Reconsideration of $x^3 - dx - a = 0$ based on the cubic equation $x^3 = 15x + 4$ solved by Rafael Bombelli

— Prelude to Extended Set Theory —

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Atsushi Koike

The fact that $\frac{\pi}{3}$ can not be trisected using ruler-and-compasses has been supposed to be established in the trisection equation shown below:

(1) $x^3 - 3x - 1 = 0$

That basic cubic equation is as follows:

(2) $x^3 - dx - a = 0$

Construct an angle ∠O to be determined by the X and Y axes as shown in Fig.1 and draw an arc centering on O. The arc and X axes an intersected point is named A, and Y axes an intersected point is named D. We trisect ∠O and on an arc intersected points is called B and C. And pull vertical lines, first, from B, second, from D with X axes intersected points are named H and G. Name $\overline{BH}$ as y and interpret it as the division number and expand it. Then, if one side of a right triangle is a

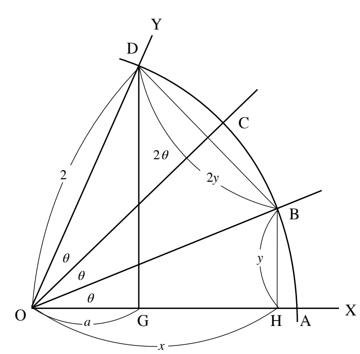

Fig.1 [Y: p.24]

as shown in Fig.2, then the hypotenuse $\overline{\mathrm{OD}}$ is $2a$ if $a = 1$ because of the well-known property of right triangle. Therefore, we get $x = 1$ or $x = -1$, however by the proof of contradiction, we will known $x^3 - 3x - 1 = 0$ has not solution of the rational number. While if it is assumed that the trisection equation of (1) is derived from the cubic equation of (2), it is possible to trisect, too. For example, $\dfrac{\pi}{2}$ as shown in (3) and Fig.3 can obtain three solutions as follows by substituting $a = 0$ [Y: pp.24 - 66]:

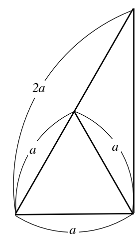

Fig.2 [Y: p.62]

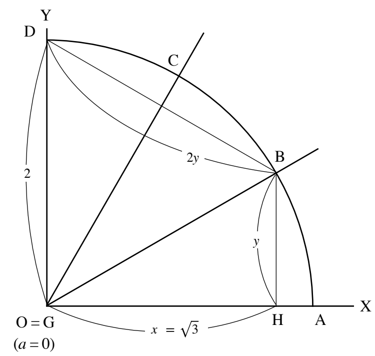

Fig.3 [Y: p.54]

$$
\begin{aligned}
&(3) \quad x^3 - 3x = 0 \\
&\qquad x(x - \sqrt{3})(x + \sqrt{3}) = 0 \\
&\qquad \therefore x = 0, \sqrt{3}, -\sqrt{3}
\end{aligned}
$$

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By the way, Rafael Bombelli is considering $x^3 = 15x + 4$ in his book "Algebra" announced in 1572 and solves it as $x = 4$. That cubic equation can be rewritten as (4) [S: p.47]:

$$(4) \quad x^3 = 15x + 4 \ \rightarrow \ x^3 - 15x - 4 = 0$$

This is obviously equal to (2) which was the trisection equation of the angle. Known as the originator of the imaginary number, Gerolamo Cardano described the solution (5) of the cubic equation $x^3 + px + q = 0$. It called Cardinal formula that is in the writing book "Ars magna de Rebus Algebraicis". Bombelli got the solution (6) of $x^3 - 15x - 4 = 0$ by referring to it. [S: p.48-49]:

$$(5) \quad x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

(6) ……

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = \left(2 + \sqrt{-1}\right) + \left(2 - \sqrt{-1}\right) = 4$$

There are two points to note in Bombelli's solution of $x^3 - 15x - 4 = 0$. First, it is $x = a$. Second, 15 and 4 are not prime numbers.

$x^3 - 15x - 4 = 0$ can be interpreted as $x^3 - dx - a = 0$. The triangular equation of the corners was also $x^3 - dx - a = 0$. Therefore, we analyze the point of attention of Bombelli's solution superimposed on (1) $x^3 - 3x - 1 = 0$ and (3) $x^3 - 3x = 0$. In the first point of attention, (1) ignores by the proof of contradiction but the interesting point is in (3), because, it is $x = a$. In the second point, 0 in (3) is not prime, but needless to say, it is the basic number, and 3 is prime number. (3) and (4) has the obvious difference. Accordingly, $d = 15$ and $a = 4$ of Bombelli's equation can be reconstructed into cubic equations like (7), (8):

