^X_{-\infty} := (\mathbb{R}_{-\infty})^X, & \quad \mathbb{R}^X_{+\infty} := (\mathbb{R}_{+\infty})^X, & \quad \mathbb{R}^X_{\infty} := (\mathbb{R}_\infty)^X. \tag{1.3}
\end{align}

A function $f \in \mathbb{R}^X_{\infty}$ is said to be extended numerical. For extended numerical functions $f$, we set

\begin{align}
\text{Dom}_{-\infty} := f^{-1}(\mathbb{R}_{-\infty}) \subset X, & \quad \text{Dom}_{+\infty} := f^{-1}(\mathbb{R}_{+\infty}) \subset X, \\
\text{Dom} f := f^{-1}(\mathbb{R}_\infty) = \text{Dom}_{-\infty} \cup \text{Dom}_{+\infty} f \subset X, & \quad \text{dom} f := f^{-1}(\mathbb{R}) = \text{Dom}_{-\infty} f \cap \text{Dom}_{+\infty} f \subset X. \tag{1.4}
\end{align}

For $f, g \in \mathbb{R}^X_{\infty}$, we write $f = g$ if $\text{Dom} f = \text{Dom} g =: D$ and $f(x) = g(x)$ for all $x \in D$, and we write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in D$. For $f \in \mathbb{R}^X_{\infty}, g \in \mathbb{R}^Y_{\infty}$ and a set $S$, we write "$f = g$ on $S$" or "$f \leq g$ on $S$" if $f|_S = g|_S$ or $f|_S \leq g|_S$ respectively.

For $f \in F \subset \mathbb{R}^X_{\infty}$, we set $f^+ : x \mapsto \max\{0, f(x)\}, x \in \text{Dom} f$, $F^+ := \{ f \geq 0 : f \in F \}$. So, $f$ is positive on $X$ if $f = f^+$, and we write "$f \geq 0$ on $X$". We will use the following construction of countable completion of $F$ up:

\begin{align}
F^\uparrow := \{ f \in \mathbb{R}^X_{\infty} : \text{there is an increasing sequence} \ (f_j)_{j \in \mathbb{N}}, f_j \in F, \\
\text{such that} \ f(x) = \lim_{j \to \infty} f_j(x) \text{ for all} x \in X \ (\text{we write} \ f_j \nearrow f) \}. \tag{1.5}
\end{align}

\[^1\text{A reference mark over a symbol of (in)equaility, inclusion, or more general binary relation, etc. means that this relation is somehow related to this reference.}\]
Proposition 1. Let \( F \subset \mathbb{R}_+^d \) be a subset closed relative to the maximum. Consider sequences 
\( F \ni f_{k_j} \uparrow f_k \uparrow f \). Then \( F \ni \max\{f_{k_j}: j \leq n, k \leq n\} \uparrow f \). In particular, 
\((F^\uparrow)^\uparrow = F^\uparrow\).

The proof is obvious.

For topological space \( X, C(X) \subset \mathbb{R}_+^X \) is the vector space over \( \mathbb{R} \) of all continuous functions.

We denote the function identically equal to resp. \( -\infty \) or \( +\infty \) on a set by the same bold symbols \( -\infty \) or \( +\infty \).

For an open set \( O \subset \mathbb{R}_+^d \), we denote by \( \text{har}(O) \) and \( \text{sbh}(O) \) the classes of all harmonic (locally affine for \( m = 1 \)) and subharmonic (locally convex for \( m = 1 \)) functions on \( O \), respectively. The class \( \text{sbh}(O) \) contains the minus-infinity function \( -\infty \);

\[
\text{sbh}^+_i(O) := \text{sbh}(O) \setminus \{-\infty\}, \quad \text{sbh}^+(O) := (\text{sbh}(O))^+.
\] (1.6)

Denote by \( \delta\text{-sbh}(O) := \text{sbh}(O) - \text{sbh}(O) \) the class of all \( \delta\)-subharmonic functions on \( O \) [2, 35, 3.1]. The class \( \delta\text{-sbh}(O) \) contains two trivial functions, \( -\infty \) and \( +\infty = -(\n -\infty)\);

\[
\delta\text{-sbh}^+_i(O) := \delta\text{-sbh}(O) \setminus \{\pm\infty\}.
\] (1.7)

If \( o \notin O \ni \infty \), then we can to use the inversion in the sphere \( \partial B(o, 1) \) centered at \( o \in \mathbb{R}^d \):

\[
\star_o: x \mapsto x^\star := \begin{cases} 
  o & \text{for } x = \infty, \\
  o + \frac{1}{|x-o|^2} (x-o) & \text{for } x \neq o, \infty, \\
  \infty & \text{for } x = o,
\end{cases}
\] (1.8*)

together with the Kelvin transform [17, Ch. 2, 6; Ch. 9]

\[
u^\star(x^\star) = |x-o|^{d-2}u(x), \quad x^\star \in O^\star := \{x^\star: x \in O\}, \quad (u \in \text{sbh}(O)) \iff (u^\star \in \text{sbh}(O^\star)).
\] (1.8u) (1.8s)

For a subset \( S \subset \mathbb{R}_+^d \), the classes \( \text{har}(S), \text{sbh}(S), \delta\text{-sbh}(S) := \text{sbh}(S) - \text{sbh}(S), \) and \( C^k(S) \) with \( k \in \mathbb{N} \cup \{\infty\} \) consist of the restrictions to \( S \) of harmonic, subharmonic, \( \delta\)-subharmonic, and \( k \) times continuously differentiable functions in some (in general, its own for each function) open set \( O \subset \mathbb{R}_+^d \) containing \( S \). Classes \( \text{sbh}^+_i(S), \delta\text{-sbh}^+_i(S) \) are defined like previous classes (1.6), (1.7),

\[
\text{sbh}^+_i(S) := \{u \in \text{sbh}(S): u \geq 0 \text{ on } S\}.
\] (1.9)

