# An Exact Formula for the Prime Counting Function

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**Abstract**

This article discusses a few main topics in Number Theory, such as the Möbius function and its generalization, leading up to the derivation of a neat power series for the prime counting function, $\pi(x)$. Among its main findings, we can cite the inversion theorem for Dirichlet series (given $F_a(s)$, we can tell what its associated function, $a(n)$, is), which enabled the creation of a formula for $\pi(x)$ in the first place, and the realization that sums of divisors and the Möbius function are particular cases of a more general concept. Another conclusion we draw is that it's unnecessary to resort to the zeros of the analytic continuation of the zeta function to obtain $\pi(x)$.

# Summary

# 1 Introduction

Many people have devoted time trying to create formulae to generate prime numbers or to count primes numbers, something that at times has bordered on obsession.

These formulae don't seem to have a lot of potential to be used as new tools for analyses, so frequently they are mere curiosities or attempts to prove oneself capable of achieving any goal or winning an intellectual challenge. That is even more true if the formula is very complicated, which seems to be the case of most that were discovered to date.

But oblivious to the bleak landscape, I was still very curious and eager to find my own, and in the process, I think I may have created a new tool to look at Dirichlet series.

In this paper we create the very first power series for the prime counting function, $\pi(x)$, aside from the Riemann prime counting function (which assumes the Riemann Hypothesis, which is still unproven). Though the principle behind it is somewhat trivial, its relative simplicity stems from sheer luck, the fact that some convoluted functions can be expressed as the product of an elementary function and a non-elementary function given by a relatively simple power series.

Though power series are not the most elegant of solutions, at times they can provide insights that may lead to the discovery of better or more useful formulae.

After I discovered the results that are discussed here, I did some research in the literature and was very surprised about coincidences between things I found and approaches that had been tried by others before me. Namely, concepts such as the Möbius function, $\mu(n)$, the Von Mangoldt lambda function, $\Lambda(n)$, and so on and so forth. Maybe these are recurring concepts on the study of the patterns of prime numbers.

# 2 Indicator Function $k$ divides $n$, $\mathbb{1}_{k|n}$

In paper [2] we introduced the indicator function $k$ divides $n$, noted $\mathbb{1}_{k|n}$ and defined as 1 if $k$ divides $n$ and 0 otherwise. This function plays a key role throughout this paper. It can be be represented by means of elementary functions:

$$\mathbb{1}_{k|n} = \frac{1}{k}\sum_{j=1}^{k}\cos\frac{2\pi nj}{k} = \frac{\cos 2\pi n - 1}{2k} + \frac{1}{2k}\sin 2\pi n \cot\frac{\pi n}{k}$$

However, that closed-form is not very practical to work with. For example, if $k$ divides $n$ we have an undefined product of the type $0 \cdot \infty$ (the sine is 0 and the cotangent is $\infty$).

Hence, it's more practical to derive a power series for $\mathbb{1}_{k|n}$, which can be accomplished by expanding the cosine on the left-hand side with Taylor series and employing Faulhaber's formula, as explained in [2], which gives us the below:

$$\frac{1}{2k}\sin 2\pi n \cot\frac{\pi n}{k} = \sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{B_{2j}k^{-2j}}{(2j)!(2i+1-2j)!} \tag{1}$$

## 2.1 Analog of the $\mathbb{1}_{k|n}$ Function

The analog of the indicator function $\mathbb{1}_{k|n}$ is the below sum:

$$\frac{1}{k}\sum_{j=1}^{k}\sin\frac{2\pi nj}{k} = \frac{\sin 2\pi n}{2k} + \frac{1}{k}\cot\frac{\pi n}{k}\sin^2 \pi n$$

As previously, we can obtain a power series for it by expanding the sine with Taylor series and making use of Faulhaber's formula:

$$\frac{1}{2k}\cot\frac{\pi n}{k}(1-\cos 2\pi n) = \sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i+1}\sum_{j=0}^{i}\frac{B_{2j}k^{-2j}}{(2j)!(2i+2-2j)!}$$

## 2.2 Transforming the Power Series of $\mathbb{1}_{k|n}$

This section is not crucial for the comprehension of the key results of this paper and can be skipped. If we remove the first term from the sum in (1), we get the below:

$$\mathbb{1}_{k|n} = \frac{\cos 2\pi n - 1}{2k} + \frac{\sin 2\pi n}{2\pi n} - \sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i+2}\sum_{j=0}^{i}\frac{B_{2j+2}k^{-2j-2}}{(2j+2)!(2i+1-2j)!}$$

