Euler numbers , Catalan's constant , Number Pi

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05/08/2019 11:03:28

abstract

We give some formulas involving Catalan's constant $G = 0.915965...$.

1. Introduction

1.1. The number Pi is defined by

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.141592... \tag{1}$$

1.2. The Catalan's constant is defined by

$$G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = 0.915965... \tag{2}$$

1.3. Euler numbers are defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty}\frac{(-1)^n E_n x^{2n}}{(2n)!} \quad ,|x| < \pi/2 \tag{3}$$

$$\{E_n : n \geq 0\} = \{1,1,5,61,1385,50521,2702785,...\} \tag{4}$$

$$E_0 = 1, \quad E_n = (-1)^n \sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k-1}\sum_{m=1}^{k}(-1)^m \binom{2k}{k-m} m^{2n} \; ,n \geq 1 \tag{5}$$

$$E_n = 2^{2n+1}\int_0^{\infty} x^{2n}\,\mathrm{sech}(\pi x)\,dx \quad ,n \geq 0 \tag{6}$$

2. Formula

2.1. If $e^{-\pi/2} < x \leq 1$, then

$$4G + \pi\ln x = 4\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)^2} - 2\sum_{n=0}^{\infty}\frac{(-1)^n E_n (\ln x)^{2n+2}}{(2n+2)!} \tag{7}$$

2.2. Examples

$$4G-\pi\ln 2=4\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{-2n-1}}{(2n+1)^2}-2\sum_{n=0}^{\infty}\frac{(-1)^n\,E_n\,(\ln 2)^{2n+2}}{(2n+2)!} \tag{8}$$

$$4G+\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n\,e^{-2n-1}}{(2n+1)^2}-2\sum_{n=0}^{\infty}\frac{(-1)^n\,E_n}{(2n+2)!} \tag{9}$$

$$8G-\pi=8e^{-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n\,e^{-n}}{(2n+1)^2}-\sum_{n=0}^{\infty}\frac{(-1)^n\,E_n\,2^{-2n}}{(2n+2)!} \tag{10}$$

$$4G-\frac{\pi}{2}=4\sum_{n=0}^{\infty}\frac{(-1)^n\,e^{-(2n+1)/2}}{(2n+1)^2}\cos\left(\frac{2n+1}{2}\right)-\sum_{n=0}^{\infty}\frac{(-1)^n\,E_n\,2^{-n}\,\mathrm{Re}\left(i^{n+1}\right)}{(2n+2)!} \tag{11}$$

$$\frac{\pi}{2}=4\sum_{n=0}^{\infty}\frac{(-1)^n\,e^{-(2n+1)/2}}{(2n+1)^2}\sin\left(\frac{2n+1}{2}\right)+\sum_{n=0}^{\infty}\frac{(-1)^n\,E_n\,2^{-n}\,\mathrm{Im}\left(i^{n+1}\right)}{(2n+2)!} \tag{12}$$

$$\begin{aligned}4G-\pi\ln 2&=2\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{-2n}}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{4}\right)\\&\quad-2\sum_{n=0}^{\infty}\frac{(-1)^n\,E_n}{(2n+2)!}\mathrm{Re}\left(\left(-\ln 2+i\frac{\pi}{4}\right)^{2n+2}\right)\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\pi^2}{4}&=2\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{-2n}}{(2n+1)^2}\sin\left(\frac{(2n+1)\pi}{4}\right)\\&\quad-2\sum_{n=0}^{\infty}\frac{(-1)^n\,E_n}{(2n+2)!}\mathrm{Im}\left(\left(-\ln 2+i\frac{\pi}{4}\right)^{2n+2}\right)\end{aligned} \tag{14}$$

Remark: $i=\sqrt{-1}$, $\mathrm{Re}(z),\mathrm{Im}(z)$ are the real and imaginary part of complex z .

References

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