By \( \text{const}_{a_1, a_2, \ldots} \in \mathbb{R} \) we denote constants, and constant functions, in general, depend on \( a_1, a_2, \ldots \) and, unless otherwise specified, only on them, where the dependence on dimension \( d \) of \( \mathbb{R}_+^d \) will be not specified and not discussed; \( \text{const}^+_i \geq 0 \).
1.3 Measures and charges

Let Borel$(S)$ be the class of all Borel subsets in $S \in \text{Borel}(\mathbb{R}^d)$. We denote by Meas$(S)$ the class of all Borel signed measures, or, charges on $S \in \text{Borel}(\mathbb{R}^d)$; Meas$_c(S)$ is the class of charges $\mu \in \text{Meas}(S)$ with a compact support supp$\mu \subset S$;

$$\text{Meas}^+ := \{\mu \in \text{Meas}(S) : \mu \geq 0\}, \quad \text{Meas}^- := \{\mu \in \text{Meas}(S) : \mu \leq 0\}, \quad \text{Meas}^+_c(S) := \text{Meas}_c(S) \cap \text{Meas}^+(S);$$

$$\text{Meas}^{1+} := \{\mu \in \text{Meas}^+ : \mu = 1\}; \quad \text{Meas}^{1-} := \{\mu \in \text{Meas}^- : \mu = -1\}; \quad \text{Meas}^+ := \{\mu \in \text{Meas}^+ : \mu = 0\};$$

For a charge $\mu \in \text{Meas}(S)$, we let $\mu^+, \mu^- := (-\mu)^+$ and $|\mu| := \mu^+ + \mu^-$ respectively denote its upper, lower, and total variations. So, $\delta_x \in \text{Meas}^1(S)$ is the Dirac measure at a point $x \in S$, i.e., supp$\delta_x = \{x\}$, $\delta_x(\{x\}) = 1$. We denote by $\mu\big|_S$ the restriction of $\mu$ to $S' \in \text{Borel}(\mathbb{R}^d)$.

If the Kelvin transform (1.8) translates the subharmonic function $u$ into another function $u'_c(1.8u)$, then its Riesz measure $\nu$ is transformed common use image under its own mapping-inversion of type 1 or 2. These rules are described in detail in L. Schwartz’s monograph [48, Vol. I, Ch. IV, § 6] and we do not dwell on them here, although here interesting questions arise, for example, for the Bernstein–Paley–Wiener–Mary Cartwright classes of entire functions [15], [41], [3], [38] etc.

Given $S \in \text{Borel}(\mathbb{R}^d)$ and $\mu \in \text{Meas}(S)$, the class $L^1_{\text{loc}}(S, \mu)$ consists of all extended numerical locally integrable functions with respect to the measure $\mu$ on $S$; $L^1_{\text{loc}}(S) := L^1_{\text{loc}}(S, \lambda_d)$. For $L \subset L^1_{\text{loc}}(S, \mu)$, we define a subclass

$$L \, d\mu := \{\nu \in \text{Meas}(S) : \text{there exists } g \in L \text{ such that } d\nu = g \, d\mu\}$$

of the class of all absolutely continuous charges with respect to $\mu$. For $\mu \in \text{Meas}(S)$, we set

$$\mu(x, r) := \mu(B(x, r)) \text{ if } B(x, r) \subset S.$$ 

Let $\triangle$ be the the Laplace operator acting in the sense of the theory of distributions, $\Gamma$ be the gamma function,

$$s_{d-1} := \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

be the surface area of the $(d-1)$-dimensional unit sphere $\partial B$ embedded in $\mathbb{R}^d$. For function $u \in \text{sbh}_c(O)$, the Riesz measure of $u$ is a Borel (or Radon [45, A.3]) positive measure

$$\Delta u := c_d \triangle u \in \text{Meas}^+(O), \quad c_d := \frac{1}{s_{d-1}(1 + (d-3)^+)} \Gamma(d/2) \max\{1, d/2\}.$$ 

In particular, $\Delta u(S) < +\infty$ for each subset $S \subset O$. By definition, $\Delta_{-\infty}(S) := +\infty$ for all $S \subset O$. 

5
We use different variants of outer Hausdorff $p$-measure $\kappa_p$ with $p \in \mathbb{N}_0$:

$$\kappa_p(S) := \lim_{0<r \to 0} \inf \left\{ \sum_{j \in \mathbb{N}} r_j^p : S \subset \bigcup_{j \in \mathbb{N}} B(x_j, r_j), 0 \leq r_j < r \right\}, \quad (1.15H)$$

$$b_p \overset{\text{(1.14)}}{=} \begin{cases} 1 & \text{if } p = 0, \\ 2 & \text{if } p = 1, \\ \frac{s_{p-1}}{p} & \text{if } p \in 1 + \mathbb{N}, \end{cases} \quad \text{is the volume of the unit ball } B \text{ in } \mathbb{R}^p. \quad (1.15b)$$

Thus, for $p = 0$, for any $S \subset \mathbb{R}^d$, its Hausdorff 0-measure $\kappa_0(S)$ is to the cardinality $\#S$ of $S$, for $p = d$ we see that $\kappa_d \overset{\text{(1.15H)}}{=} \lambda_d$ is the Lebesgue measure to Borel proper subsets $S \subset \mathbb{R}^d_\infty$, where, if $\infty \in S$, we preliminary use the inversion (1.8u), and $\sigma_{d-1} := \kappa_{d-1} \big|_{\partial B}$ is the $(d-1)$-dimensional surface measure of area on the unit sphere $\partial B$ in the usual sense.