If we want to extract one more term from the sum, we invert its indexes and extract the term where $j = 0$ on the right-hand side as illustrated below:

$$\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i+2}\sum_{j=0}^{i}\frac{B_{2j+2}k^{-2j-2}}{(2j+2)!(2i+1-2j)!} = \sum_{j=0}^{\infty}\frac{B_{2j+2}k^{-2j-2}}{(2j+2)!}\sum_{i=j}^{\infty}\frac{(-1)^i(2\pi n)^{2i+2}}{(2i+1-2j)!}$$

$$\Rightarrow \frac{B_2k^{-2}}{2!}\sum_{i=0}^{\infty}\frac{(-1)^i(2\pi n)^{2i+2}}{(2i+1)!} + \sum_{j=1}^{\infty}\frac{B_{2j+2}k^{-2j-2}}{(2j+2)!}\sum_{i=j}^{\infty}\frac{(-1)^i(2\pi n)^{2i+2}}{(2i+1-2j)!}$$

$$\Rightarrow \frac{(2\pi n)^2 B_2}{2k^2}\sum_{i=0}^{\infty}\frac{(-1)^i(2\pi n)^{2i}}{(2i+1)!}+\sum_{i=1}^{\infty}(-1)^i(2\pi n)^{2i+2}\sum_{j=1}^{i}\frac{B_{2j+2}k^{-2j-2}}{(2j+2)!(2i+1-2j)!}$$

Therefore we end up with:

$$\mathbb{1}_{k|n}=\frac{\cos 2\pi n-1}{2k}+\frac{\sin 2\pi n}{2\pi n}\left(1-\frac{(2\pi n)^2 B_2}{2k^2}\right)+\sum_{i=0}^{\infty}(-1)^{i+2}(2\pi n)^{2i+4}\sum_{j=0}^{i}\frac{B_{2j+4}k^{-2j-4}}{(2j+4)!(2i+1-2j)!}$$

If we keep extracting terms in this fashion, after extracting $m$ terms we would obtain the following equation:

$$\mathbb{1}_{k|n}=\frac{\cos 2\pi n-1}{2k}+\frac{\sin 2\pi n}{2\pi n}\sum_{i=0}^{m-1}\frac{(-1)^i(2\pi n)^{2i}B_{2i}}{(2i)!k^{2i}}+\sum_{i=0}^{\infty}(-1)^{i+m}(2\pi n)^{2i+2m}\sum_{j=0}^{i}\frac{B_{2j+2m}k^{-2j-2m}}{(2j+2m)!(2i+1-2j)!}$$

Since the first 2 terms are clearly 0 for integer $n$ we can discard them. This means that $\mathbb{1}_{k|n}$ has an infinite number of alternative power series:

$$\mathbb{1}_{k|n}=\sum_{i=0}^{\infty}(-1)^{i+m}(2\pi n)^{2i+2m}\sum_{j=0}^{i}\frac{B_{2j+2m}k^{-2j-2m}}{(2j+2m)!(2i+1-2j)!}\tag{2}$$

# 3 Sum of the Divisors of $n$

The function $\sigma_m^2(n)$ is defined as the sum of the $m$-th powers of the integer divisors of $n$ (the superscript 2 was chosen for convenience and will make sense when we reach section (4.6)). In mathematical notation, for any complex $m$:

$$\sigma_m^2(n)=\sum_{k|n}k^m$$

By using the power series we derived for $\mathbb{1}_{k|n}$, such as (1) or (2), we can obtain a power series for $\sigma_m^2(n)$. For reasons that will become apparent soon, form (1) is preferred.

## 3.1 Sum of Powers of Divisors of $n$

Let's start by deriving a power series for the number of divisors of an integer $n$, $\sigma_0^2(n)$, also known as $d(n)$. If we take equation (1) and sum $k$ over the positive integers, we get $\sigma_0^2(n)$:

$$\sigma_0^2(n)=\sum_{k=1}^{\infty}\mathbb{1}_{k|n}=\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{B_{2j}\zeta(2j)}{(2j)!(2i+1-2j)!}$$

Now, by recalling the closed-form of the zeta function at the even integers:

$$\zeta(2j)=-\frac{(-1)^j(2\pi)^{2j}B_{2j}}{2(2j)!}\Rightarrow\frac{B_{2j}}{(2j)!}=-2(-1)^j(2\pi)^{-2j}\zeta(2j),$$

we can replace $B_{2j}/(2j)!$ in equation (3) and express $\sigma_0^2(n)$ in a more useful form:

$$\sigma_0^2(n) = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}\zeta(2j)^2}{(2i+1-2j)!} \tag{3}$$

A similar rationale that also stems from equation (1) can be applied to obtain $\sigma_m^2(n)$ for any complex $m$ (here we'll leave the equation in its original form):

$$\sigma_m^2(n) = \sum_{k=1}^{\infty}\mathbb{1}_{k|n}\cdot k^m = \sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{\substack{j=0\\2j-m\neq 1}}^{i}\frac{B_{2j}\zeta(2j-m)}{(2j)!(2i+1-2j)!}$$

Note we need to be careful to avoid the zeta function pole, hence $2j-m\neq 1$.