(7) If $x^3 = (3 \cdot 5)x + (2 \cdot 2)$ and $x = 4$ then

$$x^3 = (3 \cdot 5)x + 2 \cdot 2 = 3 \cdot 5 \cdot 2 \cdot 2 + 2 \cdot 2 = 60 + 4 = 4 \cdot 4 \cdot 4$$
$$\therefore\ x = 4$$

(8) else if $x^3 = (x - 1) \cdot x \cdot (x + 1) + (2 \cdot 2)$ and then

$$x^3 = (2 \cdot 2 - 1) \cdot 2 \cdot 2 \cdot (2 \cdot 2 + 1) + 2 \cdot 2 = 3 \cdot 4 \cdot 5 + 4 = 60 + 4 = 4 \cdot 4 \cdot 4$$
$$\therefore\ x = 4$$

If we rewrite (8) into a style of $x^3 - (x - 1) \cdot x \cdot (x + 1) - a = 0$ and let $x = a$ then the following limiting equation holds:

(9) $$\lim_{x \to \infty} x^3 - (x - 1) \cdot x \cdot (x + 1) - x = 0$$

1^3	–	(1 – 1)	·	1	·	(1 + 1)	–	1	=	1	–	0	·	1	·	2	–	1	= 0,	x =	1,	d =	0
2^3	–	(2 – 1)	·	2	·	(2 + 1)	–	2	=	8	–	1	·	2	·	3	–	2	= 0,	x =	2,	d =	3
3^3	–	(3 – 1)	·	3	·	(3 + 1)	–	3	=	27	–	2	·	3	·	4	–	3	= 0,	x =	3,	d =	8
4^3	–	(4 – 1)	·	4	·	(4 + 1)	–	4	=	64	–	3	·	4	·	5	–	4	= 0,	x =	4,	d =	15
5^3	–	(5 – 1)	·	5	·	(5 + 1)	–	5	=	125	–	4	·	5	·	6	–	5	= 0,	x =	5,	d =	24
6^3	–	(6 – 1)	·	6	·	(6 + 1)	–	6	=	216	–	5	·	6	·	7	–	6	= 0,	x =	6,	d =	35
7^3	–	(7 – 1)	·	7	·	(7 + 1)	–	7	=	343	–	6	·	7	·	8	–	7	= 0,	x =	7,	d =	48
8^3	–	(8 – 1)	·	8	·	(8 + 1)	–	8	=	512	–	7	·	8	·	9	–	8	= 0,	x =	8,	d =	63
9^3	–	(9 – 1)	·	9	·	(9 + 1)	–	9	=	749	–	8	·	9	·	10	–	9	= 0,	x =	9,	d =	80
10^3	–	(10 – 1)	·	10	·	(10 + 1)	–	10	=	1000	–	9	·	10	·	11	–	10	= 0,	x =	10,	d =	99
11^3	–	(11 – 1)	·	11	·	(11 + 1)	–	11	=	1331	–	10	·	11	·	12	–	11	= 0,	x =	11,	d =	120
12^3	–	(12 – 1)	·	12	·	(12 + 1)	–	12	=	1728	–	11	·	12	·	13	–	12	= 0,	x =	12,	d =	143
⋮		⋮		⋮		⋮		⋮		⋮		⋮		⋮		⋮			⋮		⋮		⋮

Fig.4

From Fig.4 we can see the following things. If we let $\boldsymbol{d = (x - 1) \cdot (x + 1)}$ in a cubic equation of $x^3 - (x - 1) \cdot x \cdot (x + 1) - x = 0$ then $d = 3 \cdot 5$ is only for $x = 4$. Furthermore, it is only when $x = 2$ that $d = 1 \cdot 3$ is obtained. Hence, assuming that the trisection equations is $a = 2$, we obtain the solution of (10):

$$\text{(10)} \quad \text{If} \quad x^3 - 3x - a = 0 \quad \text{and} \quad a = 2 \quad \text{then}$$

$$x = \sqrt{\left(2 + \sqrt{-1}\right) + \left(2 - \sqrt{-1}\right)} = \sqrt{4} = 2$$

In the case of $\frac{\pi}{3}$, only, $2a = \overline{\mathrm{OD}} = \overline{\mathrm{OA}} = \overline{\mathrm{AD}}$ becomes the equilateral triangle $\triangle$ AOD. Therefore, $\overline{\mathrm{OD}} = 2a$ holds. Actually, we have already achieved a consensus on Fig.2.