### 1.4 Topological concepts: inward-filled hull of set

Let $O$ be a topological space, $S \subset O$, $x \in O$.

We denote by $\text{Conn}_O S$ and $\text{conn}_O(S, x) \subset \text{Conn}_O S$ a set of all connected components of $S$ and its connected component containing $x$, respectively. We write $\text{clo}_O S$, $\text{int}_O S$, and $\partial O S$ for the closure, the interior, and the boundary of $S$ in $O$. The set $S$ is $O$-precompact if $\text{clo}_O S$ is a compact subset of $O$, and we write $S \Subset O$.

**Definition 1.** An arbitrary $O$-precompact connected component of $O \setminus S$ is called a hole in $S$ with respect to $O$. The union of a subset $K \subset O$ with all holes in it will be called an inward-filled hull of this set $K$ with respect to $O$ and is denoted further as

$$\text{hull-in}_O K := K \bigcup \left( \bigcup \{ C \in \text{Conn}_O(O \setminus K) : C \Subset O \} \right). \quad (1.16)$$

Denote by $O_\infty$ the Alexandroff one-point compactification of $O$ with underlying set $O \sqcup \{ \infty \}$, where $\sqcup$ is the disjoint union of $O$ with a single point $\infty \notin O$. If this space $O$ is a topological subspace of some ambient topological space $T \supset O$, then this point $\infty$ can be identified with the boundary $\partial O \subset T$, considered as a single point $\{ \partial O \}$.

Throughout this article, we use these topological concepts only in cases when $O$ is an open non-empty proper Greenian open set $[17, \text{Ch.5, 2}]$ of $\mathbb{R}^d_\infty =: T$, i.e.,

$$\emptyset \neq O = \text{int}_{\mathbb{R}^d_\infty} O = \bigcup_{j \in N_O} D_j \neq \mathbb{R}^d_\infty, \quad j \in N_O \subset \mathbb{N}, \quad D_j = \text{conn}_{\mathbb{R}^d_\infty}(O, x_j), \quad (1.17O)$$

where points $x_j$ lie in different connected components $D_j$ of $O \subset \mathbb{R}^d_\infty$;

$$\emptyset \neq D \neq \mathbb{R}^d_\infty \quad \text{is an open connected subset, i.e., a domain.} \quad (1.17D)$$
The dependence on such an open set $O$ or such domain $D$ for constants $\text{const.}$ will not be indicated in the subscripts and is not discussed. For an open set $O$ from $(1.17O)$, we often use statements that are proved in our references only for domains $D$ from $(1.17D)$. This is acceptable since all such cases concern only to individual domains-components $D_j$. So, if $S \in O$, then $S$ meets only finite many components $D_j$. In addition, we give proofs of our statements only for cases $O, D \subset \mathbb{R}^d$. If we have $o \notin D_j = D \ni \infty$, then we can to use the inversion relative to the sphere $\partial B(o, 1)$ centered at $o \in \mathbb{R}^d$ as in (1.8).

Proposition 2 ([11, 6.3], [12]). Let $K$ be a compact set in an open set $O \subset \mathbb{R}^d$. Then

(i) $\text{hull-in}_O K$ is a compact subset in $O$;

(ii) the set $O_\infty \setminus \text{hull-in}_O K$ is connected and locally connected subset in $O_\infty$;

(iii) the inward-filled hull of $K$ with respect to $O$ coincides with the complement in $O_\infty$ of connected component of $O_\infty \setminus K$ containing the point $\infty$, i.e.,

$$\text{hull-in}_O K = O_\infty \setminus \text{conn}_{O_\infty \setminus K}(\infty);$$

(iv) if $O' \subset \mathbb{R}^d_\infty$ is an open subset and $O \subset O'$ then $\text{hull-in}_O K \subset \text{hull-in}_{O'} K$;

(v) $\mathbb{R}^d \setminus \text{hull-in}_O K$ has only finitely many components, i.e.,

$$\# \text{Conn}_{\mathbb{R}^d\setminus \text{hull-in}_O K} < \infty.$$

2 Potentials of charges and measures

Further everywhere we will assume for simplicity and brevity that

$$(O \subset \mathbb{R}^d) \leftrightarrow (\infty \notin O), \quad (D \subset \mathbb{R}^d) \leftrightarrow (\infty \notin D)$$

(2.1)

in addition to (1.17). If $\infty \in O$, $o \in \mathbb{R}^d_\infty \setminus O$, we can always easily go to cases (2.1) using a inversion $*_o$, and the Kelvin transforms (1.8).