And if we multiply $\sigma_m^2(n)$ by $n^{-m}$ we obtain $\sigma_{-m}^2(n)$. It's not so obvious to see why this works (if $k$ is a divisor of $n$, $k/n$ is the reciprocal of another divisor of $n$):

$$\sigma_{-m}^2(n) = \sum_{k=1}^{\infty}\mathbb{1}_{k|n}\cdot\left(\frac{k}{n}\right)^m = \sum_{i=0}^{\infty}(-1)^i(2\pi)^{2i}n^{2i-m}\sum_{\substack{j=0\\2j-m\neq 1}}^{i}\frac{B_{2j}\zeta(2j-m)}{(2j)!(2i+1-2j)!}$$

Another way to obtain $\sigma_{2m}^2(n)$, though this time only for integer $m$, can be achieved using (2):

$$\sigma_{2m}^2(n) = \sum_{k=1}^{\infty}\mathbb{1}_{k|n}\cdot k^{2m} = \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+m}(2\pi n)^{2i+2m}\sum_{j=0}^{i}\frac{B_{2j+2m}k^{-2j}}{(2j+2m)!(2i+1-2j)!} \Rightarrow$$

$$\sigma_{2m}^2(n) = \sum_{i=0}^{\infty}(-1)^{i+m}(2\pi n)^{2i+2m}\sum_{j=0}^{i}\frac{B_{2j+2m}\zeta(2j)}{(2j+2m)!(2i+1-2j)!}$$

And the above holds for positive or negative integer $m$, except that Bernoulli numbers are not defined for negative subscripts. But since the reciprocal of negative integer factorials is 0, that doesn't matter.

As we can see, the function $\mathbb{1}_{k|n}$ has a lot of interesting properties, and it will come in handy on our goal of studying the prime numbers.

# 4 Introducing Möbius' $\mu(n)$

The formula we created for $\sigma_0^2(n)$ in (3) begs a question: what would happen if we replaced $\zeta(2j)^2$ with $\zeta(2j)^3$? In other words, what does $\sigma_0^3(n)$ give us?

$$\sigma_0^3(n) = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}\zeta(2j)^3}{(2i+1-2j)!}$$

To answer this question, we need to rewrite the initial sum that leads to the above formula:

$$\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\mathbb{1}_{k_1\cdot k_2|n}=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\left(\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{B_{2j}(k_1\cdot k_2)^{-2j}}{(2j)!(2i+1-2j)!}\right)$$

Looking at this formula we are led to conclude $\sigma_0^3(n)$ is the number of permutations of elements from the set of divisors of $n$, taken 2 at a time, that are also divisors of $n$ when multiplied together. It's also equal to $\sum_{k|n}\sigma_0^2(k)$, as we'll see in section (4.6).

And how about $\sigma_0^1(n)$ and $\sigma_0^0(n)$? The former is a constant equal to 1 for all integer $n$, and the latter equals $\sum_{k|n}\mu(k)$, where $\mu(k)$ is the Möbius function, which we'll define in the next section. As a preliminary, $\sigma_0^0(n)$ is 1 if $n=1$ and 0 if $n$ is an integer greater than 1, which we'll prove in section (4.4).

And finally, what is $\sigma_0^{-1}(n)$? That is the Möbius' function itself, $\mu(n)$:

$$\mu(n)=-2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}\zeta(2j)^{-1}}{(2i+1-2j)!}, \tag{4}$$

which we'll prove in section (4.5), after we define square-free numbers.

## 4.1 Square-Free Numbers

A square-free number is a number that can't be divided by any squared prime. In other words, if $n$ is square-free, $p_1p_2\cdots p_k$ is its unique prime decomposition. That said, we can define a function $\mu(n)$ such that:

$$\mu(n)=\begin{cases}1, & \text{if } n\text{=1}\\ (-1)^k, & \text{if } n \text{ is square-free with } k \text{ prime factors}\\ 0, & \text{if } n \text{ is not square-free}\end{cases}$$

This function is the Möbius function from the previous section, which was named after the German mathematician who introduced it.

Back to equation (4), one of its advantages is the fact that it provides an analytic continuation of $\mu(n)$ onto $\mathbb{C}$. It can be rewritten in different forms, of which we can cite:

$$\mu(n)=-\frac{\sin 2\pi n}{\pi n}\sum_{j=0}^{\infty}\frac{n^{2j}}{\zeta(2j)}$$

The above form produces the very same power series expansion as (4), so they are the same. Even though the former has a finite radius of convergence ($|n|<1$), it is more tractable and useful for performing manipulations, while (4) is by definition its analytic continuation.

