# An Exact Solution of the Einstein Equations with Cosmological Term for Cylindrically Symmetric Space

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**Abstract**

In this article, we derived an exact solution of the Einstein equations with cosmological term for cylindrically symmetric space .

Here we derive an exact solution of the Einstein equations with cosmological term for cylindrically symmetric space.

The static condition[1] means that, with a static coordinate system, the fundamental tensors, the $g_{\mu\nu}$ are independent of the time $x^0$ or t and also $g_{0m} = 0$. The spatial coordinates may be taken to be cylindrical coordinates $x^1 = r, x^2 = \varphi, x^3 = z$. The general form for the square of invariant distance, the $ds^2$ compatible with cylindrical symmetry is

$$ds^2 = e^{2v}dt^2 - e^{2h}dr^2 - r^2d\varphi^2 - e^{2u}dz^2, \tag{1}$$

where v, h, and u are functions of r and z only.

We can read off the value of $g_{\mu\nu}$ from Eq.(1), namely,

$$g_{00} = e^{2v}, g_{11} = -e^{2h}, g_{22} = -r^2, g_{33} = -e^{2u},$$

and

$$g_{\mu\nu} = 0 for \mu \neq \nu.$$

We find

$$g^{00} = e^{-2v}, g^{11} = -e^{-2h}, g^{22} = -r^{-2}, g^{33} = -e^{-2u},$$

and

$$g^{\mu\nu} = 0 for \mu \neq \nu.$$

The Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$ can be calculated by

$$\Gamma^{\mu}_{\nu\sigma} = g^{\mu\lambda}\Gamma_{\lambda\nu\sigma}, \tag{2}$$

and

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}). \tag{3}$$

Many of them vanish.

Then we calculate the Ricci tensors by

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}. \tag{4}$$

The non vanishing components of $R_{\mu\nu}$ are

$$R_{00} = e^{2(v-h)} \times \{\frac{\partial v}{\partial r}[-\frac{\partial(v+u-h)}{\partial r} - \frac{1}{r}] - \frac{\partial^2 v}{\partial r^2}\} +$$

$$+ e^{2(v-u)} \times \{\frac{\partial v}{\partial z}[-\frac{\partial(v-u+h)}{\partial z}] - \frac{\partial^2 v}{\partial z^2}\},$$

$$R_{11} = e^{2(h-u)} \times \{\frac{\partial h}{\partial z}[\frac{\partial(v-u+h)}{\partial z}] + \frac{\partial^2 h}{\partial z^2}\} - \frac{\partial h}{\partial r}[\frac{\partial(v+u)}{\partial r} + \frac{1}{r}] +$$

$$+ \frac{\partial^2(v+u)}{\partial r^2} + (\frac{\partial v}{\partial r})^2 + (\frac{\partial u}{\partial r})^2,$$

$$R_{22} = e^{-2h} \times r \times \frac{\partial(v+u-h)}{\partial r},$$

and

$$R_{33} = e^{2(u-h)}\{\frac{\partial u}{\partial r}[\frac{\partial(v+u-h)}{\partial r} + \frac{1}{r}] + \frac{\partial^2 u}{\partial r^2}\} + (\frac{\partial v}{\partial z})^2 + (\frac{\partial h}{\partial z})^2 +$$
$$+ \frac{\partial^2(v+h)}{\partial z^2} - \frac{\partial u}{\partial z}\frac{\partial(v+h)}{\partial z}.$$

Einstein's law requires all $R_{\mu\nu}$ satisfy the following relation.

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{5}$$

for all $\mu, \nu$.