Maths are often used as magic tricks. However, mathematics must never be *tricky*. The reason for assuming that the trisection equations are impossible is that $a = 1$ is assumed. On that premise, the interpretation that *in the case of* $a = 0$*, it is* $\frac{\pi}{2}$*, so if you say* $r = 2$ *you get the solution of* $x = 0, \pm\sqrt{3}$. So that is *tricky*. Under the premise that *in the case of* $\frac{\pi}{3}$, $a = 1$ *and therefore by the theorem of right triangle,* $\overline{OD} = r = 2a = 2$ *in the case of in* $\frac{\pi}{2}$ it is said that the trisection can hold. Actually, if it is $a = 1$ in the case of $\frac{\pi}{3}$ and $a = 0$ in the case of $\frac{\pi}{2}$ then $\overline{\mathrm{OD}} = r = 2a = 2 \cdot 0 = 0$. Originally, r is a constant in $\angle$AOD, and a and $\overline{\mathrm{OH}}$ are variables in $\angle$AOD. Venture to let's assume that is $r = 2$. In the case of $\frac{\pi}{3}$, it appears to be $r = 2, a = 1$, but in the case of $\frac{\pi}{2}$ it looks like $r = 2, a = 0$ …… that's wonder ! Because when you observe the physical elements of an object has not only height and width, the depth is related and the visible range changes depending on the point of interest. In other words, if it is assumed that the X axis is always horizontal, depending on the combination of the viewpoint and the depth of field, there may be cases where the plane map overlaps the Z axis and the Y axis. It is a so-called *entanglement of ropes* in low dimensions.

The default is to assume that x is a constant that defines $x = \frac{x}{2} = \frac{r}{2} = \frac{\overline{\mathrm{OD}}}{2} = 1$. If we replace $\overline{\mathrm{OG}}$ and $\overline{\mathrm{OH}}$ with variable z and replace $x^3 - 3x = 0$ with $z^3 - 3z = 0$, then the solution is $x = 1, z = 0, \pm\sqrt{3}$. As shown on (11), not only $z = \sqrt{3}$ is suitable as one of Pythagorean theorem, but if a regular cube will draw as a wireframe model as shown in Fig.5 on the next page, if the $z = \sqrt{2}$ is the easily understood trivial diagonal, and then we can understand non-trivial diagonals of $z = \sqrt{3}$ and/or $z = 2$ is in the solid object:

$$\text{(11)} \quad \text{If } x^2 + y^2 = z^2 \text{ and } x = 1 \text{ then}$$

continue……

$$1^2+1^2=\sqrt{2^2} \qquad \therefore\ 1+1=\sqrt{2},$$
$$1^2+\sqrt{2^2}=\sqrt{3^2} \qquad \therefore\ 1+\sqrt{2}=\sqrt{3},$$
$$1^2+\sqrt{3^2}=2^2 \qquad \therefore\ 1+\sqrt{3}=2.$$

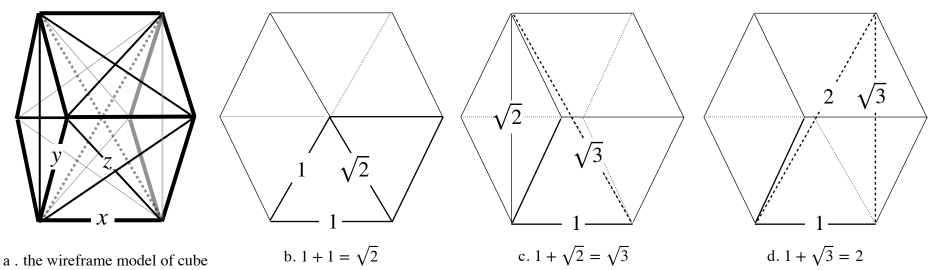

Fig.5 We will observed that there are three diagonals of different lengths by the wireframe model of cube.

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If you look at the history of mathematics from several hundred years ago to recent years, you can see that it is Pierre Wantzel who first stated in 1837 that *the trisection of the angle is impossible to draw*. And we can understand why he couldn't figure out the solution of the trisection equation. In the seventeenth century, Rene Descartes' remarks that *sense perceptions are sense deceptions* [1]. It has spread to the academy of mathematics of the era of Wantzel. A number of mathematicians influenced by Descartes all argued that it is the mathematics to think only on *Cartesian coordinates out of ignore physical figures* [1]. At that time, irrational numbers were being understood gradually. It knew that cubic equations could be solved by factoring into quadratic equations to get three solutions. Therefore, it was known that $x^3-3x=0$ would given three solutions as in (3). And Bombelli's equation $x^3=15x+4$ is also factored by $(x-4)$, and it was found that obtained three solutions as shown on (12) [S: p.50]:

$$(12) \quad x^3-15x-4=(x-4)(x^2+4x+1)$$

$$x=\frac{-4\pm\sqrt{4^2-4}}{2}=\frac{-4\pm\sqrt{12}}{2}=\frac{-4\pm\sqrt{2^2\cdot 3}}{2}=-2\pm\sqrt{3}$$
$$\therefore x=4,\ -2\pm\sqrt{3}$$

