A Proof of Sondow's Conjecture

Wiroj Homsup

Abstract

Based on the functional equation of the Dirichlet eta function, Sondow's conjecture is proved. The advantage of a method is that it only use undergraduate mathematics.

1. Introduction

The Riemann zeta function $\zeta(s)$ is the function of complex variable s = a +ib $\in$ C , which converges for any complex number having $\Re(s) > 1$ [1].

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

The Dirichlet eta function is defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s) \tag{2}$$

$\eta(s)$ converges for all s $\in$ C with $\Re(s) > 0$. The Dirichlet eta function extends the Riemann zeta function from $\Re(s) > 1$ to the larger domain $\Re(s) > 0$. The factor $(1-2^{1-s})$ has an infinity of zeros on the line $\Re(s) = 1$ given by $s = 1 + i\frac{2\pi n}{\ln 2}, n = \pm1, \pm2, \pm3$ ...

2. **A Proof of The Sondow's conjecture**

Here is Sondow's conjecture [1].

If

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \cos(b \ln n) = 0 \tag{3}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^a} \sin(b \ln n) = 0 \tag{4}$$

for some pair of real numbers a and b then a = ½ or a = 1.

Proof. As shown in [2], All zeros of $\eta(s)$ in the critical strip are located at

$$s = (\tfrac{1}{2} \pm \epsilon) \pm i b_k, \ k = 1,2,3,\ldots$$

for $0 \leq \epsilon < ½$.

Substitute $a = \frac{1}{2} \pm \epsilon$ and $b = \pm b_k$ in (3) &(4) resulted in eight equations.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}-\epsilon}} \cos(b_k \ln n) = 0 \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}-\epsilon}} \sin(b_k \ln n) = 0 \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}-\epsilon}} \cos(-b_k \ln n) = 0 \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}-\epsilon}} \sin(-b_k \ln n) = 0 \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}+\epsilon}} \cos(b_k \ln n) = 0 \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}+\epsilon}} \sin(b_k \ln n) = 0 \tag{10}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}+\epsilon}} \cos(-b_k \ln n) = 0 \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}+\epsilon}} \sin(-b_k \ln n) = 0 \tag{12}$$

Choose any pair of points with $a = \frac{1}{2} \pm \epsilon$, and using the identity sin(-x) = -sin(x) and cos(-x) = cos(x), for example by subtracting Eq. (5) and Eq.(9) we have

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^{\frac{1}{2}-\epsilon}} - \frac{(-1)^n}{n^{\frac{1}{2}+\epsilon}}\right] \cos(b_k \ln n) = 0 \tag{13}$$

and by subtracting Eq. (6) and Eq.(10), we have

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^{\frac{1}{2}-\epsilon}} - \frac{(-1)^n}{n^{\frac{1}{2}+\epsilon}}\right] \sin(b_k \ln n) = 0 \tag{14}$$

From both Eq. (13) and Eq. (14), we can see that

$$\frac{(-1)^n}{n^{\frac{1}{2}-\epsilon}} - \frac{(-1)^n}{n^{\frac{1}{2}+\epsilon}} = 0 \qquad (15)$$

A unique solution of Eq. (15) is $\epsilon = 0$. It shows that a = ½. Thus, Sondow's conjecture is true.

In [1], Sondow proves the following Proposition :
The Sondow's conjecture is true if and only if the Riemann Hypothesis (RH) is true.

References

[1] Sondow, Jonathan. A Simple Counterexample to Havil's " reformation" of the Riemann Hypothesis. arXiv:0706.2840v3 [math.NT] 30 Nov 2010.

[2] Heymann, Yuri. The admissible domain of the non-trivial zeros of the Riemann zeta function. arXiv: 1804.04700v6 [math.GM] 15 Jul 2018.

Wiroj Homsup
Email: wh5637@yahoo.com