A characterization of the golden arbelos involving an Archimedean circle

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Abstract. We consider a problem in Wasan geometry involving a golden arbelos and give a characterization of the golden arbelos involving an Archimedean circle. We also construct a self-similar circle configuration using the figure of the problem.

Keywords. arbelos, golden arbelos, Archimedean circle.

Mathematics Subject Classification (2010). 01A27, 51M04.

1. Introduction

We consider the arbelos appeared in Wasan geometry, and consider an arbelos formed by three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB (see Figure 1). We denote the arbelos and the radii of α and β by (α, β, γ) and a and b, respectively, and call the perpendicular to AB at O the axis. Circles of radius $r_{\rm A} = ab/(a+b)$ are said to be Archimedean, and the incircle of the curvilinear triangle made by α, γ and the axis is Archimedean, which is denoted by δ. Let σ be the reflection in the perpendicular bisector of AB. We consider the following problem in [11] (see Figure 2).

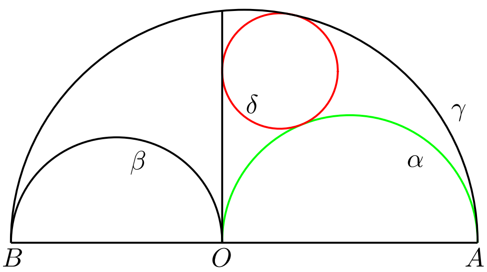

Figure 1: (α, β, γ) and the circle δ.

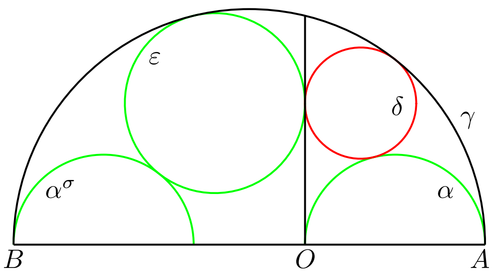

Figure 2.

Problem 1. Let ε be the circle touching α^σ externally γ internally and the axis from the side opposite to A. If ε and α have the same radius, find the radius of ε in terms of the difference of the radii of γ and δ.

The same sangaku problem proposed in 1891 [1]. If $a/b = \phi^{\pm 1}$, then (α, β, γ) is called a golden arbelos, where $\phi = (1+\sqrt{5})/2$. We will show that the figure of the problem forms a golden arbelos and the circles δ and ε touch. We will also give a condition in which the circles δ and ε touch in the case $a \neq b$, and give a characterization of the golden arbelos involving an Archimedean circle touching the axis at the point O and construct a self-similar circle configuration.

2. Circles touching a perpendicular to AB at the same point

We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a). We use the next proposition.

Proposition 1. *It two externally touching circles of radii r_1 and r_2 touch a line at two points P and Q, then $|PQ| = 2\sqrt{r_1 r_2}$.*

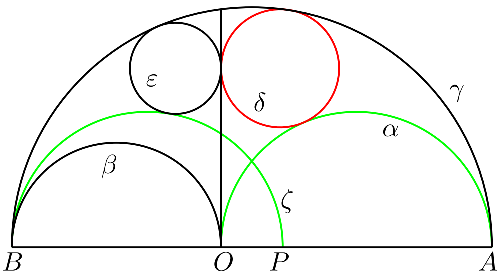

Figure 3.

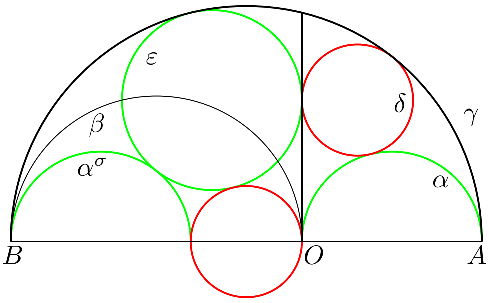

Figure 4.

Theorem 1. *Let ζ be the semicircle of diameter BP constructed on the same side of AB as γ for a point P on the segment AB, and let ε be the circle touching γ internally, ζ externally and the axis from the side opposite to A. The following statements are equivalent.*
(i) *The circles δ and ε touch.*
(ii) *The circle ε has radius $b - r_{\mathrm{A}}$.*
(iii) *The semicircle ζ coincides with α^σ.*

Proof. Let e and z be the radii of ε and ζ, respectively, and let y_e be the y-coordinate of the center of ε (see Figure 3). Then we have $(a+b-e)^2 = (-e-(a-b))^2 + y_e^2$ and $(z+e)^2 = (-e-(-2b+z))^2 + y_e^2$. Solving the equations for e and z, respectively, we get