To see why they are the same, we mentioned previously that $\sigma_1^1(n) = 1$ for all integer $n$ (in section (4.6) we'll generalize this class of functions). In this case it's possible to produce a closed-form using the generating function of $\zeta(2j)$ that we've created in [2]:

$$\sigma_1^1(n) = -\frac{\sin 2\pi n}{\pi n}\sum_{j=0}^{\infty} n^{2j}\zeta(2j) = -\frac{\sin 2\pi n}{\pi n}\left(-\frac{\pi n \cot \pi n}{2}\right) = (\cos \pi n)^2$$

The analog of $\sigma_m^2(n)$ is $\mu(n)/n^m$, which for complex $m$ is obtained by a simple modification of equation (4):

$$\frac{\mu(n)}{n^m} = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{\substack{j=0\\ \zeta(2j+m)\neq 0}}^{i}\frac{(-1)^j(2\pi)^{-2j}\zeta(2j+m)^{-1}}{(2i+1-2j)!},$$

This result is a direct consequence of the inversion theorem for Dirichlet series, discussed in section (5). As before, we need to avoid the zeros of the zeta function if they occur.

## 4.2 The Euler Product

The German mathematician Euler discovered an interesting relationship between the square-free numbers and the Riemann zeta function, known as Euler product:

$$\frac{1}{\zeta(s)} = \prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^s}\right) = \left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\cdots$$

Let's denote the set o square-free numbers by $\mathbf{S}$ and let $n$ be a member of $\mathbf{S}$.

The Euler product generates all the square-free numbers in $\mathbf{S}$, but each one comes multiplied by $(-1)^k$, where $k$ is the number of primes in the prime decomposition of $n$ $(n = p_1p_2\cdots p_k)$.

That said, it becomes evident that we can write $\zeta(s)^{-1}$ as a function of $\mu(n)$:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \tag{5}$$

## 4.3 Dirichlet Series

The right-hand side of equation (5) is one particular example of Dirichlet series. A Dirichlet series is any infinite sum of the type:

$$F_a(s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

where $a(n)$ is an arithmetic function and $F_a(s)$ is its generating function.

Given two Dirichlet series, $F_a(s)$ and $F_b(s)$, their product is a third Dirichlet series, $F_c(s)$, whose associated function, $c(n)$, is called a Dirichlet convolution of $a$ and $b$, denoted by $c = a*b$.

Of particular interest to us, the product of a Dirichlet series $F_a(s)$ and the reciprocal of the zeta function, $\zeta(s)^{-1}$, is the convolution of $a(n)$ with $\mu(n)$, which is given by the so-called Möbius inversion formula:

$$c(n) = \sum_{k|n} a(k)\mu\left(\frac{n}{k}\right)$$

We shall use this result in subsequent proofs.

## 4.4 Unit Function

Let's prove the below assertion, that we referred to in section (4):

$$\sum_{k|n} \mu(k) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

The proof for the above is pretty simple. If the prime decomposition of $n$ has $k$ primes, then its number of square-free divisors is $2^k$, half of which have an odd number of prime factors and half of which have an even number of prime factors. Since the former have a negative sign and the latter a positive, they cancel out. The exception is $n = 1$, which has no prime factor. ■

In terms of the function $\mathbb{1}_{k|n}$ from equation (1), this result implies:

$$\sum_{k=1}^{\infty} \mathbb{1}_{k|n} \cdot \mu(k) = \sum_{i=0}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=0}^{i} \frac{B_{2j}}{(2j)!(2i+1-2j)!} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^{2j}} \Rightarrow$$

$$\mu_0(n) = \sum_{i=0}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=0}^{i} \frac{(-1)^j (2\pi)^{-2j}}{(2i+1-2j)!} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

What this means is that, per the Möbius inversion formula, the convolution of the function we just called $\mu_0(n)$ with $\mu(n)$ is $\mu(n)$ itself:

$$\mu(n) = \sum_{k|n} \mu_0(k)\mu\left(\frac{n}{k}\right)$$

Actually, the convolution of $\mu_0(n)$ with any function is the function itself, and hence $\mu_0(n)$ is known as the unit function.

