Numbers:Part 2

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abstract

We give four series for Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.1415926535\ldots$$

keywords: number Pi , series , bernoulli numbers

1. Introduction.

■ Bernoulli numbers

The bernoulli numbers are defined by:

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!}B_n x^{2n} \quad , \quad |x| < 2\pi \tag{1}$$

$$B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \ldots\right\} \tag{2}$$

$$B_n = (-1)^{n-1}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k}\binom{2n+1}{k}\sum_{m=1}^{k-1}m^{2n} \quad , \; n \geq 1 \tag{3}$$

In this note we give four series for Pi.

2. Four Series for Pi

Recall that

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.1415926535\ldots \tag{4}$$

■ Pi Series

$$\pi = 2\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)B_n}{n(2n)!}\sum_{k=0}^{n-1}\binom{2n}{2k+1}(-1)^k\left(\alpha^{2n-2k-1}+\alpha^{2k+1}\right) \tag{5}$$

where

$$\alpha = 0.366937217309056220028537299625\ldots \tag{6}$$

$$\pi = 3\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)B_n}{n(2n)!}\sum_{k=0}^{n-1}\binom{2n}{2k+1}(-1)^k\left(\beta^{2n-2k-1}+\beta^{2k+1}\right) \tag{7}$$

where

$$\beta = 0.234143327172713706194494667118\ldots \tag{8}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)B_n}{n(2n)!}\sum_{k=0}^{n-1}\binom{2n}{2k+1}(-1)^k\left(\gamma^{2n-2k-1}+\gamma^{2k+1}\right) \tag{9}$$

where

$$\gamma = 0.172856370946717809149172304691\ldots \tag{10}$$

$$\pi = 6\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)B_n}{n(2n)!}\sum_{k=0}^{n-1}\binom{2n}{2k+1}(-1)^k\left(\delta^{2n-2k-1}+\delta^{2k+1}\right) \tag{11}$$

where

$$\delta = 0.113932633799902228197851855448\ldots \tag{12}$$

3. The numbers $\alpha,\beta,\gamma,\delta$

- The number α is root of the equation

$$\tan(x)\tanh(1)+\tan(1)\tanh(x)+\tan(1)\tanh(1)\tan(x)\tanh(x)=1 \tag{13}$$

- α-iteration

$$\alpha_1 = 0,\ \alpha_{n+1} = \tanh^{-1}\left(\frac{1-\tanh(1)\tan(\alpha_n)}{\tan(1)(1+\tanh(1)\tan(\alpha_n))}\right),\ n \ge 1 \Longrightarrow \alpha_n \longrightarrow \alpha \tag{14}$$

- The number β is root of the equation

$$\sqrt{3}\,(\tan(x)\tanh(1)+\tan(1)\tanh(x))+\tan(1)\tanh(1)\tan(x)\tanh(x)=1 \tag{15}$$

- β-iteration

$$\beta_1 = 0,\ \beta_{n+1} = \tanh^{-1}\left(\frac{1-\sqrt{3}\,\tanh(1)\tan(\beta_n)}{\tan(1)\left(\sqrt{3}+\tanh(1)\tan(\beta_n)\right)}\right),\ n \ge 1 \Longrightarrow \beta_n \longrightarrow \beta \tag{16}$$

- The number γ is root of the equation

$$\left(\sqrt{2}+1\right)(\tan(x)\tanh(1)+\tan(1)\tanh(x))+\tan(1)\tanh(1)\tan(x)\tanh(x)=1 \tag{17}$$

- γ-iteration

$$\gamma_1 = 0,\ \gamma_{n+1} = \tanh^{-1}\left(\frac{\sqrt{2}-1-\tanh(1)\tan(\gamma_n)}{\tan(1)\left(1+\left(\sqrt{2}-1\right)\tanh(1)\tan(\gamma_n)\right)}\right),\ n \ge 1 \Longrightarrow \gamma_n \longrightarrow \gamma \tag{18}$$

- The number δ is root of the equation

$$\left(2+\sqrt{3}\right)(\tan(x)\tanh(1)+\tan(1)\tanh(x))+\tan(1)\tanh(1)\tan(x)\tanh(x)=1 \tag{19}$$

- δ-iteration

$$\delta_1 = 0\,,\;\; \delta_{n+1} = \tanh^{-1}\left(\frac{2-\sqrt{3}\; - \tanh(1)\tan(\delta_n)}{\tan(1)\left(1+\left(2-\sqrt{3}\right)\tanh(1)\tan(\delta_n)\right)}\right),\;\; n \geq 1 \Longrightarrow \delta_n \longrightarrow \delta \tag{20}$$