Definition 2 ([34]). Let $\vartheta, \mu \in \text{Meas}(S), S \subset \text{Borel}(\mathbb{R}^d_\infty)$. Let $H \subset \mathbb{R}^S_{\pm \infty}$ be a class of Borel-measurable functions on $S$. Let us assume that the integrals $\int h \, d\vartheta$ and $\int h \, d\mu$ are well defined with values in $\mathbb{R}_{\pm \infty}$ for each function $h \in H$. We write $\vartheta \lesssim_H \mu$ and say that the charge $\mu$ is a \textit{balayage}, or, sweeping (out), of the charge $\vartheta$ for $H$, or, briefly, $\mu$ is a $H$-balayage of $\vartheta$, if

$$\int h \, d\vartheta \leq \int h \, d\mu \quad \text{for all } h \in H.$$  (2.2)
Definition 3 ([45], [16], [42]). For \( q \in \mathbb{R} \), we set
\[
    k_q(t) := \begin{cases} 
    \log t & \text{if } q = 0, \\
    -\text{sgn}(q)t^{-q} & \text{if } q \in \mathbb{R}_*, \\
    k_{d-2}(|x - y|) & \text{if } x \neq y,
    \end{cases} 
\]
for \( t \in \mathbb{R}_+^* \),
\[
    K_{d-2}(x, y) := \begin{cases} 
    -\infty & \text{if } x = y \text{ and } d \geq 2, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \\
    0 & \text{if } x = y \text{ and } d = 1.
    \end{cases}
\]

Definition 4 ([45], [28, Definition 2], [35, 3.1, 3.2]). Let \( \mu \in \text{Meas}_c(\mathbb{R}^d) \) be charge with compact support. Its potential is the function \( pt_\mu \in \delta\text{-sbh}(\mathbb{R}^d) \) defined by
\[
    pt_\mu(y) := \int K_{d-2}(x, y) \, d\mu(x),
\]
where the kernel \( K_{d-2} \) is defined in Definition 3 by the function \( k_q \) from (2.3k). The values of potential \( pt_\mu(y) \in \mathbb{R}^\pm \) is well defined for all
\[
    y \in \text{Dom}_{-\infty} pt_\mu = \left\{ y \in \mathbb{R}^d : \int_0^{\infty} \frac{\mu^-(y, t)}{t^{m-1}} \, dt < +\infty \right\}, \quad (2.4d-) \\
    y \in \text{Dom}_{+\infty} pt_\mu = \left\{ y \in \mathbb{R}^d : \int_0^{\infty} \frac{\mu^+(y, t)}{t^{m-1}} \, dt < +\infty \right\}, \quad (2.4d+) \\
    y \in \text{Dom}_{\pm\infty} pt_\mu = \text{Dom}_{-\infty} pt_\mu \cup \text{Dom}_{+\infty} pt_\mu, \quad (2.4d\pm)
\]
and their complements \( \mathbb{R}^d \setminus \text{Dom}_{-\infty} pt_\mu \) and \( \mathbb{R}^d \setminus \text{Dom}_{+\infty} pt_\mu \) are polar sets in \( \mathbb{R}^d \).

If \( \mu \in \text{Meas}_c^+(O) \) be a \( H \)-balayage of a measure \( \vartheta \in \text{Meas}_c^+(O) \), then we consider the potential
\[
    pt_{\mu - \vartheta} := pt_\mu - pt_\vartheta \in \delta\text{-sbh}(\mathbb{R}^d) \quad (2.5)
\]
where under the conditions \( d > 1 \) and \( 1 \in H \) it is natural to set \( pt_{\mu - \vartheta}(\infty) := 0 \). The latter is based on the following

**Proposition 3.** Let \( \mu \in \text{Meas}_c(\mathbb{R}^d) \). Then
\[
    pt_\mu(x) \overset{(2.3k)}{=} \mu(\mathbb{R}^d)k_{d-2}(|x|) + O\left(1/|x|^{d-1}\right), \quad x \to \infty. \quad (2.6)
\]

**Proof.** For \( d = 1 \), we have
\[
    |pt_\mu(x) - \mu(\mathbb{R})| \leq \int |x - y| \, d|\mu|(y) \leq \int |y| \, d|\mu|(y) = O(1), \quad |x| \to +\infty.
\]
See (2.6) for $d = 2$ in [45, Theorem 3.1.2].

For $d > 2$ and $|x| \geq 2 \sup \{|y| : y \in \text{supp } \mu\}$, we have

$$\left| \text{pt}_\mu(x) - \mu(\mathbb{R}^d) k_{d-2}(|x|) \right| = \left| \int \left( \frac{1}{|x|^{d-2}} - \frac{1}{|x - y|^{d-2}} \right) \text{d}\mu(y) \right|$$

$$\leq \int \left| \frac{1}{|x|^{d-2}} - \frac{1}{|x - y|^{d-2}} \right| \text{d}|\mu|(y) \leq \frac{2^{d-2}}{|x|^{2d-4}} \int |x - y|^{d-2} - |x|^{d-2} \text{d}|\mu|(y)$$

$$\leq \frac{2^{d-2}}{|x|^{2d-4}} \int |y||x|^{d-3} \sum_{k=0}^{d-3} \left( \frac{3}{2} \right)^k \text{d}|\mu|(y) \leq 2 \frac{3^{d-2}}{|x|^{d-1}} \int |y| \text{d}|\mu|(y) = O\left( \frac{1}{|x|^{d-1}} \right).$$

\[\square\]

**Proposition 4.** If

$$\mu \in \text{Meas}^+(\mathbb{R}^d), \quad L \subseteq \mathbb{R}^d, \quad o \in \mathbb{R}^d \setminus L,$$

then

$$\inf_{x \in L} \text{pt}_\mu(x) \overset{(2.3k)}{\geq} \mu(\mathbb{R}^d) k_{d-2}(\text{dist}(L, \text{supp } \mu)), \quad (2.8i)$$

$$\inf_{x \in L} \text{pt}_{\mu - \delta_o}(x) \overset{(2.4p)}{\geq} \mu(\mathbb{R}^d) k_{d-2}(\text{dist}(L, \text{supp } \mu)) - k_{d-2} \left( \sup_{x \in L} |x - o| \right) \quad (2.8o)$$

