Balayage of Measures and Their Potentials: Duality Theorems and Extended Poisson – Jensen Formula

Bulat N. Khabibullin* **Enzhe Menshikova**

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Abstract

We investigate some properties of balayage of measures and their potentials on domains or open sets in finite-dimensional Euclidean space. Main results are Duality Theorems for potentials of balayage of measures, for Arens – Singer and Jensen measures and potentials, and also a new extended and generalized variant of Poisson – Jensen formula for balayage of measure and their potentials.

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We have are considered in the survey [37] various general concepts of balayage. In this article we deal with a particular case of such balayage with respect to special classes of subharmonic functions. We use in this paper part of the results from the previous article [34]. But the main results on potentials from Sec. 2 in its main part are new, although studies on the of Jensen and Arens-Singer potentials and their special classes with applications were partially carried out in Gamelin's monograph [10, 3.1, 3.3], in articles [1], [46], [43], as well as the first of the authors together with various co-authors previously in articles [18]–[36], [5], [38], [39], [44], and also in [41, III,C], [6] etc.

1 Definitions, notations and conventions

The reader can skip this Section 1 and return to it only if necessary. We use definitions, notations and conventions from [34] with some additions.

1.1 Sets, order, topology

As usual, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all *natural, real* and *complex* numbers, respectively; $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ is French natural series, and $\mathbb{Z} := \mathbb{N}_0 \cup \mathbb{N}_0$.

For $d \in \mathbb{N}$ we denote by $\mathbb{R}^d$ the *d-dimensional real Euclidean space* with the standard *Euclidean norm* $|x| := \sqrt{x_1^2 + \dots + x_d^2}$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and the distance function $\operatorname{dist}(\cdot, \cdot)$. For the *real line* $\mathbb{R} = \mathbb{R}^1$ with *Euclidean norm-module* $|\cdot|$,

$$\mathbb{R}_{-\infty} := \{-\infty\} \cup \mathbb{R},\ \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\},\ |\pm\infty| := +\infty;\ \mathbb{R}_{\pm\infty} := \mathbb{R}_{-\infty} \cup \mathbb{R}_{+\infty} \tag{1.1$_\infty$}$$

is *extended real line* in the end topology with two ends $\pm\infty$, with the order relation $\leq$ on $\mathbb{R}$ complemented by the inequalities $-\infty \leq x \leq +\infty$ for $x \in \mathbb{R}_{\pm\infty}$, with the *positive real axis*

$$\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geq 0\},\ \mathbb{R}^+_{+\infty} := \mathbb{R}^+ \cup \{+\infty\},\ \begin{cases} x^+ & := \max\{0, x\}, \\ x^- & := (-x)^+, \end{cases} \text{for } x \in \mathbb{R}_{\pm\infty}, \tag{1.1$^+$}$$

$$S^+ := \{x \geq 0 \colon x \in S\}, \quad S_* := S \setminus \{0\} \quad \text{for } S \subset \mathbb{R}_{\pm\infty}, \quad \mathbb{R}^+_* := (\mathbb{R}^+)_*, \tag{1.1$^+_*$}$$

$$x \cdot (\pm\infty) := \pm\infty =: (-x) \cdot (\mp\infty) \quad \text{for } x \in \mathbb{R}^+_* \cup (+\infty), \tag{1.1$_\pm$}$$

$$\frac{x}{\pm\infty} := 0 \quad \text{for } x \in \mathbb{R}, \quad \text{but } 0 \cdot (\pm\infty) := 0 \tag{1.1$_0$}$$

unless otherwise specified. An open connected (sub-)set of $\mathbb{R}_{\pm\infty}$ is a *(sub-)interval* of $\mathbb{R}_{\pm\infty}$. The *Alexandroff* one-point *compactification* of $\mathbb{R}^d$ is denoted by $\mathbb{R}^d_\infty := \mathbb{R}^d \cup \{\infty\}$.

The same symbol 0 is used, depending on the context, to denote the number zero, the origin, zero vector, zero function, zero measure, etc. The *positiveness* is everywhere understood as ≥ 0 according to the context. Given $x \in \mathbb{R}^d$ and[1] $r \overset{(1.1^+)}{\in} \mathbb{R}^+_{+\infty}$, we set

$$B(x,r) := \{x' \in \mathbb{R}^d \colon |x'-x| < r\}, \quad \overline{B}(x,r) := \{x' \in \mathbb{R}^d \colon |x'-x| \leq r\}, \tag{1.2B}$$

$$B(\infty,r) := \{x \in \mathbb{R}^d_\infty \colon |x| > 1/r\}, \quad \overline{B}(\infty,r) := \{x \in \mathbb{R}^d_\infty \colon |x| \geq 1/r\}, \tag{1.2$_\infty$}$$

$$B(r) := B(0,r), \quad \mathbb{B} := B(0,1), \quad \overline{B}(r) := \overline{B}(0,r), \quad \overline{\mathbb{B}} := \overline{B}(0,1). \tag{1.2$_1$}$$

$$B_\circ(x,r) := B(x,r) \setminus \{x\}, \quad \overline{B}_\circ(x,r) := \overline{B}(x,r) \setminus \{x\}. \tag{1.2$_\circ$}$$

Thus, the basis of open (respectively closed) neighborhood of the point $x \in \mathbb{R}^d_\infty$ is *open* (respectively *closed*) *balls* $B(x,r)$ (respectively $\overline{B}(x,r)$) centered at x with radius $r > 0$.

Given a subset S of $\mathbb{R}^d_\infty$, the *closure* $\operatorname{clos} S$, the *interior* $\operatorname{int} S$ and the *boundary* ∂S will always be taken relative $\mathbb{R}^d_\infty$. For $S' \subset S \subset \mathbb{R}^d_\infty$ we write $S' \Subset S$ if $\operatorname{clos} S' \subset \operatorname{int} S$. An open connected (sub-)set of $\mathbb{R}^d_\infty$ is a *(sub-)domain* of $\mathbb{R}^d_\infty$.

1.2 Functions

Let X, Y are sets. We denote by Y^X the set of all functions $f \colon X \to Y$. The value $f(x) \in Y$ of an arbitrary function $f \in X^Y$ is not necessarily defined for all $x \in X$. The restriction of a function f to $S \subset X$ is denoted by $f\big|_S$. If $F \subset Y^X$, then $F\big|_S := \{f\big|_S \colon f \in F\}$. We set

$$\mathbb{R}^X_{-\infty} \overset{(1.1_\infty)}{:=} (\mathbb{R}_{-\infty})^X, \quad \mathbb{R}^X_{+\infty} \overset{(1.1_\infty)}{:=} (\mathbb{R}_{+\infty})^X, \quad \mathbb{R}^X_{\pm\infty} \overset{(1.1_\infty)}{:=} (\mathbb{R}_{\pm\infty})^X. \tag{1.3}$$