## 4.5 Cube-Free Numbers

A cube-free number is a number that can't be divided by any cubed prime. In other words, if $n$ is cube-free, its unique prime decomposition is $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where the $e_i$ are non-zero integer exponents less than or equal to 2. In this context, a prime factor $p_i$ is said to be single if $e_i = 1$, it's said to be double if $e_i = 2$, and so on. That said, we can introduce a modified Möbius function of order 2, $\mu_2(n)$, with the following properties:

$$\mu_2(n) = \begin{cases} 1, & \text{if } n\text{=1} \\ (-2)^k, & \text{if } n \text{ is cube-free with } k \text{ single prime factors} \\ 0, & \text{if } n \text{ is not cube-free} \end{cases}$$

This definition is different from the one proposed by Tom M. Apostol in 1970,[3] but probably equal to Popovici's function.[4]

Now, let's take the Euler product from the previous section, and see what it looks like squared:

$$\frac{1}{\zeta(s)^2} = \prod_{j=1}^{\infty} \left(1 - \frac{1}{p_j^s}\right)^2 = \left(1 - \frac{1}{2^s}\right)^2 \left(1 - \frac{1}{3^s}\right)^2 \left(1 - \frac{1}{5^s}\right)^2 \cdots \Rightarrow$$

$$\frac{1}{\zeta(s)^2} = \left(1 - 2 \cdot \frac{1}{2^s} + \frac{1}{2^{2s}}\right) \left(1 - 2 \cdot \frac{1}{3^s} + \frac{1}{3^{2s}}\right) \left(1 - 2 \cdot \frac{1}{5^s} + \frac{1}{5^{2s}}\right) \cdots$$

Looking at the expansion of $\zeta(s)^{-2}$ above, it's not very hard to conclude the following equivalence:

$$\frac{1}{\zeta(s)^2} = \sum_{n=1}^{\infty} \frac{\mu_2(n)}{n^s}$$

Now, the convolution of $\mu(n)$ with itself should give us $\mu_2(n)$, after all the latter is generated by $\zeta(s)^{-2}$:

$$\mu_2(n) = \sum_{k|n} \mu(k)\mu\left(\frac{n}{k}\right)$$

And the above result allows us to state the following theorem:

**Theorem 1** $\mu(n) = \sum_{k|n} \mu_2(k)$

**Proof 1** This result stems from the Möbius inversion formula applied to the convolution of $\mu(n)$ with itself.

It can also be proved with combinatorics, but we'll analyze just two possible scenarios. Due to the multiplicative nature of $\mu(n)$ (that is, $\mu(xy) = \mu(x)\mu(y)$, when $x$ and $y$ are co-prime), we can partition $n$ in blocks of co-prime factors where all primes are single, double, and so on, and analyze each one separately. If $n$ has a block of prime factors that are not single, then it's not square-free and hence $\sum_{k|n} \mu_2(k) = 0$, which we will show.

For the first scenario, let's assume that $n$ is square-free with $k$ prime factors, $n = p_1 p_2 \cdots p_k$. Under this scenario we have $2^k$ possible divisors. Let's also assume that $q = p_1 p_2 \cdots p_i$ is a combination of $i$ out of these $k$ primes. There are $C_{k,i}$ such divisors and they are such that $\mu_2(q) = (-2)^i$ if $i > 0$. When we plug them into the sum (plus 1, to account for divisor 1) we get:

$$\sum_{i=0}^{k} C_{k,i}(-2)^i = (1 + (-2))^k = (-1)^k = \mu(n)$$

For the second scenario, let's assume that $n$ has $k$ double prime factors, $n = p_1^2 p_2^2 \cdots p_k^2$. Under this scenario we have $3^k$ possible divisors. Let's also assume that $q = p_1 p_2 \cdots p_i$, $i > 0$, is a combination of $i$ out of these $k$ primes. There are $\sum_{j=0}^{k-i} C_{k,i,j}$ such divisors and they are such that $\mu_2(q) = (-2)^i$ if $i > 0$, and 1 otherwise. Let's see why the sum would be 0:

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} C_{k,i,j}(-2)^i = ((-2) + 1 + 1)^k = 0, \text{ where } C_{k,i,j} = \frac{k!}{i!j!(k-i-j)!} \blacksquare$$

Now that theorem 1 has been proved, we can use it to demonstrate the validity of equation (4). The demonstration is totally analogous to the previous one:

$$\sum_{k=1}^{\infty} \mathbb{1}_{k|n} \cdot \mu_2(k) = \sum_{i=0}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=0}^{i} \frac{B_{2j}}{(2j)!(2i+1-2j)!} \sum_{k=1}^{\infty} \frac{\mu_2(k)}{k^{2j}} \Rightarrow$$

$$\mu(n) = \sum_{i=0}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=0}^{i} \frac{B_{2j}\zeta(2j)^{-2}}{(2j)!(2i+1-2j)!} = \sum_{i=0}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=0}^{i} \frac{(-1)^j (2\pi)^{-2j} \zeta(2j)^{-1}}{(2i+1-2j)!}$$