[1] Marcus du Sautoy describes *Riemann had come to dislike this denial of the physical picture* as impression by him on a young at day. [The Music of the Primes, p.69; ISBN: 97841155807]

However, complex numbers were just still recognized as only *imaginary*. Surely, it is 1811 that Friedrich Gauss represented a complex number by *Gaussian plane* in a letter to Wilhelm Bessel. Jean-Robert Argand preceded Gauss in 1806, and further back in 1797 Caspar Wessel described the same [W: Complex plane]. Without except of *geometric scholars*[2] like Gauss, since mathematicians under the influence of many Descartes have matured complex numbers using complex esoteric algebraic expressions, but it is only before a few decades. Therefore, the solution of $x^3 - 3x - 2 = 0$ looks like $\frac{-2 \pm \sqrt{0}}{2} = -1$. This only look like only add the solution of rational numbers, and just as the solution of $x^3 - 3x - 1 = 0$ is as like only irrational numbers $\left(\text{because } x = 1,\ -1 \text{ means } \pm\sqrt{1^2}\right)$. But if you understand the Extended Euclidean geometry to complex numbers by Gaussian-Riemann, you can get the *tricky* calculation result of (14) which factorizes $\sqrt{0}$ based on (13):

$$(13) \quad 0^3 - (0-1)\cdot 0 \cdot (0+1) - 0 = 0 - (-1 \cdot 0 \cdot 1) - 0 = 0, \quad x = 0, \quad d = 0$$

$$(14) \quad x^3 - 3x - 2 = (x-2)(x^2 + 2x + 1)$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4}}{2} = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2 \pm \sqrt{0}}{2}$$

$$= -1, \quad \frac{\pm\sqrt{-1^2 \cdot 0^2 \cdot 1^2}}{2}$$

$$\therefore x = 2, \quad -1, \quad \pm\frac{1}{2}i, \quad \pm\frac{1}{2}$$

Actually, this paper is the prelude for the subject. Therefore I hope the results of this calculation could be accepted by extending two basic principles of the Set theory and I shall prove them by “Invitation to the Extended Set Theory I: Contribution to the principle of the Power-set based on Binary system” and “II: Contribution to the principle of the Un-limited continuum based on the Quantum logic by Birkhoff and Von Neumann”.

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Well, let's reverse the previous statement on the assumption that you have already viewed Supplement II. If we look at Fig.8 and understand immediately, the equation that prompted to

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[2] Masahito Takase describes *the word “geometrics” is often found in the introduction of D.A., but this is not a word meaning "person who studies geometry” but is synonymous with “mathematician”* [Gauss's number theory, p.62: ISBN 978-4-480-09366-0 C0141]. However I think that Gauss divided it into *the mathematician influenced by Cartesian* and *the mathematician whose starting point is geometry*.

change $z^3 - 3z = 0$ that should be $x^3 - dy - z = 0$, and the solution of $x = 0, \pm\sqrt{3}$ is the correct. This is because the Z axis not only overlaps the Y axis when $x = 0$ but also overlaps the X axis when $y = 0$. $\frac{\pi}{3}$ *can not be divided into three equal parts, but another has angles that can be divided into three parts* should be never absent on mathematics. If the one correct answer is obtained, then the answer will must be able to make *the road of the tautology*, isn't it? If we can not do that, we can never say that math is the most beautiful and elegant communication tool. In fact, when the regular expression is $x^3 - dy - z = 0$, there will be in the case of $x^3 - 3x - 0 = 0$ and $y^3 - 3y - 0 = 0$ then their solutions follow not only $x = 0, \pm\sqrt{3}$ or $y = 0, \pm\sqrt{3}$, but also the all of square roots follow Pythagorean theorem, as shown below:

$$(15) \quad \begin{array}{lll} 0^2 + \sqrt{3^2} = \sqrt{3^2} & \therefore\ 0 + \sqrt{3} = \sqrt{3} & \longrightarrow\ \forall_n\left(0 + \sqrt{n} = \sqrt{n}\right) \\ \sqrt{3^2} + 0^2 = \sqrt{3^2} & \therefore\ \sqrt{3} + 0 = \sqrt{3} & \longrightarrow\ \forall_n\left(\sqrt{n} + 0 = \sqrt{n}\right) \end{array}$$

Supplement I:
How to trisected $0 < \theta \leq \pi$ with ruler-and-compasses

The method of trisection $0 < \theta \leq \pi$ by using ruler-and-compass utilizes the property of the following isosceles triangle.

1. The equilateral triangle, which is well known to be capable of drawing with ruler-and-compass, is an isosceles triangle and already constitutes a trisection of π. Thus, all corners of an isosceles triangle are inscribed in a circle.
2. If the line segment passing through the orthocenter is the symmetric axis, constructs two right triangles that are *reflective symmetry*.
3. When the symmetric axis is extended to a semicircle, the points of intersection constitutes two isosceles triangles whose apex angles are $\frac{1}{2}$ and *reflective symmetry*.