A function $f \in \mathbb{R}^X_{\pm\infty}$ is said to be *extended numerical*. For extended numerical functions f, we set

$$\begin{aligned} \operatorname{Dom}_{-\infty} &:= f^{-1}(\mathbb{R}_{-\infty}) \subset X, \quad \operatorname{Dom}_{+\infty} f := f^{-1}(\mathbb{R}_{+\infty}) \subset X, \\ \operatorname{Dom} f &:= f^{-1}(\mathbb{R}_{\pm\infty}) = \operatorname{Dom}_{-\infty} f \bigcup \operatorname{Dom}_{+\infty} f \subset X, \\ \operatorname{dom} f &:= f^{-1}(\mathbb{R}) = \operatorname{Dom}_{-\infty} f \bigcap \operatorname{Dom}_{+\infty} f \subset X, \end{aligned} \tag{1.4}$$

For $f, g \in \mathbb{R}^X_{\pm\infty}$ we write $f = g$ if $\operatorname{Dom} f = \operatorname{Dom} g =: D$ and $f(x) = g(x)$ for all $x \in D$, and we write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in D$. For $f \in \mathbb{R}^X_{\pm\infty}$, $g \in \mathbb{R}^Y_{\pm\infty}$ and a set S, we write "*$f = g$ on S*" or "*$f \leq g$ on S*" if $f\big|_{S\cap D} = g\big|_{S\cap D}$ or $f\big|_{S\cap D} \leq g\big|_{S\cap D}$ respectively.

For $f \in F \subset \mathbb{R}^X_{\pm\infty}$, we set $f^+ \colon x \mapsto \max\{0, f(x)\}$, $x \in \operatorname{Dom} f$, $F^+ := \{f \geq 0 \colon f \in F\}$. So, f is *positive* on X if $f = f^+$, and we write "*$f \geq 0$ on X*".We will use the following construction of *countable completion of F up:*

$$F^\uparrow := \Bigl\{f \in \mathbb{R}^X_{\pm\infty} \colon \textit{there is an increasing sequence } (f_j)_{j\in\mathbb{N}},\ f_j \in F, \\ \textit{such that } f(x) = \lim_{j\to\infty} f_j(x) \textit{ for all } x \in X \text{ (we write } f_j \underset{j\to\infty}{\nearrow} f)\Bigr\}. \tag{1.5}$$

[1] A reference mark over a symbol of (in)equality, inclusion, or more general binary relation, etc. means that this relation is somehow related to this reference.

Proposition 1. *Let $F \subset \mathbb{R}^X_{\pm}$ be a subset closed relative to the maximum. Consider sequences $F \ni f_{kj} \underset{j\to\infty}{\nearrow} f_k \underset{k\to\infty}{\nearrow} f$. Then $F \ni \max\{f_{kj} \colon j \leq n, k \leq n\} \underset{n\to\infty}{\nearrow} f$. In particular, $(F^{\uparrow})^{\uparrow} = F^{\uparrow}$.*

The proof is obvious.

For topological space X, $C(X) \subset \mathbb{R}^X$ is the vector space over $\mathbb{R}$ of all continuous functions.

We denote the function identically equal to resp. $-\infty$ or $+\infty$ on a set by the same bold symbols $\boldsymbol{-\infty}$ or $\boldsymbol{+\infty}$.

For an open set $O \subset \mathbb{R}^d_\infty$, we denote by $\operatorname{har}(O)$ and $\operatorname{sbh}(O)$ the classes of all *harmonic* (locally affine for m = 1) and *subharmonic* (locally convex for $m = 1$) functions on O, respectively. The class $\operatorname{sbh}(O)$ contains the *minus-infinity function* $\boldsymbol{-\infty}$;

$$\operatorname{sbh}_*(O) := \operatorname{sbh}(O) \setminus \{\boldsymbol{-\infty}\}, \quad \operatorname{sbh}^+(O) := (\operatorname{sbh}(O))^+. \tag{1.6}$$

Denote by $\delta\text{-}\operatorname{sbh}(O) := \operatorname{sbh}(O) - \operatorname{sbh}(O)$ the class of all *δ-subharmonic* functions on O [2], [35, 3.1]. The class $\delta\text{-}\operatorname{sbh}(O)$ contains two trivial functions, $\boldsymbol{-\infty}$ and $\boldsymbol{+\infty} := -(\boldsymbol{-\infty})$;

$$\delta\text{-}\operatorname{sbh}_*(O) \overset{(1.6)}{:=} \delta\text{-}\operatorname{sbh}(O) \setminus \{\boldsymbol{\pm\infty}\}. \tag{1.7}$$

If $o \notin O \ni \infty$, then we can to use the *inversion* in the sphere $\partial B(o, 1)$ centered at $o \in \mathbb{R}^d$:

$$\star_o \colon x \longmapsto x^{\star_o} := \begin{cases} o & \text{for } x = \infty, \\ o + \frac{1}{|x-o|^2}(x - o) & \text{for } x \neq o, \infty, \\ \infty & \text{for } x = o, \end{cases} \qquad \star := \star_0 =: \star_\infty \tag{1.8$\star$}$$

together with the *Kelvin transform* [17, Ch. 2, 6; Ch. 9]

$$u^{\star_o}(x^{\star_o}) = |x - o|^{d-2} u(x), \quad x^{\star_o} \in O^{\star_o} := \{x^{\star_o} \colon x \in O\}, \tag{1.8u}$$

$$\Bigl(u \in \operatorname{sbh}(O)\Bigr) \iff \Bigl(u^{\star_o} \in \operatorname{sbh}(O^{\star_o})\Bigr). \tag{1.8s}$$

For a subset $S \subset \mathbb{R}^d_\infty$, the classes $\operatorname{har}(S)$, $\operatorname{sbh}(S)$, $\delta\text{-}\operatorname{sbh}(S) := \operatorname{sbh}(S) - \operatorname{sbh}(S)$, and $C^k(S)$ with $k \in \mathbb{N} \cup \{\infty\}$ consist of the restrictions to S of *harmonic, subharmonic, δ-subharmonic,*and *k times continuously differentiable functions* in some (in general, its own for each function) open set $O \subset \mathbb{R}^d_\infty$ containing S. Classes $\operatorname{sbh}_*(S)$, $\delta\text{-}\operatorname{sbh}_*(S)$ are defined like previous classes (1.6), (1.7),

$$\operatorname{sbh}^+(S) \overset{(1.6)}{:=} \bigl\{u \in \operatorname{sbh}(S) \colon u \geq 0 \text{ on } S\bigr\}. \tag{1.9}$$

By $\operatorname{const}_{a_1,a_2,\dots} \in \mathbb{R}$ we denote constants, and constant functions, in general, depend on $a_1, a_2, \dots$ and, unless otherwise specified, only on them, where the dependence on dimension d of $\mathbb{R}^d_\infty$ will be not specified and not discussed; $\operatorname{const}^+_{\dots} \geq 0$.