(1) $$e = b - \frac{y_e^2}{4a}$$

and

(2) $$z = b - e + \frac{y_e^2}{4b}.$$

While (i) is equivalent to $y_e = 2\sqrt{ar_{\mathrm{A}}}$ by Proposition 1. Therefore (1) implies that $y_e = 2\sqrt{ar_{\mathrm{A}}}$ is equivalent to $e = b - r_{\mathrm{A}}$, i.e., (i) and (ii) are equivalent. Substituting (1) in (2), we get

(3) $$y_e^2 = 4zr_{\mathrm{A}}.$$

The equation gives that $y_e = 2\sqrt{ar_{\mathrm{A}}}$ if and only if $z = a$, i.e., (i) and (iii) are equivalent. □

We now consider the figure of Problem 1 and assume that ε and ζ have radius a in Theorem 1 (see Figure 4). Then by the equivalence of (ii) and (iii), we have

(4) $$a + r_{\mathrm{A}} = b.$$

Then $2a = a + b - r_{\mathrm{A}} =$, i.e., $a = (a + b - r_{\mathrm{A}})/2$, which is an answer of Problem 1. On the other hand (4) is equivalent to $b = \phi a$. Therefore (α, β, γ) is a golden arbelos, and r_{A}, a, b, c form a geometric progression with common ratio ϕ. Also

(4) implies that there is an Archimedean circle concentric to γ touching the axis and the circles α, α^σ and ε externally.

The next theorem shows that the Archimedean circle touching the circle ε externally and the axis at the point O can also be obtained in the case $b \neq \phi a$, and gives a characterization of the golden arbelos using the Archimedean circle touching the axis at O.

Theorem 2. *Let ζ and ε be the semicircle and the circle as in Theorem 1, and let η be the circle touching ε externally and the axis at O from the side opposite to A. Then η is Archimedean if and only if ζ and ε have the same radius. In this event, (α, β, γ) is a golden arbelos with $b = \phi a$ if and only if ζ and η touch.*

Proof. We use the same notations as in the proof of Theorem 1. The radius of the circle η equals $y_e^2/(4e) = (z/e)r_{\mathrm{A}}$ by Proposition 1 and (3). Therefore η is Archimedean if and only if $z = e$ (see Figure 5). We now assume $z = e$. The semicircle ζ and the circle η touch if and only if $z + r_{\mathrm{A}} = b$. The last equation is equivalent to Theorem 1(ii), which is equivalent to $z = a$ by the equivalence of (ii) and (iii) in the same theorem. Therefore ζ and η touch if and only if (4) holds, which is equivalent to $b = \phi a$. □

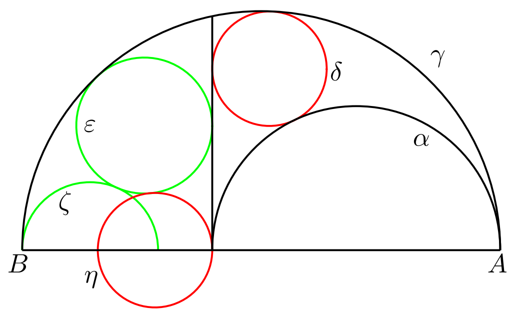

Figure 5.

We have considered two circles touching a perpendicular to AB from the opposite side at the same point in a general way in [5]. Theorem 1 gives a special case in which we get such a pair of circles. Another condition using the reflection in the axis can also be found in [6].

3. Application of division by zero

We consider the relations (1), (2) with the recent definition of division by zero: $z/0 = 0$ for any real number z [3].

We consider (1). Notice that this relation is derived only from the assumption that the circle ε touches γ internally and the axis from the side opposite to A. If $a = 0$, then the semicircle α degenerates to the point A, β and γ coincide, and $y_e^2/(4a) = y_e^2/0 = 0$ by the definition of division by zero. Hence (1) implies $e = b$. Therefore the half part of the circle ε coincides with γ (see Figure 6).

We consider (2). If $b = 0$, then β and ε degenerate to the point B, i.e., $e = z = 0$, and $y_e^2/(4b) = 0$. Therefore (2) still holds (see Figure 6).

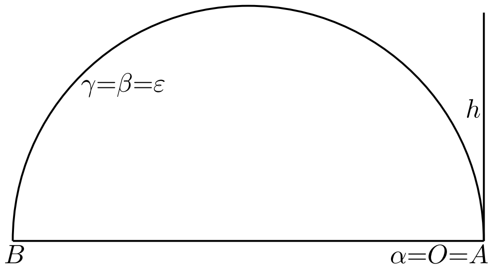

Figure 6: $a = 0$.

