A Proof of Sondow’s Conjecture

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Abstract

Based on the functional equation of the Dirichlet eta function, Sondow’s conjecture is proved. The advantage of a method is that it only use undergraduate mathematics.

1. Introduction

The Riemann zeta function \( \zeta(s) \) is the function of complex variable \( s = a + ib \in \mathbb{C} \), which converges for any complex number having \( \Re(s) > 1 \) [1].

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

(1)

The Dirichlet eta function is defined as

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s}) \zeta(s)
\]

(2)

\( \eta(s) \) converges for all \( s \in \mathbb{C} \) with \( \Re(s) > 0 \). The Dirichlet eta function extends the Riemann zeta function from \( \Re(s) > 1 \) to the larger domain \( \Re(s) > 0 \). The factor \((1-2^{1-s})\) has an infinity of zeros on the line \( \Re(s) = 1 \) given by \( s = 1 + i \frac{2\pi n}{\ln 2}, n = \pm 1, \pm 2, \pm 3 \ldots \)

2. A Proof of The Sondow’s conjecture

Here is Sondow’s conjecture [1].

If

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \cos(b \ln n) = 0
\]

(3)

and

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \sin(b \ln n) = 0
\]

(4)

for some pair of real numbers \( a \) and \( b \) then \( a = \frac{1}{2} \) or \( a = 1 \).

Proof. As shown in [2], All zeros of \( \eta(s) \) in the critical strip are located at

\[ s = \left( \frac{1}{2} \pm \epsilon \right) \pm ib \quad k=1,2,3, \ldots \]

for \( 0 \leq \epsilon < \frac{1}{2} \).

Substitute \( a = \frac{1}{2} \pm \epsilon \) and \( b = \pm b_k \) in (3) & (4) resulted in eight equations.

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2^k \epsilon}} \cos(b_k \ln n) = 0
\]

(5)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2^k \epsilon}} \sin(b_k \ln n) = 0
\]

(6)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2^k \epsilon}} \cos(-b_k \ln n) = 0
\]

(7)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2^k \epsilon}} \sin(-b_k \ln n) = 0
\]

(8)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2^k \epsilon}} \cos(-b_k \ln n) = 0
\]

(9)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2^k \epsilon}} \sin(-b_k \ln n) = 0
\]

(10)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2^k \epsilon}} \cos(-b_k \ln n) = 0
\]

(11)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2^k \epsilon}} \sin(-b_k \ln n) = 0
\]

(12)

Choose any pair of points with \( a = \frac{1}{2} \pm \epsilon \), and using the identity \( \sin(-x) = -\sin(x) \) and \( \cos(-x) = \cos(x) \), for example by subtracting Eq. (5) and Eq. (9) we have

\[
\sum_{n=1}^{\infty} \left[ \frac{(-1)^n}{n^{2^k \epsilon}} - \frac{(-1)^n}{n^{2^k \epsilon}} \right] \cos(b_k \ln n) = 0
\]

(13)

and by subtracting Eq. (6) and Eq. (10), we have

\[
\sum_{n=1}^{\infty} \left[ \frac{(-1)^n}{n^{2^k \epsilon}} - \frac{(-1)^n}{n^{2^k \epsilon}} \right] \sin(b_k \ln n) = 0
\]

(14)

From both Eq. (13) and Eq. (14), we can see that
\[
\frac{(-1)^n}{n^2} + \frac{(-1)^n}{n^2 + 1} = 0
\]  
(15)

A unique solution of Eq. (15) is \( \epsilon = 0 \). It shows that \( a = \frac{1}{2} \). Thus, Sondow’s conjecture is true.

In [1], Sondow proves the following Proposition:
The Sondow’s conjecture is true if and only if the Riemann Hypothesis (RH) is true.

**References**


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