Cauchy's Integral formula

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1 Introduction

In this article we will introduce Cauchy's integral formula. This formula is one of the most important formulas in complex analysis,named after great mathematician Augustin-Louis Cauchy. It's very useful in evaluating complex integrals. It simply states that the values of a holomorphic(we will give definition of homorphicity below) function inside a disk are determined by the values of that function on the boundary of the disk.

2 Cauchy's integral formula

Definition 2.1. *Let $U \subset \mathbb{C}$ be open set and let $f : U \mapsto \mathbb{C}$ be a complex function. Then f is analytic in U if and only if f is* **differentiable** *at each point in U*

Theorem 1. *Suppose C is a simple closed curve and the function $f(z)$ is analytic on a region containing C and its interior. We assume C is oriented counterclockwise. Then for any z_0 inside C:*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \tag{1}$$

Proof. We know that if the function is analytic at some point then this function is also continuous at this point. We use this fact and integral of $\frac{1}{z - z_0}$ over the curve C in order to prove our theorem.
Firstly, let's evaluate integral. In order to integrate $\frac{1}{z - z_0}$ we'll parametrize it. Let $z = z_0 + re^i\theta$ where $0 \leq \theta \leq 2\pi$. Then $dz = ire^{i\theta}d\theta$.Then we have:

$$\int_{C_r} \frac{1}{z - z_0} dz = \int_0^{2\pi} id\theta = 2\pi i$$

Using this and fact that integral over the circle C_r and C have the same values,equation (1) can be written as:

$$\int_C \frac{f(z)}{z - z_0} = f(z_0) \int_C \frac{1}{z - z_0} = \int_C \frac{f(z_0)}{z - z_0} \tag{2}$$

Since the function is continuous at the point z_0, by the definition of continuity at the point:
$\forall \epsilon > 0 \qquad \exists \delta = \delta(\epsilon) > 0 \qquad |z - z_0| < \delta \qquad |f(z) - f(z_0)| < \epsilon$
Now we just have to show equation (2) is true.Let's estimate following absolute value:

$$\left| \int_{C_r} \frac{f(z)}{z - z_0} dz - \int_{C_r} \frac{f(z_0)}{z - z_0} dz \right| \leq \int_{C_r} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| dz < \int_{C_r} \frac{\epsilon}{|z - z_0|} dz$$

Now let's pick $r < \delta$.Now we get:

$$\int_{C_r} \frac{\epsilon}{|z - z_0|} dz = \frac{\epsilon}{r} \int_{C_r} dz \tag{3}$$

Since the integral of the right hand side of (3) gives us length of circle C_r, which is $2\pi r$ we get estimate:

$$\left| \int_{C_r} \frac{f(z)}{z - z_0} dz - \int_{C_r} \frac{f(z_0)}{z - z_0} dz \right| < 2\pi\epsilon$$

Now theorem is proved since ϵ can be made arbitrarily small. □

Example. Compute

$$\int_C \frac{(z - 2)^2}{z + i} dz$$

where C is the circle of radius 2 centered at origin.

Let $f(z) = (z - 2)^2$, clearly f is analytic everywhere in the interior of C. Hence, by the Cauchy's integral formula:

$$\int_C \frac{(z - 2)^2}{z + i} dz = 2\pi i f(-i) = -8\pi + 6\pi i$$

Theorem 2. ***(Cauchy's integral theorem for derivatives)*** *If $f(z)$ and C satisfy the same hypotheses for Cauchy's integral formula,then for all z_0 inside C we have*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{(n+1)}}$$

where, C is simple closed curve, oriented counterclockwise, z_0 is inside C and $f(z)$ is analytic on and inside C.