4. Graphics

- α-iteration

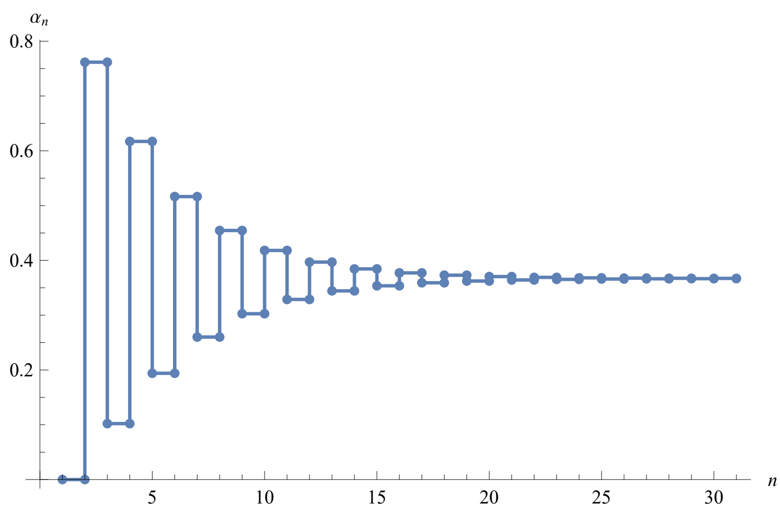

Figure 1.

- β-iteration

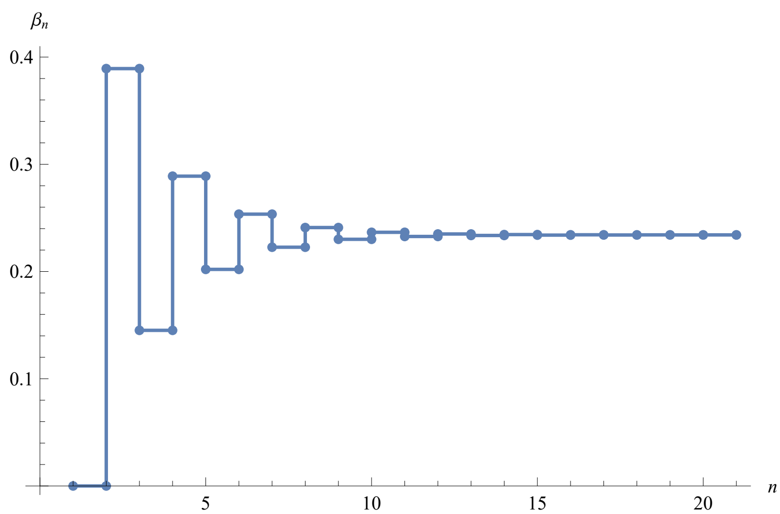

Figure 2.

- γ-iteration

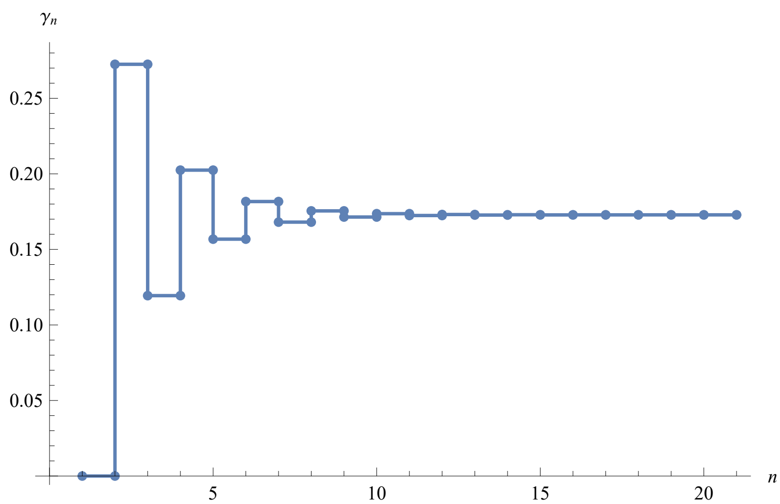

Figure 3.

■ δ-iteration

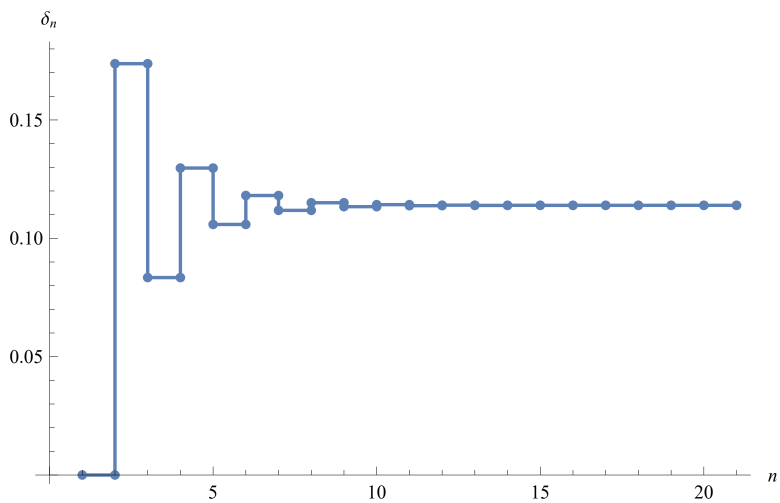

Figure 4.

References

A. Arndt, J., and Haenel, C.: π unleashed. Springer-Verlag, 2001.

B. Blatner, D.: The Joy of π . Walker Publ., 1999.

C. Ebbinghaus, H., et al.: Numbers. Springer-Verlag, Reading in Mathematics, 1991.

D. Olver, F.W.J., et al.: NIST Handbook of Mathematical Functions. Cambridge University Press, 2010.