**Proof.** If $\text{dist}(L, \text{supp } \mu) = 0$, then the right-hand sides in the inequalities (2.8) are equal to $-\infty$, and the inequalities (2.8) are true. Otherwise, by Definition 4, we obtain

$$\text{pt}_\mu(x) = \int k_{d-2}(|x - y|) \text{d}\mu(y) \geq \inf_{y \in \text{supp } \mu} k_{d-2}(|x - y|) \mu(\mathbb{R}^d)$$

$$\geq \inf_{y \in \text{supp } \mu} \left( \inf_{y \in \text{supp } \mu} |x - y| \right) \mu(\mathbb{R}^d) = \mu(\mathbb{R}^d) k_{d-2}(\text{dist}(x, \text{supp } \mu)), \quad (2.9)$$

since the function $k_{d-o}$ from (2.3k) is increasing, which implies the inequality (2.8i) after applying the operation $\inf_{x \in L}$ to both sides of inequality (2.9). Using (2.8i), we have

$$\inf_{x \in L} \text{pt}_{\mu - \delta_o}(x) \overset{(2.4p)}{=} \inf_{x \in L} \left( \text{pt}_\mu(x) - k_{d-2}(|x - o|) \right) \geq \inf_{x \in L} \text{pt}_\mu(x) - \sup_{x \in L} k_{d-2}(|x - o|)$$

$$\overset{(2.8)}{\geq} \mu(\mathbb{R}^d) k_{d-2}(\text{dist}(L, \text{supp } \mu)) - k_{d-2} \left( \sup_{x \in L} |x - o| \right)$$

which gives the inequality (2.8o). \[\square\]
2.1 Duality Theorem for har(O)-balayage

**Duality Theorem 1** (for har(O)-balayage). If a measure \( \mu \in \text{Meas}^+_c(O) \) is a har(O)-balayage of a measure \( \vartheta \in \text{Meas}^+_c(O) \), then

\[
\begin{align*}
\text{pt}_\mu &\in \text{sbh}_*(\mathbb{R}^d) \cap \text{har}(\mathbb{R}^d \setminus \text{supp} \mu), \\
\text{pt}_\mu &= \text{pt}_\vartheta \text{ on } \mathbb{R}^d \setminus \text{hull-in}_O(\text{supp} \vartheta \cup \text{supp} \mu \cup). 
\end{align*}
\tag{2.10p}
\]

Conversely, suppose that there is a subset \( S \subseteq O \), and a function \( p \) such that

\[
\begin{align*}
p^{(2.10p)} &\in \text{sbh}(O) \cap \text{har}(O \setminus S), \\
p^{(2.10=)} &= \text{pt}_p \text{ on } O \setminus S.
\end{align*}
\tag{2.11=}
\]

Then the Riesz measure

\[
\mu := \Delta_p^{(1.14)} := c_d \Delta p^{(2.11)} \in \text{Meas}^+(\text{clos } S) \subset \text{Meas}^+_c(O)
\tag{2.12}
\]

of this function \( p \) is a har(O)-balayage of \( \vartheta \).

**Proof.** The first property \((2.10p)\) is evidently. For each \( y \in \mathbb{R}^d \), the kernel \( K_{d-2}(\cdot, y) \) is harmonic on \( \mathbb{R}^d \setminus \{y\} \). By

**Proposition 5** ([34]). Let \( \mu \in \text{Meas}^+_c(O) \) be a balayage of \( \vartheta \in \text{Meas}^+_c(O) \) for har(O). Then

\[
\int h \, d\vartheta = \int h \, d\mu \text{ for any } h \in \text{har}(\text{hull-in}_O(\text{supp} \mu \cup \text{supp} \vartheta))
\tag{2.13}
\]

(see Subsec. 1.4, Definition 1 of inward-filled hull of compact subset \( \text{supp} \mu \cup \text{supp} \vartheta \) in \( O \)).

for \( h := K_{d-2}(\cdot, y) \) in \((2.13)\), we have

\[
\text{pt}_\vartheta(y) = \int K_{d-2}(x, y) \, d\vartheta(x) \overset{(2.13)}{=} \int K_{d-2}(x, y) \, d\mu(x) = \text{pt}_\mu(y)
\tag{2.14}
\]

for all \( y \in \text{hull-in}_O(\text{supp} \mu \cup \text{supp} \vartheta) \). This gives \((2.10=)\).

In the opposite direction, we can extend the function \( p \) to \( \mathbb{R}^d \) so that \( p = \text{pt}_\vartheta \) on \( \mathbb{R}^d \setminus S \). In view of \((2.37)\), we have \( p \in \text{sbh}(\mathbb{R}^d) \cap \text{har}(\mathbb{R}^d \setminus S) \), and

\[
p(x) - \vartheta(O)k_{d-2}(|x|) = p(x) - \text{pt}_\vartheta(x) + O(1/|x|^{d-1}) \overset{(2.11=)}{=} O(1/|x|^{d-1}), \quad x \to \infty.
\tag{2.15}
\]

Hence the function \( p \) is a potential with the Riesz measure \((2.12)\), and \( \mu(O) = \vartheta(O) \), i.e., \( p = \text{pt}_\mu \). Further, we can use the following
Lemma 1 ([11, Lemma 1.8]). Let $F$ be a compact subset of $\mathbb{R}^d$, let $h \in \text{har}(F)$, and $\varepsilon > 0$. Then there are points $y_1, y_2, \ldots, y_k$ in $\mathbb{R}^d \setminus F$ such that

$$
|h(x) - \sum_{j=1}^{k} k_{d-2}(|x - y_j|)| < \varepsilon \quad \text{for all } x \in F. \quad (2.16)
$$