1. Procedure to divide $0 < \angle A < \pi$ into three equal parts

The drawing procedure from 1 to 12 is shown in Fig.6 on page 8.

1. Determine an arbitrary point A, subtract two half lines $\overline{\mathrm{B}}$ and $\overline{\mathrm{C}}$ starting from A, and determine an arbitrary a small $\angle$A than $\dfrac{\pi}{2}$.
2. Draw an arbitrary arc centered on A, name the point of intersection with $\overline{\mathrm{B}}$ as B, and the point of intersection with $\overline{\mathrm{C}}$ as C. Draw a straight line passing through B, C and call it $\bar{x}$.
3. Draw an arbitrary arc whose radius is larger than $\dfrac{\overline{\mathrm{BC}}}{2}$ centering on B, C, and draw a half line $\bar{y}$ that intersects $\bar{x}$ at a right angle with A as the starting point. We denote the intersection of $\bar{x}$ and $\bar{y}$ as O.
4. We call $\overline{\mathrm{OA}}$ as r, draw a perfect circle $\bigcirc\alpha$ of radius r centered on O, and name the point of intersection with $\bar{y}$ as Q.
5. Draw an arc $\frown\beta_1$ of radius r centered on A, draw a straight line passing through the left and right intersections with $\bigcirc\alpha$, and name the intersection with $\bar{y}$ as P.
6. Draw an arc $\frown\beta_2$ of radius r centered on Q, draw a straight line passing through the left and right intersection with $\bigcirc\alpha$, and name the intersection with $\bar{y}$ as Q.
7. Draw an arc $\frown\omega$ whose radius is $2r$ centered on A, and name the point of intersection with $\overline{\mathrm{B}}$ as B'.
8. Draw a straight line passing Q, B' and name the point of intersection with $\bigcirc\alpha$ as D (D is a half straight line passing through the vertical center of $\triangle$ AB'Q starting with A that is also the point at which the bottom intersects).
9. Draw an arc $\frown\gamma_1$ whose radius is $\dfrac{3r}{2}\left(=\overline{\mathrm{PQ}}\right)$ centering on P. Starting from A draw a half line passing to $\frown\gamma_1$ via D name the point of intersection with $\frown\gamma_1$ as R.

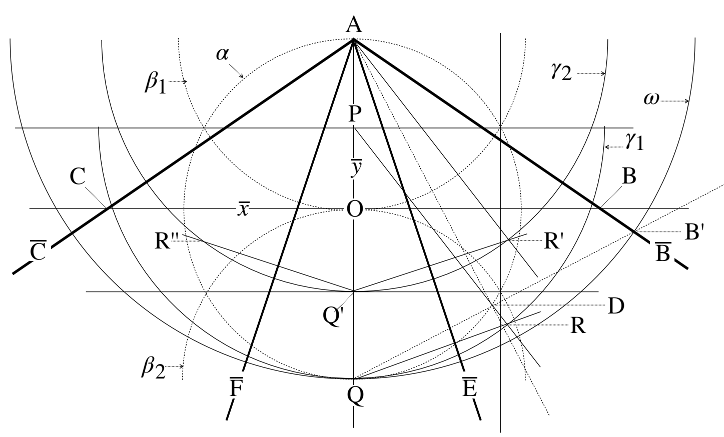

Fig.6 Trisection of any angle

10. Draw an arc $\frown\gamma_2$ whose radius is $\frac{3r}{2}$ $\left(=\overline{\mathrm{AQ'}}\right)$ centering on A. Take the distance of Q, R by the compass. At centering on Q' the distance of Q, R is moved onto $\frown\gamma_2$ to mark the intersection R', R''. $\triangle$ PQR, $\triangle$ AR'Q' and $\triangle$ AQ'R'' are congruent isosceles triangles.
11. Draw an arbitrary arc outside of $\frown\gamma_2$ centering on Q', R'. The half line passing the intersection of the two arcs starting from A is named $\overline{\mathrm{E}}$.
12. Draw an arbitrary arc outside of $\frown\gamma_2$ centering on Q', R''. The half line passing the intersection of the two arcs starting from A is named $\overline{\mathrm{F}}$.

    $\overline{\mathrm{E}}$, $\overline{\mathrm{F}}$ has been dividing $\angle$A into three equal angles.

2. The proof using vector

The proof model is $\frac{\pi}{3}$, which is established to be impossible. This proof uses a drawing procedure that draws a triangle and two hexagons that fit within a perfect circle. The names of the positions used are shown in Fig.7 of page 10, but the way of drawing the auxiliary lines is well known and therefore abbreviated.