1.3 Measures and charges

Let $\mathrm{Borel}(S)$ be the class of all Borel subsets in $S \in \mathrm{Borel}(\mathbb{R}^d_\infty)$. We denote by $\mathrm{Meas}(S)$ the class of all Borel signed measures, or, *charges* on $S \in \mathrm{Borel}(\mathbb{R}^d_\infty)$; $\mathrm{Meas}_{\mathrm{c}}(S)$ is the class of charges $\mu \in \mathrm{Meas}(S)$ with a compact support $\operatorname{supp}\mu \Subset S$;

$$\mathrm{Meas}^+(S) := \{\mu \in \mathrm{Meas}(S) \colon \mu \geq 0\}, \ \mathrm{Meas}^+_{\mathrm{c}}(S) := \mathrm{Meas}_{\mathrm{c}}(S) \cap \mathrm{Meas}^+(S); \tag{1.10$^+$}$$

$$\mathrm{Meas}^{1+}(S) := \{\mu \in \mathrm{Meas}^+(S) \colon \mu(S) = 1\}, \quad \textit{probability measures.} \tag{1.10^1}$$

For a charge $\mu \in \mathrm{Meas}(S)$, we let μ^+, $\mu^- := (-\mu)^+$ and $|\mu| := \mu^+ + \mu^-$ respectively denote its *upper, lower,* and *total variations.* So, $\delta_x \in \mathrm{Meas}^{1+}_{\mathrm{c}}(S)$ is the *Dirac measure* at a point $x \in S$, i.e., $\operatorname{supp}\delta_x = \{x\}$, $\delta_x(\{x\}) = 1$. We denote by $\mu\big|_{S'}$ the restriction of μ to $S' \in \mathrm{Borel}(\mathbb{R}^d_\infty)$.

If the Kelvin transform (1.8) translates the subharmonic function u into another function $u^\star_o$ (1.8u), then its Riesz measure υ is transformed common use image under its own mapping-inversion of type 1 or 2. These rules are described in detail in L. Schwartz's monograph [48, Vol. I,Ch.IV, § 6] and we do not dwell on them here, although here interesting questions arise, for example, for the Bernstein – Paley – Wiener –Mary Cartwright classes of entire functions [15], [41], [3], [38] etc.

Given $S \in \mathrm{Borel}(\mathbb{R}^d_\infty)$ and $\mu \in \mathrm{Meas}(S)$, the class $L^1_{\mathrm{loc}}(S,\mu)$ consists of all extended numerical locally integrable functions with respect to the measure μ on S; $L^1_{\mathrm{loc}}(S) := L^1_{\mathrm{loc}}(S,\lambda_d)$. For $L \subset L^1_{\mathrm{loc}}(S,\mu)$, we define a subclass

$$L\,\mathrm{d}\mu := \Bigl\{\nu \in \mathrm{Meas}(S) \colon \textit{there exists } g \in L \textit{ such that } \mathrm{d}\nu = g\,\mathrm{d}\mu\Bigr\} \tag{1.11}$$

of the class of all absolutely continuous charges with respect to μ. For $\mu \in \mathrm{Meas}(S)$, we set

$$\mu(x,r) := \mu\bigl(B(x,r)\bigr) \text{ if } B(x,r) \overset{(1.2)}{\subset} S. \tag{1.12}$$

Let $\triangle$ be the the *Laplace operator* acting in the sense of the theory of distributions, Γ be the *gamma function,*

$$s_{d-1} := \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{1.13}$$

be the *surface area* of the $(d-1)$*-dimensional unit sphere* $\partial\mathbb{B}$ embedded in $\mathbb{R}^d$. For function $u \in \mathrm{sbh}_*(O)$, the *Riesz measure of* u is a Borel (or Radon [45, A.3]) *positive measure*

$$\varDelta_u := c_d \triangle u \in \mathrm{Meas}^+(O), \quad c_d \overset{(1.13)}{:=} \frac{1}{s_{d-1}(1+(d-3)^+)} = \frac{\Gamma(d/2)}{2\pi^{d/2}\max\bigl\{1, d-2\bigr\}}. \tag{1.14}$$

In particular, $\Delta_u(S) < +\infty$ for each subset $S \Subset O$. By definition, $\Delta_{-\infty}(S) := +\infty$ for all $S \subset O$.

We use different variants of *outer Hausdorff p-measure* $\varkappa_p$ with $p \in \mathbb{N}_0$:

$$\varkappa_p(S) := b_p \lim_{0<r\to 0} \inf\Biggl\{\sum_{j\in\mathbb{N}} r_j^p : S \subset \bigcup_{j\in\mathbb{N}} B(x_j, r_j), 0 \leq r_j < r\Biggr\}, \tag{1.15H}$$

$$b_p \overset{(1.14)}{:=} \begin{cases} 1 & \text{if } p = 0, \\ 2 & \text{if } p = 1, \\ \dfrac{s_{p-1}}{p} & \text{if } p \in 1 + \mathbb{N}, \end{cases} \quad \text{is the } \textit{volume of the unit ball } \mathbb{B} \textit{ in } \mathbb{R}^p. \tag{1.15b}$$

Thus, for $p = 0$, for any $S \subset \mathbb{R}^d$, its Hausdorff 0-measure $\varkappa_0(S)$ is to the cardinality $\#S$ of S, for $p = d$ we see that $\varkappa_d \overset{(1.15H)}{=:} \lambda_d$ is the *Lebesgue measure* to Borel proper subsets $S \subset \mathbb{R}^d_\infty$, where, if $\infty \in S$, we preliminary use the inversion(1.8u), and $\sigma_{d-1} := \varkappa_{d-1} \big|_{\partial\mathbb{B}}$ is the $(d-1)$-dimensional surface measure of area on the unit sphere $\partial\mathbb{B}$ in the usual sense.

1.4 Topological concepts: inward-filled hull of set

Let O be a topological space, $S \subset O$, $x \in O$.

We denote by $\operatorname{Conn}_O S$ and $\operatorname{conn}_O(S, x) \in \operatorname{Conn}_O S$ a set of all connected components of S and its connected component containing x, respectively. We write $\operatorname{clos}_O S$, $\operatorname{int}_O S$, and $\partial_O S$ for the *closure*, the *interior*, and the *boundary* of S in O. The set S is *O-precompact* if $\operatorname{clos}_O S$ is a compact subset of O, and we write $S \Subset O$.

Definition 1. An arbitrary O-precompact connected component of $O \setminus S$ is called a *hole* in S with respect to O. The union of a subset $K \subset O$ with all holes in it will be called an *inward-filled hull* of this set K with respect to O and is denoted further as

$$\text{hull-in}_O K := K \bigcup \Bigl(\bigcup \{C \in \operatorname{Conn}_O(O \setminus K) \colon C \Subset O\}\Bigr). \tag{1.16}$$

Denote by O_∞ the *Alexandroff one-point compactification of* O with underlying set $O \sqcup \{\infty\}$, where $\sqcup$ is the *disjoint union* of O with a single point $\infty \notin O$. If this space O is a topological subspace of some ambient topological space $T \supset O$, then this point ∞ can be identified with the boundary $\partial O \subset T$, considered as a single point $\{\partial O\}$.