The equations $R_{00} = \Lambda g_{00}, R_{11} = \Lambda g_{11}, R_{22} = \Lambda g_{22}$, and $R_{33} = \Lambda g_{33}$ will be,

$$e^{2(v-h)} \times \{\frac{\partial v}{\partial r}[-\frac{\partial(v+u-h)}{\partial r} - \frac{1}{r}] - \frac{\partial^2 v}{\partial r^2}\} +$$
$$+ e^{2(v-u)} \times \{\frac{\partial v}{\partial z}[-\frac{\partial(v-u+h)}{\partial z}] - \frac{\partial^2 v}{\partial z^2}\} = \Lambda e^{2v}, \tag{6}$$

$$e^{2(h-u)} \times \{\frac{\partial h}{\partial z}[\frac{\partial(v-u+h)}{\partial z}] + \frac{\partial^2 h}{\partial z^2}\} - \frac{\partial h}{\partial r}[\frac{\partial(v+u)}{\partial r} + \frac{1}{r}] +$$
$$+ \frac{\partial^2(v+u)}{\partial r^2} + (\frac{\partial v}{\partial r})^2 + (\frac{\partial u}{\partial r})^2 = -\Lambda e^{2h}, \tag{7}$$

$$e^{-2h} \times r \times \frac{\partial(v+u-h)}{\partial r} = -\Lambda r^2, \tag{8}$$

and

$$e^{2(u-h)}\{\frac{\partial u}{\partial r}[\frac{\partial(v+u-h)}{\partial r} + \frac{1}{r}] + \frac{\partial^2 u}{\partial r^2}\} + (\frac{\partial v}{\partial z})^2 + (\frac{\partial h}{\partial z})^2 +$$
$$+ \frac{\partial^2(v+h)}{\partial z^2} - \frac{\partial u}{\partial z}\frac{\partial(v+h)}{\partial z} = -\Lambda e^{2u}. \tag{9}$$

From the $eq(8)$, we can expect that $h = h(r)$ which is independent of variable z.

Here we limit our solution when all the functions are variables separable. $v(r,z) = v_1(r) + v_2(z)$, $u(r,z) = u_1(r) + u_2(z)$, and $h(r,z) = h(z)$,

$$-e^{2u_2} \times (-(\frac{dv_2}{dz})^2 - \frac{d^2v_2}{dz^2} + \frac{du_2}{dz}\frac{dv_2}{dz}) =$$

$$e^{2u_1} \times (-\Lambda + r\Lambda\frac{dv_1}{dr} - (\frac{1}{r}\frac{dv_1}{dr} + \frac{d^2v_1}{dr^2})e^{-2h}) = k_1, \tag{10}$$

$$-(\frac{1}{r} + \frac{d(v_1+u_1)}{dr})\frac{dh}{dr} + (\frac{dv_1}{dr})^2 + (\frac{du_1}{dr})^2 + \frac{d^2(v_1+u_1)}{dr^2}) = -\Lambda e^{2h} \tag{11}$$

$$e^{-2h} \times \frac{d(v_1+u_1-h)}{dr} = -\Lambda r, \tag{12}$$

and

$$-e^{-2u_2}((\frac{dv_2}{dz})^2 - \frac{dv_2}{dz}\frac{du_2}{dz} + \frac{d^2v_2}{dz^2}\} =$$

$$e^{2u_1}(\Lambda - r\Lambda\frac{du_1}{dr} + (\frac{1}{r}\frac{du_1}{dr} + \frac{d^2u_1}{dr^2})e^{-2h}) = k_3. \tag{13}$$

From eq(10) and eq(13) for varibals z, we have $k_1 = -k_3 = k$. and $u_2 = -v_2$ with the exact solution;

$$e^{2v_2} = kz^2 + \epsilon z + 1, \tag{14}$$

here $\epsilon$ is an integration constant and $e^{2v_2} \longrightarrow 1$, when $z \longrightarrow 0$.