Applying Lemma 1 to the compact set $F$ (2.11p) $= \text{clos} S \cup \text{supp } \vartheta \subset O$ and a function $h \in \text{har}(O)$, we obtain

$$
\left| \int_{F} h \, d(\mu - \vartheta) \right| = \left| \int_{F} h \, d(\mu - \vartheta) - \sum_{j=1}^{k} (\text{pt}_\mu(y_j) - \text{pt}_\vartheta(y_j)) \right| 
\leq \sup_{x \in F} \left| h(x) - \sum_{j=1}^{k} k_{d-2}(|x - y_j|) \right| (\mu(O) + \vartheta(O)) \leq \varepsilon (\mu(O) + \vartheta(O))
$$

for any $\varepsilon > 0$. Hence the measure $\mu$ is a $\text{har}(O)$-balayage of $\vartheta$. \hfill \Box

Corollary 1. Let $\vartheta, \mu \in \text{Meas}_c(O)$, $\text{supp } \vartheta \cup \text{supp } \mu \subset S \subset O$. If $\mu$ is a balayage of $\vartheta$ for the class

$$
H = \{ \pm k_{d-2}(|y - \cdot|) : y \in \mathbb{R}^d \setminus \text{clos } S \}, \quad (2.17)
$$

then $\mu$ is a $\text{har}(O)$-balayage of $\vartheta$.

Proof. We have (2.14) for all $y \in \mathbb{R}^d \setminus \text{clos } S$. By Duality Theorem 1, $\vartheta \preceq_{\text{har}(O)} \mu$. \hfill \Box

Corollary 2. Let $\mu \in \text{Meas}_c^+(O)$ be a $\text{har}(O)$-balayage of measure $\vartheta \in \text{Meas}_c^+(O)$, and $\varsigma \in \text{Meas}_c^+(O)$ also be a $\text{har}(O)$-balayage of the same measure $\vartheta$. If

$$
\text{hull-in}_O(\text{supp } \vartheta \cup \text{supp } \varsigma) \subset \text{hull-in}_O(\text{supp } \vartheta \cup \text{supp } \mu), \quad (2.18)
$$

then the measure $\mu$ is a $\text{har}(O)$-balayage of the measure $\varsigma$.

### 2.2 Arens–Singer measures and their potentials

Example 1 ([10], [28]). Let $x \in O$. If $\mu \in \text{Meas}_c^+(O)$ is a balayage of $\delta_x$ for $\text{har}(O)$, then such measure $\mu$ is called an Arens–Singer measure for $x$. The class of such measures is denoted by $\text{AS}_x(O) \supset J_x(O)$. Arens–Singer measures are often referred to as representing measures.

By Example 1, if we choose $x \in O$ and $\vartheta := \delta_x \preceq_{\text{har}(O)} \mu \in \text{Meas}_c^+(O)$, i.e., $\mu$ is an Arens–Singer measure for $x \in O$, then potential

$$
\text{pt}_{\mu - \delta_x}(y) = \text{pt}_\mu(y) - K_{d-2}(x, y), \quad y \in \mathbb{R}^d \setminus \{x\} \quad (2.19)
$$

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satisfies conditions
\[ pt_{\mu - \delta x} \in \text{sbh}(\mathbb{R}_\infty^d), \quad pt_{\mu - \delta x}(\infty) := 0, \]
\[ pt_{\mu - \delta x} \equiv 0 \quad \text{on } \mathbb{R}_\infty^d \setminus \text{hull-in}(\{x\} \cup \text{supp } \mu) \]
\[ pt_{\mu - \delta x}(y) \leq -Kd-2(x,y) + O(1) \quad \text{for } x \neq y \rightarrow x. \] (2.20)

Remember, that the function \( V \in \text{sbh}^\ast(\mathbb{R}_\infty^d \setminus \{x\}) \) is called a Arens–Singer potential on \( O \) with pole at \( x \in O \) [28], [30, Definition 6] (partially in [10, 3.3,3.4], [1], [46]), if this function \( V \) satisfies conditions
\[ V \equiv 0 \quad \text{on } \mathbb{R}_\infty^d \setminus S(V) \]
\[ V(y) \leq -Kd-2(x,y) + O(1) \quad \text{for } x \neq y \rightarrow x. \] (2.21)

The class of all Arens–Singer potential on \( O \) with pole at \( x \in O \) denote by \( \text{PAS}_x(O) \). In this class \( \text{PAS}_x(O) \) we will consider a special subclass
\[ \text{PAS}_x^1(O) := \{ V \in \text{PAS}_x(O) : V(y) = -Kd-2(x,y) + O(1) \text{ for } x \neq y \rightarrow x \} \] (2.22)

By Duality Theorem 1, we have

**Duality Theorem A** ([28, Proposition 1.4, Duality Theorem]). The mapping
\[ P_x : \mu \mapsto pt_{\mu - \delta x} \] (2.23)
is the affine bijection from \( \text{AS}_x(O) \) onto \( \text{PAS}_x(O) \) with inverse mapping
\[ P^{-1}_x : V \mapsto c_d \Delta V \bigg|_{\mathbb{R}^d \setminus \{x\}} + \left(1 - \limsup_{x \neq y \rightarrow x} \frac{V(y)}{-Kd-2(x,y)}\right) \cdot \delta_x. \] (2.24)