Throughout this article, we use these topological concepts only in cases when O is an *open non-empty proper Greenian open set* [17, Ch.5, 2] of $\mathbb{R}^d_\infty =: T$, i. e.,

$$\varnothing \neq O = \operatorname{int}_{\mathbb{R}^d_\infty} O = \bigsqcup_{j\in N_O} D_j \neq \mathbb{R}^d_\infty, \quad j \in N_O \subset \mathbb{N}, \quad D_j = \operatorname{conn}_{\mathbb{R}^d_\infty}(O, x_j), \tag{1.17O}$$

where points x_j lie in *different connected components* D_j of $O \subset \mathbb{R}^d_\infty$;

$$\varnothing \neq D \neq \mathbb{R}^d_\infty \quad \text{is an open connected subset, i. e., a } \textit{domain}. \tag{1.17D}$$

The dependence on such an open set O or such domain D for constants $\mathrm{const}_{\dots}$ will not be indicated in the subscripts and is not discussed. For an open set O from (1.17O), we often use statements that are proved in our references only for domains D from (1.17D). This is acceptable since all such cases concern only to individual domains-components D_j. So, if $S \Subset O$, then S meets only finite many components D_j. In addition, we give proofs of our statements only for cases $O, D \subset \mathbb{R}^d$. If we have $o \notin D_j = D \ni \infty$, then we can to use the *inversion* relative to the sphere $\partial B(o, 1)$ centered at $o \in \mathbb{R}^d$ as in (1.8).

Proposition 2 ([11, 6.3], [12]). *Let K be a compact set in an open set $O \subset \mathbb{R}^d$. Then*

(i) $\operatorname{hull-in}_O K$ *is a compact subset in O;*

(ii) *the set $O_\infty \setminus \operatorname{hull-in}_O K$ is connected and locally connected subset in O_∞;*

(iii) *the inward-filled hull of K with respect to O coincides with the complement in O_∞ of connected component of $O_\infty \setminus K$ containing the point ∞, i. e.,*

$$\operatorname{hull-in}_O K = O_\infty \setminus \operatorname{conn}_{O_\infty \setminus K}(\infty);$$

(iv) *if $O' \subset \mathbb{R}^d_\infty$ is an open subset and $O \subset O'$ then* $\operatorname{hull-in}_O K \subset \operatorname{hull-in}_{O'} K$*;*

(v) $\mathbb{R}^d \setminus \operatorname{hull-in}_O K$ *has only finitely many components, i. e.,*

$$\# \operatorname{Conn}_{\mathbb{R}^d_\infty}(\mathbb{R}^d \setminus \operatorname{hull-in}_O K) < \infty.$$

2 Potentials of charges and measures

Further everywhere we will assume for simplicity and brevity that

$$(O \subset \mathbb{R}^d) \Leftrightarrow (\infty \notin O), \quad (D \subset \mathbb{R}^d) \Leftrightarrow (\infty \notin D) \tag{2.1}$$

in addition to (1.17). If $\infty \in O$, $o \in \mathbb{R}^d_\infty \setminus O$, we can always easily go to cases (2.1) using a inversion $\star_o$, and the Kelvin transforms (1.8).

Definition 2 ([34]). Let $\vartheta, \mu \in \operatorname{Meas}(S)$, $S \subset \operatorname{Borel}(\mathbb{R}^d_\infty)$. Let $H \subset \mathbb{R}^S_{\pm\infty}$ be a class of Borel-measurable functions on S. Let us assume that the integrals $\int h \, \mathrm{d}\vartheta$ and $\int h \, \mathrm{d}\mu$ are well defined with values in $\mathbb{R}_{\pm\infty}$ for each function $h \in H$. We write $\vartheta \preceq_H \mu$ and say that the charge μ is a *balayage,* or, sweeping (out), of the charge ϑ *for* H, or, briefly, μ is a H-balayage of ϑ, if

$$\int h \, \mathrm{d}\vartheta \leq \int h \, \mathrm{d}\mu \quad \textit{for all } h \in H. \tag{2.2}$$

Definition 3 ([45], [16], [42])**.** For $q \in \mathbb{R}$, we set

$$k_q(t) := \begin{cases} \log t & \text{if } q = 0, \\ -\operatorname{sgn}(q)t^{-q} & \text{if } q \in \mathbb{R}_*, \end{cases} \qquad t \in \mathbb{R}^+_*, \tag{2.3k}$$

$$K_{d-2}(x,y) := \begin{cases} k_{d-2}\bigl(|x-y|\bigr) & \text{if } x \neq y, \\ -\infty & \text{if } x = y \text{ and } d \geq 2, \\ 0 & \text{if } x = y \text{ and } d = 1, \end{cases} \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.3K}$$

Definition 4 ([45], [28, Definition 2], [35, 3.1, 3.2])**.** Let $\mu \in \operatorname{Meas}_{\mathrm{c}}(\mathbb{R}^d)$ be charge with compact support. Its *potential* is the function $\operatorname{pt}_\mu \in \delta\text{-}\operatorname{sbh}_*(\mathbb{R}^d)$ defined by

$$\operatorname{pt}_\mu(y) \overset{(2.3K)}{:=} \int K_{d-2}(x,y)\,\mathrm{d}\mu(x), \tag{2.4p}$$

where the kernel K_{d-2} is defined in Definition 3 by the function k_q from (2.3k). The values of potential $\operatorname{pt}_\mu(y) \in \mathbb{R}_{\pm\infty}$ is well defined for all

$$y \in \operatorname{Dom}_{-\infty} \operatorname{pt}_\mu = \left\{ y \in \mathbb{R}^d \colon \int_0 \frac{\mu^-(y,t)}{t^{m-1}}\,\mathrm{d}t < +\infty \right\} \tag{2.4d−}$$

$$y \in \operatorname{Dom}_{+\infty} \operatorname{pt}_\mu = \left\{ y \in \mathbb{R}^d \colon \int_0 \frac{\mu^+(y,t)}{t^{m-1}}\,\mathrm{d}t < +\infty \right\} \tag{2.4d+}$$

$$y \in \operatorname{Dom}_{\pm\infty} \operatorname{pt}_\mu = \operatorname{Dom}_{-\infty} \operatorname{pt}_\mu \bigcup \operatorname{Dom}_{+\infty} \operatorname{pt}_\mu \tag{2.4d±}$$

$$y \in \operatorname{dom} \operatorname{pt}_\mu = \operatorname{Dom}_{-\infty} \operatorname{pt}_\mu \bigcap \operatorname{Dom}_{+\infty} \operatorname{pt}_\mu, \tag{2.4d}$$

and their complements $\mathbb{R}^d \setminus \operatorname{Dom}_{-\infty} \operatorname{pt}_\mu$ and $\mathbb{R}^d \setminus \operatorname{Dom}_{+\infty} \operatorname{pt}_\mu$ are *polar sets* in $\mathbb{R}^d$.

