Espil short proof of generalized Cauchy's residue theorem

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Abstract: shortly we can derive the Cauchy's residue theorem (its general form) just by direct integration of a Taylor series “without” making any radius go to zero, even without the limit circumference idea take place.

H) Let D be a simply connected open subset of the complex plane, where \( z = a \in D \), enclosed by a rectifiable positively oriented simple curve \((C^+)\) in D, and \( f \) a function defined and holomorphic on D

\[
\oint_{C^+} \frac{f(z)}{(z-a)^n} \, dz = \lim_{z \to a} \frac{2\pi i}{(n-1)!} \frac{d^{n-1}f(z)}{dz^{n-1}}.
\]

D) Being \( f \) holomorphic on D, its infinitely differentiable and equal to its own Taylor series at \( z=a \) and in the neighborhood.

\[
f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f(a)}{dz^k} (z-a)^k
\]

\[
\oint_{C^+} \frac{f(z)}{(z-a)^n} \, dz = \oint_{C^+} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f(a)}{dz^k} (z-a)^k \frac{1}{(z-a)^n} \, dz = \oint_{C^+} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f(a)}{dz^k} (z-a)^{k-n} \, dz
\]

\[
\oint_{C^+} \left\{ \sum_{k=0}^{n-2} \frac{1}{k!} \frac{d^k f(a)}{dz^k} (z-a)^{k-n} + \frac{1}{(n-1)!} \frac{d^{n-1} f(a)}{dz^{n-1}} \frac{1}{(z-a)} + \sum_{k=n}^{\infty} \frac{1}{k!} \frac{d^k f(a)}{dz^k} (z-a)^{k-n} \right\} \, dz
\]

Let \( z_0 = a + \rho_0 e^{i\theta_0} \in \partial D \), being \( \theta_0 = \arg (z-a) \) when travelling counterclockwise over \( \partial D \) around \( z = a \), being the start point: \( z_0 \) and the end point: \( z_1 = a + \rho_0 e^{i(\theta_0+2\pi)} \in \partial D \), then integrating ...
\[
\sum_{k=0}^{n-2} \frac{1}{k!} \frac{d^k f(a)}{dz^k} \frac{(z-a)^{k-n+1}}{k-n+1} + \frac{1}{(n-1)!} \frac{d^{n-1} f(a)}{dz^{n-1}} \ln(z-a) + \sum_{k=n}^{\infty} \frac{1}{k!} \frac{d^k f(a) (z-a)^{k-n+1}}{dz^k (k-n+1)z_0^{k-n+1}}
\]

Both lateral sums are canceled, remaining the middle term

\[
\frac{1}{(n-1)!} \frac{d^{n-1} f(a)}{dz^{n-1}} \ln\left(\frac{z_1-a}{z_0-a}\right) = \frac{1}{(n-1)!} \frac{d^{n-1} f(a)}{dz^{n-1}} \ln\left(\frac{\rho_0 e^{i(\theta_0+2\pi)}}{\rho_0 e^{i\theta_0}}\right) = \frac{1}{(n-1)!} \frac{d^{n-1} f(a)}{dz^{n-1}} \ln(e^{2\pi i})
\]

\[
= \frac{2\pi i}{(n-1)!} \frac{d^{n-1} f(a)}{dz^{n-1}}
\]

thus

\[
\oint_{C^+} \frac{f(z)}{(z-a)^n} dz = \lim_{z \to a} \frac{2\pi i}{z-a-n!} \frac{d^{n-1} f(z)}{dz^{n-1}}
\]