Let \( x \in \text{int } Q = Q \subseteq O \). The restriction of \( P_x \) to the class
\[ \{ \mu \in \text{AS}_x(O) : \text{supp } \mu \cap Q = \emptyset \} \] (2.25)
define a bijection from class (2.25) onto class (see (2.22))
\[ \text{PAS}_x^1(O) \bigcap \text{har}(Q \setminus \{x\}). \] (2.26)

The restriction of \( P_x \) to the class
\[ \{ \mu \in \text{AS}_x(O) : \text{supp } \mu \cap Q = \emptyset \} \bigcap (C^\infty(O) \text{ d}\lambda_d) \] (2.27)
define also a bijection from class (2.27) onto class
\[ \text{PAS}_x^1(O) \bigcap \text{har}(Q \setminus \{x\}) \bigcap C^\infty(O \setminus \{x\}). \] (2.28)

This transition from the main bijection \( P_x \) to the bijection from (2.25) onto (2.26) or from (2.27) onto (2.28) by restriction of \( P_x \) to (2.25) or (2.27) is quite obvious.
2.3 A generalization of Poisson–Jensen formula

**Theorem 1** (extended Poisson–Jensen formula for $\text{har}(O)$-balayage). Let $\mu \in \text{Meas}^+(O)$ be a $\text{har}(O)$-balayage of $\vartheta \in \text{Meas}^+(O)$. If $u \in \text{sbh}(O)$ is a function with the Riesz measure $\Delta_u^{[1,14]} := \triangle_u \in \text{Meas}(O)$, then

$$
\int u \, d\vartheta + \int_K \text{pt}_u \, d\Delta_u = \int_K \text{pt}_\vartheta \, d\Delta_u + \int u \, d\mu, \quad K := \text{hull-in}_O(\text{supp } \vartheta \cup \text{supp } \mu). \quad (2.29)
$$

In particular, if

$$
\int u \, d\vartheta > -\infty, \quad (2.30)
$$

then (2.29) can be written as

$$
\int u \, d\vartheta = \int u \, d\mu - \int_K \text{pt}_{\vartheta - u} \, d\Delta_u. \quad (2.31)
$$

**Proof.** Consider first the case (2.30). Choose an open set $O'$ such that $K \subseteq O' \subseteq O$. By the Riesz decomposition theorem $u = \text{pt}_{\vartheta'} + h$ on $O'$, where $\vartheta' := \Delta_u \mid_{O'}$ and $h \in \text{har}(O')$. Integrating this representation with respect to $d\vartheta$ and $d\mu$, we obtain

$$
\int u \, d\mu = \int \text{pt}_{\vartheta'} \, d\mu + \int h \, d\mu, \quad (2.32\mu)
$$

$$
\int u \, d\vartheta = \int \text{pt}_{\vartheta'} \, d\vartheta + \int h \, d\vartheta, \quad (2.32\vartheta)
$$

where the three integrals in (2.32\vartheta) are finite, although in the equality (2.32\mu) the first two integrals can take simultaneously the value of $-\infty$, but the last integral in (2.32\mu) is finite. Therefore, the difference (2.32\mu) – (2.32\vartheta) of these two equalities is well defined:

$$
\int u \, d\mu - \int u \, d\vartheta = \int \text{pt}_{\vartheta'} \, d\mu - \int \text{pt}_{\vartheta'} \, d\vartheta + \int h \, d(\mu - \vartheta), \quad (2.33)
$$

where the first and third integrals can simultaneously take the value of $-\infty$, and the remaining integrals are finite. By Proposition 5, the last integral in (2.33) vanishes. Using Fubini’s theorem, in view of the symmetry property of kernel in (2.4p), we have

$$
\int \text{pt}_{\vartheta'} \, d\vartheta = \int \int K_{d-2}(y, x) \, d\vartheta'(y) \, d\vartheta(x)
$$

$$
= \int \int K_{d-2}(x, y) \, d\vartheta(x) \, d\vartheta'(y) = \int_{O'} \text{pt}_{\vartheta} \, d\Delta_u. \quad (2.34)
$$

and the same way

$$
\int \text{pt}_{\vartheta'} \, d\mu = \int \int K_{d-2}(y, x) \, d\vartheta'(y) \, d\mu(x)
$$

$$
= \int \int K_{d-2}(x, y) \, d\mu(x) \, d\vartheta'(y) = \int_{O'} \text{pt}_{\mu} \, d\Delta_u \quad (2.35)
$$
even if the integral on the left side of equalities (2.35) takes the value $-\infty$ because the integrand $K_{d-2}(\cdot, \cdot)$ is bounded from above on the compact set $\text{clos} O' \times \text{clos} O'$ [16, Theorem 3.5]. Hence equality (2.33) can be rewritten as

$$\int u \, d\mu - \int u \, d\vartheta = \int_{O'} pt_\mu \, d\Delta_u - \int_{O'} pt_\vartheta \, d\Delta_u = \int_K pt_\mu \, d\Delta_u - \int_K pt_\vartheta \, d\Delta_u$$

since $pt_\mu = pt_\vartheta$ on $O' \setminus K$. This gives equality (2.29) in the case (2.30).