If $\mu \in \operatorname{Meas}^+_{\mathrm{c}}(O)$ be a H-balayage of a measure $\vartheta \in \operatorname{Meas}^+_{\mathrm{c}}(O)$, then we consider the potential

$$\operatorname{pt}_{\mu-\vartheta} \overset{(2.4p)}{:=} \operatorname{pt}_\mu - \operatorname{pt}_\vartheta \in \delta\text{-}\operatorname{sbh}(\mathbb{R}^d) \tag{2.5}$$

where under the conditions $d > 1$ and $1 \in H$ it is natural to set $\operatorname{pt}_{\mu-\vartheta}(\infty) := 0$. The latter is based on the following

Proposition 3. *Let* $\mu \in \operatorname{Meas}_{\mathrm{c}}(\mathbb{R}^d)$. *Then*

$$\operatorname{pt}_\mu(x) \overset{(2.3k)}{=} \mu(\mathbb{R}^d)k_{d-2}\bigl(|x|\bigr) + O\bigl(1/|x|^{d-1}\bigr), \quad x \to \infty. \tag{2.6}$$

Proof. For $d = 1$, we have

$$\bigl|\operatorname{pt}_\mu(x) - \mu(\mathbb{R})|x|\bigr| \leq \int \bigl||x-y| - |x|\bigr|\,\mathrm{d}|\mu|(y) \leq \int |y|\,\mathrm{d}|\mu|(y) = O(1), \quad |x| \to +\infty.$$

See (2.6) for $d=2$ in [45, Theorem 3.1.2].
For $d>2$ and $|x|\geq 2\sup\bigl\{|y|\colon y\in\operatorname{supp}\mu\bigr\}$, we have

$$\begin{aligned}\bigl|\mathrm{pt}_\mu(x)-\mu(\mathbb{R}^d)k_{d-2}\bigl(|x|\bigr)\bigr| &= \left|\int\left(\frac{1}{|x|^{d-2}}-\frac{1}{|x-y|^{d-2}}\right)\,\mathrm{d}\mu(y)\right|\\ &\leq \int\left|\frac{1}{|x|^{d-2}}-\frac{1}{|x-y|^{d-2}}\right|\,\mathrm{d}|\mu|(y)\leq\frac{2^{d-2}}{|x|^{2d-4}}\int\bigl||x-y|^{d-2}-|x|^{d-2}\bigr|\,\mathrm{d}|\mu|(y)\\ &\leq\frac{2^{d-2}}{|x|^{2d-4}}\int|y||x|^{d-3}\sum_{k=0}^{d-3}\Bigl(\frac{3}{2}\Bigr)^k\,\mathrm{d}|\mu|(y)\leq 2\frac{3^{d-2}}{|x|^{d-1}}\int|y|\,\mathrm{d}|\mu|(y)=O\Bigl(\frac{1}{|x|^{d-1}}\Bigr).\end{aligned}$$

□

Proposition 4. *If*

$$\mu\in\mathrm{Meas}^+_{\mathrm{c}}(\mathbb{R}^d),\quad L\Subset\mathbb{R}^d,\quad o\in\mathbb{R}^d\setminus L, \tag{2.7}$$

then

$$\inf_{x\in L}\mathrm{pt}_\mu(x)\overset{(2.3\mathrm{k})}{\geq}\mu(\mathbb{R}^d)k_{d-2}\bigl(\operatorname{dist}(L,\operatorname{supp}\mu)\bigr), \tag{2.8i}$$

$$\inf_{x\in L}\mathrm{pt}_{\mu-\delta_o}(x)\overset{(2.4\mathrm{p})}{\geq}\mu(\mathbb{R}^d)k_{d-2}\bigl(\operatorname{dist}(L,\operatorname{supp}\mu)\bigr)-k_{d-2}\left(\sup_{x\in L}|x-o|\right) \tag{2.8o}$$

Proof. If $\operatorname{dist}(L,\operatorname{supp}\mu)=0$, then the right-hand sides in the inequalities (2.8) are equal to $-\infty$, and the inequalities (2.8) are true. Otherwise, by Definition 4, we obtain

$$\begin{aligned}\mathrm{pt}_\mu(x)&=\int k_{d-2}\bigl(|x-y|\bigr)\,\mathrm{d}\mu(y)\geq\inf_{y\in\operatorname{supp}\mu}k_{d-2}\bigl(|x-y|\bigr)\mu(\mathbb{R}^d)\\ &\geq\inf_{y\in\operatorname{supp}\mu}k_{d-2}\left(\inf_{y\in\operatorname{supp}\mu}|x-y|\right)\mu(\mathbb{R}^d)=\mu(\mathbb{R}^d)k_{d-2}\bigl(\operatorname{dist}(x,\operatorname{supp}\mu)\bigr),\end{aligned} \tag{2.9}$$

since the function k_q from (2.3k) is *increasing,* which implies the inequality (2.8i) after applying the operation $\inf_{x\in L}$ to both sides of inequality (2.9). Using (2.8i), we have

$$\begin{aligned}\inf_{x\in L}\mathrm{pt}_{\mu-\delta_o}(x)&\overset{(2.4\mathrm{p})}{=}\inf_{x\in L}\bigl(\mathrm{pt}_\mu(x)-k_{d-2}\bigl(|x-o|\bigr)\bigr)\geq\inf_{x\in L}\mathrm{pt}_\mu(x)-\sup_{x\in L}k_{d-2}\bigl(|x-o|\bigr)\\ &\overset{(2.8\mathrm{i})}{\geq}\mu(\mathbb{R}^d)k_{d-2}\bigl(\operatorname{dist}(L,\operatorname{supp}\mu)\bigr)-k_{d-2}\left(\sup_{x\in L}|x-o|\right)\end{aligned}$$

which gives the inequality (2.8o). □

2.1 Duality Teorem for $\mathrm{har}(O)$-balayage

Duality Theorem 1 (for $\mathrm{har}(O)$-balayage)**.** *If a measure $\mu\in \mathrm{Meas}_{\mathrm c}^+(O)$ is a $\mathrm{har}(O)$-balayage of a measure $\vartheta\in \mathrm{Meas}_{\mathrm c}^+(O)$, then*

$$\mathrm{pt}_\mu \in \mathrm{sbh}_*(\mathbb R^d)\cap \mathrm{har}(\mathbb R^d\setminus \operatorname{supp}\mu), \tag{2.10p}$$

$$\mathrm{pt}_\mu = \mathrm{pt}_\vartheta \text{ on } \mathbb R^d\setminus \text{hull-in}_O(\operatorname{supp}\vartheta\cup \operatorname{supp}\mu\cup). \tag{2.10=}$$

Conversely, suppose that there is a subset $S\Subset O$, and a function p such that

$$p \overset{(2.10\text{p})}{\in} \mathrm{sbh}(O)\cap \mathrm{har}(O\setminus S), \tag{2.11p}$$

$$p \overset{(2.10=)}{=} \mathrm{pt}_\vartheta \quad \text{on } O\setminus S. \tag{2.11=}$$

Then the Riesz measure

$$\mu:=\Delta_p \overset{(1.14)}{:=} c_d \,\triangle\, p \overset{(2.11)}{\in} \mathrm{Meas}^+(\operatorname{clos} S)\subset \mathrm{Meas}_{\mathrm c}^+(O) \tag{2.12}$$

of this function p is a $\mathrm{har}(O)$-balayage of ϑ.