## 4.6 Duality Between $\mu_q(n)$ and $\sigma_0^q(n)$

From the previous expositions, we can define a generalized Möbius function of order $q$, $\mu_q(n)$, which coincides with Popovici's definition:[4] $\mu_q(n) = \mu * \cdots * \mu$ is the $q$-fold Dirichlet convolution of the Möbius function with itself. And again, because $\zeta(s)^{-1}$ is the generating function of $\mu(n)$, its convolution with $\mu_q(n)$ justifies the below recurrence:

$$\mu_{q+1}(n) = \sum_{k|n} \mu_q(k)\mu\left(\frac{n}{k}\right) \Rightarrow \mu_q(n) = \sum_{k|n} \mu_{q+1}(k) \text{ (where } \mu_1(n) = \mu(n))$$

Therefore, it follows from this and previous results that:

$$\mu_q(n) = -2 \sum_{i=0}^{\infty} (-1)^i (2\pi n)^{2i} \sum_{j=0}^{i} \frac{(-1)^j (2\pi)^{-2j} \zeta(2j)^{-q}}{(2i+1-2j)!}$$

The above expression for $\mu_q(n)$ is insightful, if we think about negative values of $q$. For example:

$$\sigma_0^3(n) = \sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}\zeta(2j)^3}{(2i+1-2j)!} = \sum_{k|n}\sigma_0^2(k)$$

(Again, notice that $\sigma_0^2(k)$ is the number of divisors of $k$, also referred to as $d(k)$.)

So, we conclude that there is a duality between $\mu_q(n)$ and $\sigma_0^q(n)$, they are equivalent and can be used interchangeably, more precisely, $\mu_q(n) = \sigma_0^{-q}(n)$.

To make this finding more intuitive, let's say that for positive $q$ the following identities hold:

$$\frac{1}{\zeta(s)^q} = \sum_{n=1}^{\infty}\frac{\mu_q(n)}{n^s}, \text{ and } \zeta(s)^q = \sum_{n=1}^{\infty}\frac{\sigma_0^q(n)}{n^s}$$

The second equation is obvious for case $q = 0$ (since $\mu_0(n) = \sigma_0^0(n) = 0$ for all integer $n$ except 1) and $q = 1$ (since $\sigma_0^1(n) = 1$ for all integer $n$).

## 4.7 Sum of the Square-Free Divisors of $n$

Going back to the topic of integer divisors of $n$, now that square-free numbers and the Euler product have been introduced, we can obtain the sum of powers of square-free divisors of $n$ using the following result from the literature:

$$\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$$

Therefore, through the same rationale as before, we conclude that for any complex $m$:

$$\sum_{k=1}^{\infty}\mathbb{1}_{k|n}\cdot|\mu(k)|\cdot k^m = \sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{\substack{j=0\\2j-m\neq 1\\\zeta(4j-2m)\neq 0}}^{i}\frac{B_{2j}}{(2j)!(2i+1-2j)!}\frac{\zeta(2j-m)}{\zeta(4j-2m)}$$

In particular, the number of distinct prime factors of $n$ is:

$$\log_2\left(\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{B_{2j}}{(2j)!(2i+1-2j)!}\frac{\zeta(2j)}{\zeta(4j)}\right)$$

# 5 Inversion Theorem for Dirichlet Series

We now enunciate a little theorem that relates a Dirichlet series to its coefficients.

**Theorem 2** Suppose that $F_a(s)$ is a Dirichlet series and $a(n)$ is its associated arithmetic function. Then $a(n)$ is given by:

$$a(n) = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}F_a(2j)}{(2i+1-2j)!}$$

**Proof 2** The formula is obviously true for $\zeta(2j)^q$, for any integer $q$, which we've already proven in previous sections.

For the general case, the proof is kind of trivial:

$$-2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}F_a(2j)}{(2i+1-2j)!} \Rightarrow$$

$$-2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}}{(2i+1-2j)!}\sum_{k=1}^{\infty}\frac{a(k)}{k^{2j}} \Rightarrow$$

$$\sum_{k=1}^{\infty}a(k)\left(-2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}k^{-2j}}{(2i+1-2j)!}\right)$$

The theorem then follows from the following equation:

$$-2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi k)^{-2j}}{(2i+1-2j)!} = \begin{cases} 1, & \text{if } n = k \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

And the above equation is justified for being the convolution of $\mu_0(n)$ and the associated function of the series $k^{-s}$, $b(n)$, since from the convolution formula:

$$c(n) = (\mu_0 * b)(n) = \sum_{d|n}\mu_0(d)b\left(\frac{n}{d}\right) = b(n) = \begin{cases} 1, & \text{if } n = k \\ 0, & \text{otherwise} \end{cases}$$

Now, by using this result along with the same reasoning employed in the proof from section (4.4), we derive equation (6). ■

It's quite remarkable that Dirichlet series have coefficients given by Taylor series. One of the advantages of this formula is that if you know $F_a(s)$ at the even integers, you know the coefficients of its series expansion. Another advantage is that it extends $a(n)$ to the real or complex numbers. Notice it works even using the analytic continuation of $F_a(s)$ at 0 if it's undefined, but if we skip summing $j$ over 0 it works too.