From the equations of eq(10) and eq (13) for varibles r, we have $v_1 = u_1$,

The equations can be simplfied as

$$e^{2u_1} \times (-\Lambda + r\Lambda\frac{du_1}{dr} - (\frac{1}{r}\frac{du_1}{dr} + \frac{d^2u_1}{dr^2})e^{-2h}) = k, \tag{15}$$

$$-(\frac{1}{r}+2\frac{du_1}{dr})\frac{dh}{dr}+2(\frac{du_1}{dr})^2+2\frac{d^2u_1}{dr^2}=-\Lambda e^{2h} \quad (16)$$

$$e^{-2h}\times\frac{d(2u_1-h)}{dr}=-\Lambda r, \quad (17)$$

and

To solve the equations (15) to (17), we have,

$$2\frac{u_1'}{r}+\Lambda e^{2h}+(u_1')^2=-ke^{2h-2u_1}, \quad (18)$$

and

$$u_1''-\frac{u_1'}{r}+(u_1')^2-u_1'h'=0, \quad (19)$$

here we use ' as $\frac{d}{dr}$,

we have the solution,

$$ln(\frac{u_1'}{r})=h-u_1+h_0, \quad (20)$$

here $h_0$ is an integration constant. and we have,

$$u_1'=re^{h-u_1+h_0}, \quad (21)$$

Insert this result to the eq(18),

Then we will have,

$$2e^{h-u_1+h_0}+\Lambda e^{2h}+r^2e^{2h+2h_0-2u_1}=-ke^{2h-2u_1}, \quad (22)$$

let $y=e^{u_1}$,and $e^h=\frac{y'}{r}e^{-h_0}$, we will have,

$$y'(\Lambda e^{-h_0}y^2+ke^{-h_0}+r^2e^{h_0})=-2re^{h_0}y, \quad (23)$$

we will have,

$$(\frac{1}{3}\Lambda e^{-h_0}y^3+(ke^{-h_0}+r^2e^{h_0})y)'=0, \tag{24}$$

$$\frac{1}{3}\Lambda e^{-h_0}y^3+(ke^{-h_0}+r^2e^{h_0})y=y_0, \tag{25}$$

$$y^3+\frac{3}{\Lambda}(k+r^2e^{2h_0})y-\frac{3y_0}{\Lambda}e^{h_0}=0, \tag{26}$$

This is an algebra equation,

$$y^3+py+q=0, \tag{27}$$

we have $y \longrightarrow 1$ when $r \longrightarrow 0$ , we have,

$$1+\frac{3k}{\Lambda}-\frac{3y_0}{\Lambda}e^{h_0}=0, \tag{28}$$

let $e^{h_0}=\frac{\Lambda}{3}=\frac{1}{R^2}$,

when $y_0=0$, we have $k=-\frac{\Lambda}{3}$ and eq(27) will have non zero solution of $y^2+p=0$,

$$y^2=1-\frac{r^2}{R^2}=e^{2u_1}=e^{2v_1}, \tag{29}$$

and we have,

$$e^{2h}=\frac{1}{1-\frac{r^2}{R^2}}, \tag{30}$$

In general $y_0 \neq 0$,

we have,

$$y=\{-\frac{q}{2}+\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}\}^{\frac{1}{3}}+\{-\frac{q}{2}-\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}\}^{\frac{1}{3}}, \tag{31}$$

For $y_0 = 0$, we will have,

$$e^{2v} = (1 - \frac{r^2}{R^2})(1 - \frac{z^2}{R^2} + \epsilon z), \tag{32}$$

$$e^{2u} = (1 - \frac{r^2}{R^2})(1 - \frac{z^2}{R^2} + \epsilon z)^{-1}, \tag{33}$$

and we have,

$$e^{2h} = \frac{1}{1 - \frac{r^2}{R^2}}, \tag{34}$$

An interesting phenomina is that when a light is sent by a remote star in the z axis to an observer at the origin point, this physics can be happen in the cylindrical description. The red shift factor is $e^{-v_2}$ which becomes infinite when $z \longrightarrow R$ if we take $\epsilon = 0$ .

The red shift is depending on the square of the distance, $z^2$ of the remote star to the observer at the original point. This can explain the so called "speeding expansion of the universe".

# References

[1] P.A.M. Dirac,General Theory of Relativity, Wiley, 1975.