Proof. The first property (2.10p) is evidently. For each $y\in\mathbb R^d$, the kernel $K_{d-2}(\cdot,y)$ is harmonic on $\mathbb R^d\setminus\{y\}$. By

Proposition 5 ([34])**.** *Let $\mu\in \mathrm{Meas}_{\mathrm c}(O)$ be a balayage of $\vartheta\in\mathrm{Meas}_{\mathrm c}(O)$ for $\mathrm{har}(O)$. Then*

$$\int h\,\mathrm d\vartheta=\int h\,\mathrm d\mu \quad \text{for any } h\in \mathrm{har}\bigl(\text{hull-in}_O(\operatorname{supp}\mu\cup\operatorname{supp}\vartheta)\bigr) \tag{2.13}$$

(see Subsec. 1.4, Definition 1 of inward-filled hull of compact subset $\operatorname{supp}\mu\cup\operatorname{supp}\vartheta$ in O).

for $h:=K_{d-2}(\cdot,y)$ in (2.13), we have

$$\mathrm{pt}_\vartheta(y)=\int K_{d-2}(x,y)\,\mathrm d\vartheta(x) \overset{(2.13)}{=} \int K_{d-2}(x,y)\,\mathrm d\mu(x)=\mathrm{pt}_\mu(y) \tag{2.14}$$

for all $y\in \text{hull-in}_O(\operatorname{supp}\mu\cup\operatorname{supp}\vartheta)$. This gives (2.10=).

In the opposite direction, we can extend the function p to $\mathbb R^d$ so that $p=\mathrm{pt}_\vartheta$ on $\mathbb R^d\setminus S$. In view of (2.37), we have $p\in \mathrm{sbh}(\mathbb R^d)\cap\mathrm{har}(\mathbb R^d\setminus S)$, and

$$p(x)-\vartheta(O)k_{d-2}(|x|)=p(x)-\mathrm{pt}_\vartheta(x)+O\bigl(1/|x|^{d-1}\bigr) \overset{(2.11=)}{=} O\bigl(1/|x|^{d-1}\bigr),\quad x\to\infty. \tag{2.15}$$

Hence the function p is a potential with the Riesz measure (2.12), and $\mu(O)=\vartheta(O)$, i. e., $p=\mathrm{pt}_\mu$. Further, we can use the following

Lemma 1 ([11, Lemma 1.8])**.** *Let F be a compact subset of $\mathbb{R}^d$, let $h \in \operatorname{har}(F)$, and $\varepsilon > 0$. Then there are points $y_1, y_2, \dots, y_k$ in $\mathbb{R}^d \setminus F$ such that*

$$\Bigl| h(x) - \sum_{j=1}^{k} k_{d-2}\bigl(|x - y_j|\bigr) \Bigr| < \varepsilon \quad \text{for all } x \in F. \tag{2.16}$$

Applying Lemma 1 to the compact set $F \overset{(2.11\text{p})}{:=} \operatorname{clos} S \cup \operatorname{supp} \vartheta \Subset O$ and a function $h \in \operatorname{har}(O)$, we obtain

$$\begin{aligned}\Bigl| \int_F h \,\mathrm{d}(\mu - \vartheta) \Bigr| &\overset{(2.11=)}{=} \Bigl| \int_F h \,\mathrm{d}(\mu - \vartheta) - \sum_{j=1}^{k} \bigl(\operatorname{pt}_\mu(y_j) - \operatorname{pt}_\vartheta(y_j)\bigr) \Bigr| \\ &\le \sup_{x \in F} \Bigl| h(x) - \sum_{j=1}^{k} k_{d-2}\bigl(|x - y_j|\bigr) \Bigr| \bigl(\mu(O) + \vartheta(O)\bigr) \le \varepsilon \bigl(\mu(O) + \vartheta(O)\bigr)\end{aligned}$$

for any $\varepsilon > 0$. Hence the measure μ is a $\operatorname{har}(O)$-balayage of ϑ. □

Corollary 1. *Let $\vartheta, \mu \in \operatorname{Meas}_{\mathrm{c}}(O)$, $\operatorname{supp} \vartheta \cup \operatorname{supp} \mu \subset S \Subset O$. If μ is a balayage of ϑ for the class*

$$H = \bigl\{ \pm k_{d-2}\bigl(|y - \cdot|\bigr) \colon y \in \mathbb{R}^d \setminus \operatorname{clos} S \bigr\}, \tag{2.17}$$

then μ is a $\operatorname{har}(O)$-balayage of ϑ.

Proof. We have (2.14) for all $y \in \mathbb{R}^d \setminus \operatorname{clos} S$. By Duality Theorem 1, $\vartheta \preceq_{\operatorname{har}(O)} \mu$. □

Corollary 2. *Let $\mu \in \operatorname{Meas}^+_{\mathrm{c}}(O)$ be a $\operatorname{har}(O)$-balayage of measure $\vartheta \in \operatorname{Meas}^+_{\mathrm{c}}(O)$, and $\varsigma \in \operatorname{Meas}^+_{\mathrm{c}}(O)$ also be a $\operatorname{har}(O)$-balayage of the same measure ϑ. If*

$$\operatorname{hull-in}_O(\operatorname{supp} \vartheta \cup \operatorname{supp} \varsigma) \subset \operatorname{hull-in}_O(\operatorname{supp} \vartheta \cup \operatorname{supp} \mu), \tag{2.18}$$

then the measure μ is a $\operatorname{har}(O)$-balayage of the measure ς.

2.2 Arens – Singer measures and their potentials

Example 1 ([10], [28])**.** Let $x \in O$. If $\mu \in \operatorname{Meas}^+_{\mathrm{c}}(O)$ is a balayage of δ_x for $\operatorname{har}(O)$, then such measure μ is called a *Arens – Singer measure for x*. The class of such measures is denoted by $AS_x(O) \supset J_x(O)$. Arens – Singer measures are often referred to as representing measures.

By Example 1, if we choose $x \in O$ and $\vartheta := \delta_x \preceq_{\operatorname{har}(O)} \mu \in \operatorname{Meas}^+_{\mathrm{c}}(O)$, i. e., μ is a Arens – Singer measure for $x \in O$, then potential

$$\operatorname{pt}_{\mu - \delta_x}(y) = \operatorname{pt}_\mu(y) - K_{d-2}(x, y), \quad y \in \mathbb{R}^d \setminus \{x\} \tag{2.19}$$

satisfies conditions

$$\begin{aligned} \mathrm{pt}_{\mu-\delta_x} &\in \mathrm{sbh}(\mathbb{R}^d_\infty), \quad \mathrm{pt}_{\mu-\delta_x}(\infty) := 0,\\ \mathrm{pt}_{\mu-\delta_x} &\equiv 0 \quad \text{on } \mathbb{R}^d_\infty \setminus \text{hull-in}_O\bigl(\{x\}\cup \operatorname{supp}\mu\bigr)\\ \mathrm{pt}_{\mu-\delta_x}(y) &\leq -K_{d-2}(x,y)+O(1) \quad \text{for } x\neq y\to x. \end{aligned} \tag{2.20}$$

Remember, that the function $V\in \mathrm{sbh}_*\bigl(\mathbb{R}^d_\infty\setminus\{x\}\bigr)$ is called a *Arens – Singer potential on O with pole at $x\in O$* [28], [30, Definition 6] (partially in [10, 3.3,3.4], [1], [46]), if this function V satisfies conditions

$$\begin{aligned} V&\equiv 0 \quad \text{on } \mathbb{R}^d_\infty\setminus S(V)) \text{ for a subset } S(V)\Subset O\\ V(y)&\leq -K_{d-2}(x,y)+O(1) \quad \text{for } x\neq y\to x. \end{aligned} \tag{2.21}$$