From the proof we conclude that we'd get the same result if we used the partial series instead of the infinite series (e.g., $H_{2j}(n)$ instead of $\zeta(2j)$).

Unfortunately, this inversion formula doesn't apply to the analytic continuation of the Riemann zeta function, it would be really interesting if it did.

# 6 Applications

Even though the possibilities are endless, let's see a few examples.

## 6.1 Square Root of the Zeta

The function $a(n)$ seems to have a predilection for rational outputs when $F_a(s)$ is some variation of the zeta function:

$$\sqrt{\zeta(s)} = 1+\frac{1}{2\cdot 2^s}+\frac{1}{2\cdot 3^s}+\frac{3}{8\cdot 4^s}+\frac{1}{2\cdot 5^s}+\frac{1}{4\cdot 6^s}+\frac{1}{2\cdot 7^s}+\frac{5}{16\cdot 8^s}+\frac{3}{8\cdot 9^s}+\frac{1}{4\cdot 10^s}+\frac{1}{2\cdot 11^s}+\cdots$$

It's not hard to guess the patterns of $a(n)$: numbers with the same type of prime decomposition have the same coefficients.

## 6.2 Zeta Raised to $\boldsymbol{i}$

To foray into complex realm, the following coefficients were calculated using the inversion formula:

$$\zeta(s)^{\boldsymbol{i}} = 1+\frac{\boldsymbol{i}}{2^s}+\frac{\boldsymbol{i}}{3^s}+\frac{-1+\boldsymbol{i}}{2\cdot 4^s}+\frac{\boldsymbol{i}}{5^s}+\frac{-1}{6^s}+\frac{\boldsymbol{i}}{7^s}+\frac{-3+\boldsymbol{i}}{6\cdot 8^s}+\frac{-1+\boldsymbol{i}}{2\cdot 9^s}+\frac{-1}{10^s}+\frac{\boldsymbol{i}}{11^s}+\cdots$$

## 6.3 Prime Indicator Function, $\mathbb{1}_{n\in\mathbb{P}}$

If $P(x)$ is the prime zeta function, then for integer $n$:

$$\mathbb{1}_{n\in\mathbb{P}} = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=1}^{i}\frac{(-1)^j(2\pi)^{-2j}P(2j)}{(2i+1-2j)!}$$

We can skip $P(0)$, since its value is unknown.

## 6.4 Modulus of $\mu(n)$

Based on the closed-form of the Dirichlet series whose associated function is $|\mu(n)|$, which appeared in section (4.7), we can deduce that:

$$|\mu(n)| = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}}{(2i+1-2j)!}\frac{\zeta(2j)}{\zeta(4j)}$$

## 6.5 Liouville Function

If $\lambda(n)$ is the Liouville function, which counts the number of prime factors (with multiplicity) of $n$, then:

$$\lambda(n) = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}}{(2i+1-2j)!}\frac{\zeta(4j)}{\zeta(2j)},\text{ since }\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}=\frac{\zeta(2s)}{\zeta(s)}$$

## 6.6 Mertens Function

We can extend the Mertens function to the complex domain, based on equation (4):

$$M(n) = \sum_{k=1}^{n} \mu(k) = -2\sum_{i=0}^{\infty}(-1)^i(2\pi)^{2i}H_{-2i}(n)\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}\zeta(2j)^{-1}}{(2i+1-2j)!},$$

where $H_{-2i}(n)$ is the sum of the $2i$-th powers of the first $n$ positive integers (harmonic numbers of negative orders).

## 6.7 Square Root of an Integer

Using Theorem 2, we can derive power series expansions for functions that are not analytic at zero (and therefore don't admit a Taylor series at 0), which hold only at the positive integers.

For example, if $n$ is a positive integer then:

$$\sqrt{n} = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}}{(2i+1-2j)!}\zeta\left(-\frac{1}{2}+2j\right), \text{ since } \zeta\left(-\frac{1}{2}+s\right) = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^s}$$

The above allows us to create a power series for the sum of the square root of the first $n$ positive integers:

$$\sum_{k=1}^{n}\sqrt{k} = -2\sum_{i=0}^{\infty}(-1)^i(2\pi)^{2i}H_{-2i}(n)\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}}{(2i+1-2j)!}\zeta\left(-\frac{1}{2}+2j\right)$$

The downside here is that we don't know closed-forms for the zeta function at the half-integers.