2.1. Preparation of the proof model

1. Draw a perfect circle $\bigcirc\alpha$ of radius r centered on O. The left and right intersections of the horizontal line passing through O are called R, S, and the upper and lower intersections of the vertical line are called P, Q.
2. Draw an arc $\frown\beta_1$, $\frown\beta_2$, $\frown\beta_3$, $\frown\beta_4$ of radius r with R, S, P, Q as each center.
3. Draw a straight line passing through the intersection points of $\bigcirc\alpha$ and $\frown\beta_1$, and name the point of $\frac{r}{2}$ on $\overline{\mathrm{PQ}}$ as A.
4. The intersections of $\bigcirc\alpha$ and $\frown\beta_2$ are named R', S' and connected by a straight line, and the intersection of $\frac{r}{2}$ on $\overline{\mathrm{OQ}}$ is named Q'.
5. Let the lower intersection of $\bigcirc\alpha$ and $\frown\beta_3$ be called B.
6. Let the lower intersection of $\bigcirc\alpha$ and $\frown\beta_4$ be called C.
7. Draw an arc $\frown\omega$ of radius $2r$ centered on P, and name the point of intersection with the extension line of $\overline{\mathrm{PR'}}$ as R''.
8. Draw an arc $\frown\gamma$ of radius $3\frac{r}{2}$ centered on A, and name the point of intersection with the extension of $\overline{\mathrm{PB}}$ as B'

2.2. The proof

1. $\triangle$ PR'S' is an equilateral triangle with all angles inscribed in $\bigcirc\alpha$ being $\frac{\pi}{3}$. Therefore $\triangle$ PQ'R' and $\triangle$ PQ'S' constitute a *reflective symmetry* whose axis of symmetry is $\overline{\text{PQ'}}$ passing through the vertical orthocenter O. So that these two are congruent right triangles and $\angle$A (Q'R') is $\frac{\pi}{6}$.
2. $\triangle$ PQR'' is an isosceles triangle because the apex angle $\angle$P (QR'') is the center of $\frown\omega$ and the base is both an intersection with $\frown\omega$. Therefore if the half $\angle$P (QB) is $\frac{\pi}{12}$ and if $\overline{\text{PB}}$ is extended to get the intersection with $\frown\omega$ be B'' then $\angle$P (QB'') is an isosceles triangle.
3. $\triangle$ OQB is an isosceles triangle because the apex angle is the center of $\bigcirc\alpha$ and the base is the intersection with $\bigcirc\alpha$. $\angle$O (QB) is $\frac{\pi}{6}$ because $\triangle$ OQB is a similar form of $\frac{1}{4}$ of $\triangle$ PQR''.
4. $\triangle$ AQB' is an isosceles triangle because the apex angle $\angle$A is the center of $\frown\gamma$ and the base $\overline{\text{OB'}}$ is an intersection with $\frown\gamma$.
5. Now assuming that there is a vector starting from P, A, O and all ends to the same Q, let the vector length of $\overrightarrow{\text{PQ}}$ as named $\mathfrak{a}$, $\overrightarrow{\text{AQ}}$ as named $\mathfrak{x}$, and $\overrightarrow{\text{OQ}}$ as named $\mathfrak{b}$.
   And if other vectors start from P, A, O and ends at all the same distance to Q, then $\triangle$ PQB'', $\triangle$ AQB', $\triangle$ OQB are isosceles triangles. For this it is possible to assume that the vector length and the angle is the one-to-one correspondence according to the radian principle. Therefore since $\mathfrak{a}$ corresponds to $\frac{\pi}{12}$ the division number corresponds to $a = 12$ and $\mathfrak{b}$ corresponds to $\frac{\pi}{6}$ so $b = 6$. Since the number of

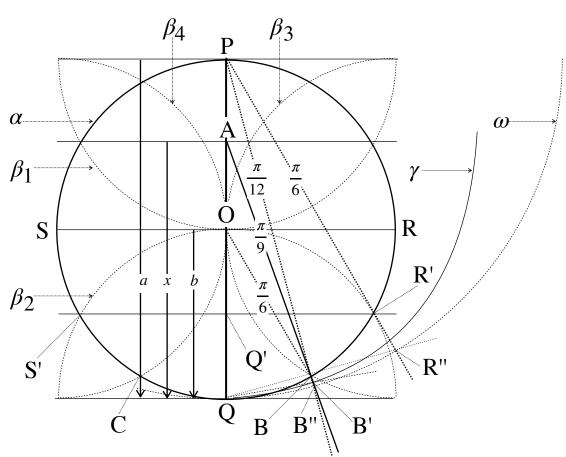

Fig.7 Proof model of trisection of angles

divisions corresponding to $\mathfrak{x}$ is unknown, we use x. However although the division number of $\mathfrak{x}$ is unknown, the vector length of $\mathfrak{a}$, $\mathfrak{x}$, $\mathfrak{b}$ can be measured.