The class of all Arens – Singer potential on O with pole at $x\in O$ denote by $PAS_x(O)$. In this class $PAS_x(O)$ we will consider a special subclass

$$PAS^1_x(O) := \bigl\{V\in PAS_x(O)\colon V(y) = -K_{d-2}(x,y)+O(1) \text{ for } x\neq y\to x\bigr\} \tag{2.22}$$

By Duality Theorem 1, we have

Duality Theorem A ([28, Proposition 1.4, Duality Theorem]). *The mapping*

$$\mathcal{P}_x\colon \mu\longmapsto \mathrm{pt}_{\mu-\delta_x} \tag{2.23}$$

is the affine bijection from $AS_x(O)$ onto $PAS_x(O)$ with inverse mapping

$$\mathcal{P}_x^{-1}\colon V \overset{(1.14)}{\longmapsto} c_d\triangle V\,\Bigr|_{\mathbb{R}^d\setminus\{x\}} + \left(1-\limsup_{x\neq y\to x}\frac{V(y)}{-K_{d-2}(x,y)}\right)\cdot\delta_x. \tag{2.24}$$

Let $x\in \operatorname{int} Q = Q\Subset O$. The restriction of $\mathcal{P}_x$ to the class

$$\bigl\{\mu\in AS_x(O)\colon\ \operatorname{supp}\mu\cap Q=\varnothing\bigr\} \tag{2.25}$$

define a bijection from class (2.25) *onto class* (see (2.22))

$$PAS^1_x(O)\bigcap \mathrm{har}\bigl(Q\setminus\{x\}\bigr). \tag{2.26}$$

The restriction of $\mathcal{P}_x$ to the class

$$\bigl\{\mu\in AS_x(O)\colon\ \operatorname{supp}\mu\cap Q=\varnothing\bigr\}\bigcap\bigl(C^\infty(O)\,\mathrm{d}\lambda_d\bigr) \tag{2.27}$$

define also a bijection from class (2.27) *onto class*

$$PAS^1_x(O)\bigcap \mathrm{har}\bigl(Q\setminus\{x\}\bigr)\bigcap C^\infty\bigl(O\setminus\{x\}\bigr). \tag{2.28}$$

This transition from the main bijection $\mathcal{P}_x$ to the bijection from (2.25) onto (2.26) or from (2.27) onto (2.28) by restriction of $\mathcal{P}_x$ to (2.25) or (2.27) is quite obvious.

2.3 A generalization of Poisson – Jensen formula

Theorem 1 (extended Poisson – Jensen formula for $\operatorname{har}(O)$-balayage)**.** *Let $\mu \in \operatorname{Meas}_{\mathrm c}^{+}(O)$ be a $\operatorname{har}(O)$-balayage of $\vartheta \in \operatorname{Meas}_{\mathrm c}^{+}(O)$. If $u \in \operatorname{sbh}(O)$ is a function with the Riesz measure $\varDelta_u \overset{(1.14)}{:=} c_d \bigtriangleup u \in \operatorname{Meas}^{+}(O)$, then*

$$\int u \,\mathrm{d}\vartheta + \int_K \operatorname{pt}_\mu \,\mathrm{d}\varDelta_u = \int_K \operatorname{pt}_\vartheta \,\mathrm{d}\varDelta_u + \int u \,\mathrm{d}\mu, \quad K := \text{hull-in}_O(\operatorname{supp}\vartheta \cup \operatorname{supp}\mu). \tag{2.29}$$

In particular, if

$$\int u \,\mathrm{d}\vartheta > -\infty, \tag{2.30}$$

then (2.29) can be written as

$$\int u \,\mathrm{d}\vartheta = \int u \,\mathrm{d}\mu - \int_K \operatorname{pt}_{\mu-\vartheta} \,\mathrm{d}\varDelta_u. \tag{2.31}$$

Proof. Consider first the case (2.30). Choose an open set O' such that $K \Subset O' \Subset O$. By the Riesz decomposition theorem $u = \operatorname{pt}_{\nu'} + h$ on O', where $\nu' := \varDelta_u \big|_{O'}$ and $h \in \operatorname{har}(O')$. Integrating this representation with respect to $\mathrm{d}\vartheta$ and $\mathrm{d}\mu$, we obtain

$$\int u \,\mathrm{d}\mu = \int \operatorname{pt}_{\nu'} \,\mathrm{d}\mu + \int h \,\mathrm{d}\mu, \tag{2.32μ}$$

$$\int u \,\mathrm{d}\vartheta = \int \operatorname{pt}_{\nu'} \,\mathrm{d}\vartheta + \int h \,\mathrm{d}\vartheta, \tag{2.32ϑ}$$

where the three integrals in (2.32ϑ) are finite, although in the equality (2.32μ) the first two integrals can take simultaneously the value of $-\infty$, but the last integral in (2.32μ) is finite. Therefore, the difference (2.32μ)$-$(2.32ϑ) of these two equalities is well defined:

$$\int u \,\mathrm{d}\mu - \int u \,\mathrm{d}\vartheta = \int \operatorname{pt}_{\nu'} \,\mathrm{d}\mu - \int \operatorname{pt}_{\nu'} \,\mathrm{d}\vartheta + \int h \,\mathrm{d}(\mu - \vartheta), \tag{2.33}$$

where the first and third integrals can simultaneously take the value of $-\infty$, and the remaining integrals are finite. By Proposition 5, the *last integral* in (2.33) *vanishes.* Using Fubini's theorem, in view of the symmetry property of kernel in (2.4p), we have

$$\int \operatorname{pt}_{\nu'} \,\mathrm{d}\vartheta = \int\int K_{d-2}(y,x) \,\mathrm{d}\nu'(y) \,\mathrm{d}\vartheta(x)$$
$$= \int\int K_{d-2}(x,y) \,\mathrm{d}\vartheta(x) \,\mathrm{d}\nu'(y) = \int_{O'} \operatorname{pt}_\vartheta \,\mathrm{d}\varDelta_u. \tag{2.34}$$

and the same way

$$\int \operatorname{pt}_{\nu'} \,\mathrm{d}\mu = \int\int K_{d-2}(y,x) \,\mathrm{d}\nu'(y) \,\mathrm{d}\mu(x)$$
$$= \int\int K_{d-2}(x,y) \,\mathrm{d}\mu(x) \,\mathrm{d}\nu'(y) = \int_{O'} \operatorname{pt}_\mu \,\mathrm{d}\varDelta_u \tag{2.35}$$

even if the integral on the left side of equalities (2.35) takes the value $-\infty$ because the integrand $K_{d-2}(\cdot,\cdot)$ is bounded from above on the compact set $\operatorname{clos} O'\times \operatorname{clos} O'$ [16, Theorem 3.5]. Hence equality (2.33) can be rewritten as

$$\int u\,\mathrm{d}\mu-\int u\,\mathrm{d}\vartheta=\int_{O'}\mathrm{pt}_\mu\,\mathrm{d}\varDelta_u-\int_{O'}\mathrm{pt}_\vartheta\,\mathrm{d}\varDelta_u=\int_K\mathrm{pt}_\mu\,\mathrm{d}\varDelta_u-\int_K\mathrm{pt}_\vartheta\,\mathrm{d}\varDelta_u$$

since $\mathrm{pt}_\mu=\mathrm{pt}_\vartheta$ on $O'\setminus K$. This gives equality (2.29) in the case (2.30).