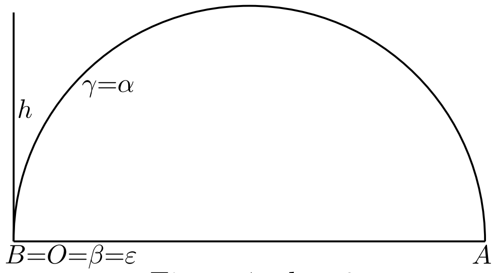

Figure 7: $b = 0$.

For more applications of division by zero and division by zero calculus to Wasan geometry see [2], [4], [7, 8], [9, 10].

4. A self-similar circle configuration arising from the golden arbelos

We construct a self-similar circle configuration using the figure in Problem 1. Let τ be the product of σ and the homothety of center A and ratio ϕ^{-1}. Let p be the x-coordinate of a point P on AB. Then we have $(p + p^\sigma)/2 = a - b$ and $(p^\sigma - 2a)/\phi = p^\tau - 2a$, where p^σ and p^τ are the x-coordinates of the points P^σ and P^τ, respectively. Then $p^\tau = 2a + (p^\sigma - 2a)/\phi = 2a + (-2b - p)/\phi = -p/\phi$. Therefore τ coincides with the homothety of center O with ratio $-1/\phi$. Hence $p^{\tau^n} = (-1)^n p/\phi^n$, i.e., P^{τ^n} has x-coordinate $(-1)^n p/\phi^n$, and the axis is fixed by τ. Notice that γ^τ passes through the point of tangency of δ and ε by Proposition 1, because $(2\sqrt{ar_\mathrm{A}})^2 = 2a \cdot 2\phi a = |B^\tau O||A^\tau O|$ (see Figure 8).

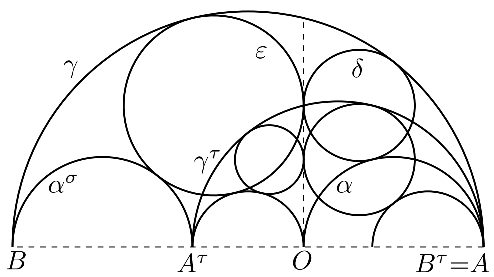

Figure 8: $\mathcal{K} \cup \mathcal{K}^\tau = \mathcal{K}_1 \cup \mathcal{K}_2$.

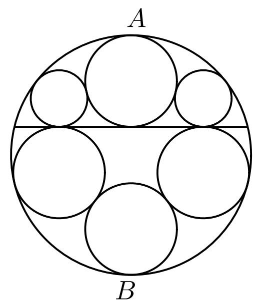

Figure 9.

Let $\mathcal{K}$ be the figure consisting of γ, α, α^σ, δ and ε in the case $b = \phi a$, which is obtained from Figure 2 by removing AB and the axis. Let $\mathcal{K}_i = \mathcal{K}^{\tau^{i-1}}$ for $i = 1, 2, 3, \cdots$, and $\mathcal{K}_0 = \bigcup_{i \geq 1} \mathcal{K}_i$. It is a custom of Wasan geometry to describe the arbelos by three circles so that their centers lie on a vertical line. The original figure of Problem 1 is also described by $\mathcal{K}$ with the axis and its reflection in AB so that AB is a vertical segment as in Figure 9. Following to this custom, we also describe $\mathcal{K}_0$ so that AB is a vertical line with its reflection in AB (see Figure 10).

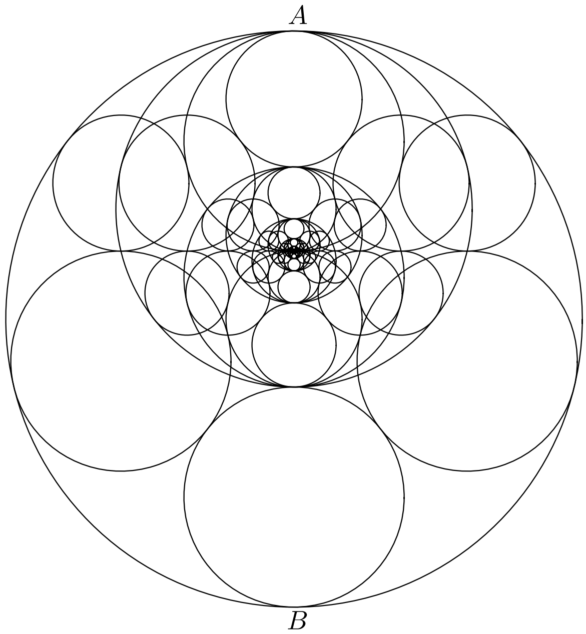

Figure 10: $\mathcal{K}_0$ with it reflection in AB.

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