## 6.8 Logarithm of an Integer

Like before, if $n$ and $k$ are positive integers, then:

$$(\log n)^k = -2(-1)^k\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}\zeta^{(k)}(2j)}{(2i+1-2j)!}, \text{ since } \zeta^{(k)}(s) = (-1)^k\sum_{n=1}^{\infty}\frac{(\log n)^k}{n^s}$$

## 6.9 Von Mangoldt Function

The Von Mangoldt function is defined as:

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime } p \text{ and integer k} \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

We can obtain $\Lambda(n)$ by:

$$\Lambda(n) = 2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}\zeta'(2j)}{(2i+1-2j)!\zeta(2j)}, \text{ since } \frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

From $\Lambda(n)$ we can derive another arithmetic function:

$$\frac{\Lambda(n)}{\log n} = \begin{cases} 1/k, & \text{if } n = p^k \text{ for some prime } p \text{ and integer k} \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence, we are able to write $\log\zeta(s)$ as a Dirichlet series as follows:

$$\frac{1}{\zeta(s)} = \prod_{k=1}^{\infty}\left(1-\frac{1}{p_k^s}\right) \Rightarrow \log\zeta(s) = -\sum_{k=1}^{\infty}\log\left(1-\frac{1}{p_k^s}\right) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{i\left(p_k^i\right)^s} = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n}\frac{1}{n^s}$$

Therefore, by the inversion theorem:

$$\frac{\Lambda(n)}{\log n} = -2\sum_{i=0}^{\infty}(-1)^i(2\pi n)^{2i}\sum_{j=0}^{i}\frac{(-1)^j(2\pi)^{-2j}\log\zeta(2j)}{(2i+1-2j)!},$$

and this function, combined with $\mu(n)$, is sufficient for us to create an exact formula for the prime counting function, $\pi(x)$.

Let's rewrite the last power series:

$$\frac{\Lambda(n)}{\log n} = -\frac{\sin 2\pi n}{\pi n}\sum_{j=0}^{\infty}n^{2j}\log\zeta(2j)$$

# 7 Prime Counting Function, $\pi(x)$

We don't even have to know the zeros of the analytic continuation of the zeta function to be able to derive a formula for $\pi(x)$.

A number is prime if it's square-free and it's a prime power. So, now that we have power series for both $\mu(n)$ and $\Lambda(n)/\log n$, we can easily create a function that is 1 whenever $n$ is prime, and 0 otherwise, which is given simply by:

$$\mathbb{1}_{n\in\mathbb{P}} = -\mu(n)\frac{\Lambda(n)}{\log n} = \frac{-1+\cos 4\pi n}{2\pi^2 n^2}\sum_{j=0}^{\infty}\frac{n^{2j}}{\zeta(2j)}\sum_{j=0}^{\infty}n^{2j}\log\zeta(2j)$$

Perhaps this is not the only or even the best way to derive $\mathbb{1}_{n\in\mathbb{P}}$, but it's the most obvious.

If we expand the above function using Taylor series, we arrive at the below equation (note that since the result is a real number, we can skip summing over $\log\zeta(0)$):

$$\mathbb{1}_{n\in\mathbb{P}} = -\frac{1}{2\pi^2}\sum_{h=1}^{\infty} n^{2h-2}\sum_{j=1}^{h}\frac{\log\zeta(2j)}{\zeta(2h-2j)} + \frac{1}{2\pi^2}\sum_{h=1}^{\infty} n^{2h-2}\sum_{i=1}^{h}\sum_{j=0}^{h-i}\frac{\log\zeta(2i)}{\zeta(2j)}\frac{(-1)^{h-i-j}(4\pi)^{2h-2i-2j}}{(2h-2i-2j)!}$$

Fortunately, the above power series can be simplified into a better looking series (which is also more efficient for numeric computation):

$$\mathbb{1}_{n\in\mathbb{P}} = -8\sum_{h=1}^{\infty} n^{2h}\sum_{i=1}^{h}\log\zeta(2i)\sum_{v=i}^{h}\frac{(-1)^{h-v}(4\pi)^{2h-2v}}{\zeta(2v-2i)(2h+2-2v)!}$$

And finally, the prime counting function, $\pi(x)$, is the sum of $\mathbb{1}_{n\in\mathbb{P}}$ over $n$:

$$\pi(x) = -8\sum_{h=1}^{\infty} H_{-2h}(x)\sum_{i=1}^{h}\log\zeta(2i)\sum_{v=i}^{h}\frac{(-1)^{h-v}(4\pi)^{2h-2v}}{\zeta(2v-2i)(2h+2-2v)!},$$

where $H_{-2h}(x)$ is the sum of the $2h$-th powers of the first $x$ positive integers. Even though it's difficult to compute this power series for large $x$, the zeros of the zeta function are even harder to compute.

It's interesting to note that in a way this function resembles the Riemann prime counting function under the Riemann Hypothesis (both are the difference of two sums):

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho})$$

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