If condition(2.30) is not fulfilled, then from the representation (2.32ϑ) it follows that the integral on the left-hand side of (2.34) also takes the value $-\infty$. The equalities (2.34) is still true [16, Theorem 3.5]. Hence, the first integral on the right side of the formula (2.29) also takes the value $-\infty$ and this formula (2.29) remains true. □

Remark 1. If $\vartheta:=\delta_x$ and $\mu:=\omega_D(x,\cdot)$ for $x\in D\Subset O$, then the formula (2.31) is the classical Poisson – Jensen formula [16, Theorem 5.27]

$$u(x)=\int_{\partial D}u\,\mathrm{d}\omega_D(x,\cdot)-\int_{\operatorname{clos} D}g_D(\cdot,x)\,\mathrm{d}\varDelta_u,\quad x\in D, \tag{2.36a}$$

$$\delta_x\preceq_{\mathrm{sbh}(O)}\omega_D(x,\cdot),\quad \mathrm{pt}_{\omega_D(x,\cdot)}-\mathrm{pt}_{\delta_x}=\mathrm{pt}_{\omega_D(x,\cdot)-\delta_x}=g_D(\cdot,x). \tag{2.36b}$$

2.4 Duality Theorem for $\mathrm{sbh}(O)$-balayage

Duality Theorem 2 (for $\mathrm{sbh}(O)$-balayage)**.** *If a measure $\mu\in\mathrm{Meas}_{\mathrm{c}}^+(O)$ is a $\mathrm{sbh}(O)$-balayage of a measure $\vartheta\in\mathrm{Meas}_{\mathrm{c}}^+(O)$, then we have (2.10), and*

$$\mathrm{pt}_\mu\geq\mathrm{pt}_\vartheta\quad \text{on } \mathbb{R}^d. \tag{2.37}$$

Conversely, suppose that there is a subset $S\Subset O$, and a function p such that we have (2.11), and $p\geq\mathrm{pt}_\vartheta$ on $\operatorname{clos} S$. Then the Riesz measure (2.12) of p is a $\mathrm{sbh}(O)$-balayage of ϑ.

Proof. If $\vartheta\preceq_{\mathrm{sbh}(O)}\mu$, then $\vartheta\preceq_{\mathrm{har}(O)}\mu$ and we have properties (2.10) by Duality Theorem 1. For each $y\in\mathbb{R}^d$, the function $K_{d-2}(\cdot,y)$ is subharmonic on $\mathbb{R}^d$ and (2.37) follows from Definitions 2 and 4. Conversely, if a function p is such as in (2.11), then, by Duality Theorem 1, this function is a potential $\mathrm{pt}_\mu=p$ with the Riesz measure (2.12), this measure $\mu\in\mathrm{Meas}_{\mathrm{c}}^+(O)$ is a $\mathrm{har}(O)$-balayage for ϑ, and $K:=\text{hull-in}(\operatorname{supp}\vartheta\cup\operatorname{supp}\mu)\subset\operatorname{clos} S$. Let $u\in\mathrm{sbh}_*(O)$. It follows from $\mathrm{pt}_\mu\geq\mathrm{pt}_\vartheta$ on K that $\int_K\mathrm{pt}_\vartheta\,\mathrm{d}\varDelta_u\leq\int_K\mathrm{pt}_\mu\,\mathrm{d}\varDelta_u$. Hence, by the extended Poisson – Jensen formula (2.29) from Theorem 1, we obtain $\int u\,\mathrm{d}\vartheta\leq\int u\,\mathrm{d}\mu$. □

2.5 Jensen measures and their potentials

Example 2 ([10], [7], [8], [47])**.** Let $x\in O$. If a measure $\mu\in\mathrm{Meas}_{\mathrm{c}}^+(O)$ is a balayage of the Dirac measure δ_x for $\mathrm{sbh}(O)$, then this measure μ is called a *Jensen measure for x*. The class of such measures is denoted by $J_x(O)$.

By Example 2, if we choose $x \in O$ and $\vartheta := \delta_x \preceq_{\mathrm{sbh}(O)} \mu \in \mathrm{Meas}_{\mathrm{c}}^+(O)$, i. e., μ is a Jensen measure for $x \in O$, then potential

$$\mathrm{pt}_{\mu-\delta_x}(y) = \mathrm{pt}_\mu(y) - K_{d-2}(x,y), \quad y \in \mathbb{R}^d \setminus \{x\} \tag{2.38}$$

satisfies conditions (2.20) and $\mathrm{pt}_{\mu-\delta_x} \geq 0$ on $\mathbb{R}^d_\infty \setminus \{x\}$. Remember, that a *positive* function $V \in \mathrm{sbh}^+\bigl(\mathbb{R}^d_\infty \setminus \{x\}\bigr)$ is called a *Jensen potential on O with pole at $x \in O$* [28], [30, Definition 8], if this function V satisfies conditions (2.21) The class of all Jensen potential on O with pole at $x \in O$ denote by $PJ_x(O) \subset AS_x(O)$. In this class $J_x(O)$ we will consider a special subclass

$$PJ^1_x(O) \overset{(2.22)}{:=} PJ_x(O) \bigcap PAS^1_x(O) \subset PAS^1_x(O). \tag{2.39}$$

By Duality Theorem 2, we have

Duality Theorem B ([28, Proposition 1.4, Duality Theorem]). *The mapping* (2.23) *is the affine bijection from $J_x(O)$ onto $PJ_x(O)$ with inverse mapping* (2.24).

Let $x \in \operatorname{int} Q = Q \Subset O$. The restriction of $\mathcal{P}_x$ to the class (cf. (2.25))

$$\bigl\{\mu \in J_x(O) \colon \operatorname{supp}\mu \cap Q = \varnothing\bigr\} \tag{2.40}$$

define a bijection from class (2.40) *onto class* (see (2.39), cf. (2.26))

$$PJ^1_x(O) \bigcap \mathrm{har}\bigl(Q \setminus \{x\}\bigr). \tag{2.41}$$

Let $x \in \operatorname{int} Q = Q \Subset O$. The restriction of $\mathcal{P}_x$ to the class (cf. (2.27))

$$\bigl\{\mu \in J_x(O) \colon \operatorname{supp}\mu \cap Q = \varnothing\bigr\} \bigcap \bigl(C^\infty(O)\,\mathrm{d}\lambda_d\bigr) \tag{2.42}$$

define a bijection from class (2.42) *onto class* (cf. (2.28))

$$PJ^1_x(O) \bigcap \mathrm{har}\bigl(Q \setminus \{x\}\bigr) \bigcap C^\infty\bigl(O \setminus \{x\}\bigr). \tag{2.43}$$

This transition from the main bijection $\mathcal{P}_x$ to the bijection from (2.40) onto (2.41) or from (2.42) onto (2.43) by restriction of $\mathcal{P}_x$ to (2.40) or to (2.42) is quite obvious.

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