If condition (2.30) is not fulfilled, then from the representation (2.32) it follows that the integral on the left-hand side of (2.34) also takes the value $-\infty$. The equalities (2.34) is still true [16, Theorem 3.5]. Hence, the first integral on the right side of the formula (2.29) also takes the value $-\infty$ and this formula (2.29) remains true. \hfill \Box

**Remark 1.** If $\vartheta := \delta_x$ and $\mu := \omega_D(x, \cdot)$ for $x \in D \subseteq O$, then the formula (2.31) is the classical Poisson–Jensen formula [16, Theorem 5.27]

\begin{align}
&u(x) = \int_{\partial D} u \, d\omega_D(x, \cdot) - \int_{\text{clos} D} g_D(\cdot, x) \, d\Delta_u, \quad x \in D, \\
&\delta_x \preceq_{\text{sbh}(O)} \omega_D(x, \cdot), \quad pt_{\omega_D(x, \cdot)} - pt_{\delta_x} = pt_{\omega_D(x, \cdot) - \delta_x} = g_D(\cdot, x). \quad (2.36b)
\end{align}

### 2.4 Duality Theorem for \text{sbh}(O)-balayage

**Duality Theorem 2** (for \text{sbh}(O)-balayage). If a measure $\mu \in \text{Meas}^+_c(O)$ is a \text{sbh}(O)-balayage of a measure $\vartheta \in \text{Meas}^+_c(O)$, then we have (2.10), and

$$pt_\mu \geq pt_\vartheta \quad \text{on} \quad \mathbb{R}^d. \quad (2.37)$$

Conversely, suppose that there is a subset $S \subseteq O$, and a function $p$ such that we have (2.11), and $p \geq pt_\vartheta$ on $\text{clos} S$. Then the Riesz measure (2.12) of $p$ is a \text{sbh}(O)-balayage of $\vartheta$.

**Proof.** If $\vartheta \preceq_{\text{sbh}(O)} \mu$, then $\vartheta \preceq_{\text{har}(O)} \mu$ and we have properties (2.10) by Duality Theorem 1. For each $y \in \mathbb{R}^d$, the function $K_{d-2}(\cdot, y)$ is subharmonic on $\mathbb{R}^d$ and (2.37) follows from Definitions 2 and 4. Conversely, if a function $p$ is such as in (2.11), then, by Duality Theorem 1, this function is a potential $pt_\mu = p$ with the Riesz measure (2.12), this measure $\mu \in \text{Meas}^+_c(O)$ is a \text{har}(O)-balayage for $\vartheta$, and $K := \text{hull-in}(\text{supp} \vartheta \cup \text{supp} \mu) \subseteq \text{clos} S$. Let $u \in \text{sbh}_c(O)$. It follows from $pt_\mu \geq pt_\vartheta$ on $K$ that $\int_K pt_\vartheta \, d\Delta_u \leq \int_K pt_\mu \, d\Delta_u$. Hence, by the extended Poisson–Jensen formula (2.29) from Theorem 1, we obtain $\int u \, d\vartheta \leq \int u \, d\mu$. \hfill \Box

### 2.5 Jensen measures and their potentials

**Example 2** ([10], [7], [8], [47]). Let $x \in O$. If a measure $\mu \in \text{Meas}^+_c(O)$ is a balayage of the Dirac measure $\delta_x$ for \text{sbh}(O), then this measure $\mu$ is called a **Jensen measure for $x$**. The class of such measures is denoted by $J_x(O)$. 

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By Example 2, if we choose \( x \in O \) and \( \vartheta := \delta_x \leq_{\text{shh}(O)} \mu \in \text{Meas}_c^+(O) \), i.e., \( \mu \) is a Jensen measure for \( x \in O \), then potential

\[
pt_{\mu-\delta_x}(y) = pt_{\mu}(y) - K_{d-2}(x, y), \quad y \in \mathbb{R}^d \setminus \{x\}
\]  

(2.38)
satisfies conditions (2.20) and \( pt_{\mu-\delta_x} \geq 0 \) on \( \mathbb{R}^d \setminus \{x\} \). Remember, that a positive function \( V \in \text{shh}^+(\mathbb{R}^d \setminus \{x\}) \) is called a Jensen potential on \( O \) with pole at \( x \in O \) [28], [30], Definition 8, if this function \( V \) satisfies conditions (2.21) The class of all Jensen potential on \( O \) with pole at \( x \in O \) denote by \( PJ_x(O) \subseteq \text{AS}_x(O) \). In this class \( J_x(O) \) we will consider a special subclass

\[
P J^1_x(O) := PJ_x(O) \bigcap PAS_x^1(O) \subseteq PAS_x^1(O).
\]  

(2.39)

By Duality Theorem 2, we have

**Duality Theorem B** ([28, Proposition 1.4, Duality Theorem]). The mapping (2.23) is the affine bijection from \( J_x(O) \) onto \( PJ_x(O) \) with inverse mapping (2.24).

Let \( x \in \text{int} Q = Q \subseteq O \). The restriction of \( P_x \) to the class (cf. (2.25))

\[
\{ \mu \in J_x(O): \text{supp } \mu \cap Q = \emptyset \}
\]  

(2.40)
define a bijection from class (2.40) onto class (see (2.39), cf. (2.26))

\[
P J^1_x(O) \bigcap \text{har}(Q \setminus \{x\}).
\]  

(2.41)

Let \( x \in \text{int} Q = Q \subseteq O \). The restriction of \( P_x \) to the class (cf. (2.27))

\[
\{ \mu \in J_x(O): \text{supp } \mu \cap Q = \emptyset \} \bigcap (C^\infty(O) \circ \lambda_d)
\]  

(2.42)
define a bijection from class (2.42) onto class (cf. (2.28))

\[
P J^1_x(O) \bigcap \text{har}(Q \setminus \{x\}) \bigcap C^\infty(O \setminus \{x\}).
\]  

(2.43)

This transition from the main bijection \( P_x \) to the bijection from (2.40) onto (2.41) or from (2.42) onto (2.43) by restriction of \( P_x \) to (2.40) or to (2.42) is quite obvious.

**References**