6. Assuming that the longest vector length $\mathfrak{a}$ is 1, $\mathfrak{b}$ is half of it and $\mathfrak{x}$ is $\frac{3}{4}$. Therefore the basic vector length is $l = \frac{\overrightarrow{PA} + \overrightarrow{AO}}{2} = \frac{\overrightarrow{PQ} - \overrightarrow{OQ}}{2} = 1$, the vector length of $\mathfrak{x}$ is $d = 3l$, and it can be obtained by the following simple calculation that we get $\angle A$ is $\frac{\pi}{9}$:

$$\angle A = \frac{\pi}{x} = \frac{\pi}{\frac{a-b}{2} \times d} \quad \therefore \frac{\pi}{\frac{12-6}{2} \times 3} = \frac{\pi}{9}$$

Q.E.D.

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What means of this supplement which even if $\angle A$ is a category generally called transcendental number, the trisection of all the corners can be constructed with ruler-and-compass. Even if the reflex angle, trisection can be easily made by combining the straight angle with the acute angle or the obtuse angle.

Supplement II:
The basis which can interpret $x^3 - 3x = 0$ as $z^3 - 3z = 0$

We stated that $x^3 - 3x - a = 0$ is a cubic equation of $x^3 - dx - a = 0$, and that the solution is obtained only in the case of $x = a$. Therefore if $a = 0$ then the rational solution of $x^3 - 3x = 0$ is $x = 0$ because it is $0^3 - 3 \cdot 0 - 0 = 0$. However it is at first glance tricky to say that $x = 0$. If we allow $x = 0$, then d allows any square root solution to be $x = 0, \pm\sqrt{d}$. That is $x^3 - 3x = 0$ will be not an equation under the condition of $x = a$ and $a = 0$ of $x^3 - dx - a = 0$. It will be an equation under the condition of $x \neq a$ and $a = 0$. Therefore under the condition of $x = 0$ and $x \neq z$, we think that if it is rewritten to $x^3 - dy - z = 0$ then both a linear equation $x - dy = 0$ and a quadratic equation $x^2 - dy = 0$ will hold without contradiction. Because z is the third algebra that first appears under cubic equations. Therefore if x is the default unit circle, then under the condition $x = 1$ and $x \neq z \rightarrow z = 0$, $z^3 - 3z = 0$ will be $z = 0, \pm\sqrt{3}$.

On the other hand, making y a line segment $\overline{\mathrm{BH}}$ is correct in the range of quadratic equations. This is because in a quadratic equation in which the range of numerical values in x is $-\omega \leq x \leq \omega$, y must be a variable in the range $-x \leq y \leq x$. If the X axis is a horizontal line, then $y = \pm\,\omega$ for $x = 0$, and the Y axis intersects the O perpendicularly to the X axis at the intersection point.

In the cubic equation, $z = \mathrm{O}\frown\mathrm{AB}$. The Pythagorean theorem is based on isosceles triangles, but can be extended to the cubic equation theorem. The Thales theorem is a theorem of staying in quadratic equations based on right triangles. In the quadratic equation, the line segment $\overline{\mathrm{OH}}$ of $\triangle\mathrm{OBH}$ according to the right triangle theorem constitutes *reflective symmetry* which does not appear in Fig.1 as the symmetry axis of the isosceles triangle. Thales theorem can be applied to cubic equations because quadratic equations inherit linear equations and cubic equations inherit quadratic equations. If all higher equations do not contain lower equations among all the equations, the logic is inconsistent. Therefore $y = \overline{\mathrm{BH}}$ is not a mistake if taken up in the range of the quadratic equation. If $z = \mathrm{O}\frown\mathrm{AB}$, then if $2z = \mathrm{O}\frown\mathrm{AC}$ is the intersection of $\overline{\mathrm{AC}}$ and $\overline{\mathrm{OB}}$, then $\frac{\mathrm{AH}}{2} \cong \overline{\mathrm{AH'}} + \overline{\mathrm{H'C}}$ and $\overline{\mathrm{AH'}} \cong \overline{\mathrm{BH}}$, so that it can be interpreted as $y \approx \overline{\mathrm{BH}}$ assuming that it is y that opposes x in the range of the quadratic equation in which z does not appear yet.

However if we consider $\overline{\mathrm{OH}}$, which is one of the neighboring sides that make up a right triangle as the middle line of an isosceles triangle, as x, the cubic equation does not hold. Repeatedly, x is the radius for drawing the definition domain $[x]$, as Gauss proves it in both quadratic and cubic equations, and also is predicting in multidimensional equations[3]. Therefore if $\overline{\mathrm{OA}}$ is the diameter, the middle point is G, and the radius is $\overline{\mathrm{OG}}$ and $\overline{\mathrm{GA}}$, that is, $\overline{\mathrm{OA}} = 2\,|\,x\,|$. This indicates that O *and* A *are in a one-to-one relationship that is the reflective symmetry with* G *as the symmetry point*.

Now give A a second name ω. To the intersection point, of the arc $\frown z$ with radius $\overline{\mathrm{OA}}\ (= 2x)$ and the Y axis, give two names O' and ωi. The intersection point G of the vertical line drawn down from D and the X axis changes to the two names d and z_0. The intersection between the vertical line drawn from C and the X axis is named c, and the intersection H between the vertical line drawn from B and the X axis is changed b. Starting

[3] It's my opinion that Gauss's world is the Extended Euclidean geometry, which is not equivalent to what is called Non-Euclidean geometry.

from D, draw a half line parallel to the X axis, and call the intersection with the Y axis as di. Similarly, the intersection point of a parallel half line starting at C with the Yaxis is ci, and the intersection point of a parallel half line starting at B with the Y axis is bi. Name it as the intersection of $\overline{\mathrm{OB}}$ and $\overline{bib}$ is called z_1, the intersection of $\overline{\mathrm{OC}}$ and $\overline{cic}$ is called z_2, and the intersection of $\overline{\mathrm{OD}}$ and $\overline{did}$ is z_3. Name the middle point of $\overline{\mathrm{OO'}}$ as z_4. At this time, $\overline{\mathrm{OA}} = \overline{\mathrm{OB}} = \overline{\mathrm{OC}} = \overline{\mathrm{OD}} = \overline{\mathrm{OO'}} = \overline{bib} = \overline{cic} = \overline{did}$ have the center z_0, z_1, z_2, z_3, z_4, that will draw a perfect circle $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ by $r = x = 1$. In addition, $\overline{\mathrm{OB}}$ and $\overline{bib}$, $\overline{\mathrm{OC}}$ and $\overline{cic}$, $\overline{\mathrm{OD}}$ and $\overline{did}$ have z_1, z_2, z_3 as symmetry points, respectively $\triangle z_1\mathrm{O}b$ and $\triangledown z_1\mathrm{B}bi$, $\triangleright z_1bi\mathrm{O}$ and $\triangleleft z_1b\mathrm{B}$, $\triangle z_2\mathrm{O}c$ and $\triangledown z_2\mathrm{C}ci$, $\triangleright z_2ci\mathrm{O}$ and $\triangleleft z_2c\mathrm{C}$, $\triangle z_3\mathrm{O}d$ and $\triangledown z_3\mathrm{D}di$, $\triangleright z_3di\mathrm{O}$ and $\triangleleft z_3d\mathrm{D}$ form isosceles triangles that are *reflective symmetries*.

If z_0, z_1, z_2, z_3, z_4 are regarded as the starting points of vectors, all the directed segments that make up point symmetry with them as the target point correspond one to one. As shown in Fig.8, we can easily understand if it is superimposed on a quarter of a circle that is, a half of *Gaussian plane* that represents a cross section of an eighth of a sphere. Here, if a perfect circle whose unit circle is $r = 2x = 1$ is the object of discussion, one of the vectors that is line symmetries of z_0, z_1, z_2, z_3, z_4 of symmetric points is $\dfrac{r}{2}$.

Therefore the starting points of all vectors can be regarded as the zero points that make up a circle of complex numbers and if we extend the Gaussian plane to the Cartesian coordinate system, the zero points $z_0, z_1, z_2, z_3, z_4, \ldots\ldots$ will be constructed the perfect circle centered on O with $r = \dfrac{1}{2}$.

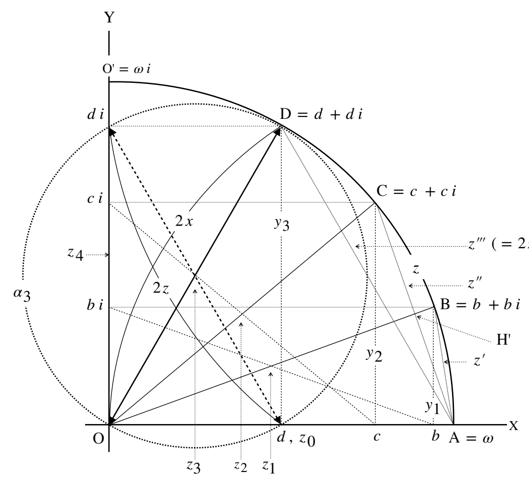

Fig.8 $x^3 - 3y - z = 0$ on the Gaussian plane

References

S: Shimeno, Nobukaz; What is a complex number, ISBN 978-4-06-257788-5

Y: Yano, Kentaro; Trisection of the angle, ISBN 4-480-09003-7 C01414

and thanks to Wikipedia.