# On the Limits of a Generalized Harmonic Progression

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**Abstract**

This is the fourth paper I'm releasing on the topic of harmonic progressions. Here we address a more complicated problem, namely, the determination of the limiting function of a generalized harmonic progression. It underscores the utility of the formula we derived for $\sum_{j=1}^{n} 1/(a\boldsymbol{i}j+b)^k$ in *Complex Harmonic Progression* and of results we presented in *Generalized Harmonic Numbers Revisited*. Our objective is to create a generating function for $\sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty} 1/(j+b)^k$, with complex $x$ and $b$, whose derivatives at 0 give us the limit of the harmonic progressions (of order 2 and higher) as $n$ approaches infinity.

## 1 Introduction

When I wrote *Generalized Harmonic Progression*,[3] I mentioned in passing that I would not try to obtain the limit of that expression, as I imagined that it would be nearly impossible to achieve that. I had no clue how take that formula and come up with an approach that would allow me to find out those limits. But after I wrote *Complex Harmonic Progression*,[4] with a more general formula, the insight on how to use it to figure out the limit of the harmonic progression (of order 2 and higher) came almost instantly.

This is one of those problems that become simple to solve with the proper tools and a little creativity, unlike the hard nuts that no matter how hard one tries they never crack. Not that it was very easy to figure out all the steps here, but the characteristics of the formula helped.

The solution presented here constitutes another proof that the harmonic progression (of order 1) diverges.

First, let's recall the formula we created for the generalized harmonic progression in [4]. For simplicity, let's assume that $a = 1$ and that $b$ is real, so we know which part of the formula is real and which part is complex. Let $\phi$ be the Hurwitz-Lerch transcendent function:

$$\phi(z,s,\alpha) = \sum_{k=0}^{\infty} \frac{z^k}{(k+\alpha)^s}$$

Then, if $\boldsymbol{i}b \notin \mathbb{Z}$:

$$\sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n+b)^k} +$$
$$\mathrm{e}^{-2\pi b}(2\pi)^k \int_0^1 \sum_{j=1}^{k} \frac{\phi\left(\mathrm{e}^{-2\pi b}, -j+1, 0\right)(1-u)^{k-j}}{(j-1)!(k-j)!} \mathrm{e}^{\pi(\boldsymbol{i}n+2b)u} \sin \pi n u \cot \pi u \, du$$

The integral can be simplified by using Euler's formula for the exponential of a complex argument, and by replacing $\cos \pi n u \sin \pi n u$ and $\sin^2 \pi n u$ with equivalent expressions:

$$\sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n+b)^k} +$$
$$\mathrm{e}^{-2\pi b}(2\pi)^k \int_0^1 \sum_{j=1}^{k} \frac{\phi\left(\mathrm{e}^{-2\pi b}, -j+1, 0\right)(1-u)^{k-j}}{(j-1)!(k-j)!} \mathrm{e}^{2\pi b u} \left(\frac{\sin 2\pi n u}{2} + \boldsymbol{i}\frac{1-\cos 2\pi n u}{2}\right) \cot \pi u \, du$$

# 2 Direct Calculation of the Real Part

Let's recall a result that appeared in reference [2], whose proof depends on formulae that feature in Abramowitz and Stegun:[1]

**Theorem 1** $\lim_{n\to\infty} \int_0^1 u^k \sin 2\pi n(1-u) \cot \pi(1-u) \, du = \begin{cases} 1, & \text{if } k=0 \\ \frac{1}{2}, & \text{if integer } k>0 \end{cases}$

Due to the above theorem, it becomes convenient to rewrite the previous formula:

$$\sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n+b)^k}$$
$$+\frac{(2\pi)^k}{2} \int_0^1 \sum_{j=1}^{k} \frac{\phi\left(\mathrm{e}^{-2\pi b}, -j+1, 0\right) u^{k-j}}{(j-1)!(k-j)!} \mathrm{e}^{-2\pi b u} \left(\sin 2\pi n(1-u) + \boldsymbol{i}(1-\cos 2\pi n(1-u))\right) \cot \pi(1-u) \, du$$

If we look at the real part of the formula only, we can split the sum within the integral into a polynomial in $u$ and a constant, since the limits for each are different. And since $\mathrm{e}^{-2\pi b u}$ is a polynomial in $u$ itself, when they are multiplied together, as below, in only one of the four possible cases the limit as $n$ approaches infinity is 1, all others being $1/2$:

$$\lim_{n\to\infty} \int_0^1 \left(\frac{\phi\left(\mathrm{e}^{-2\pi b}, -k+1, 0\right)}{(k-1)!} + \sum_{j=1}^{k-1} \frac{\phi\left(\mathrm{e}^{-2\pi b}, -j+1, 0\right) u^{k-j}}{(j-1)!(k-j)!}\right) \left(1 + \sum_{v=1}^{\infty} \frac{(-2\pi b)^v u^v}{v!}\right) \sin 2\pi n(1-u) \cot \pi(1-u) \, du$$

Therefore, in light of Theorem 1, we can draw our first conclusion:

$$\Re\left(\sum_{j=1}^{\infty} \frac{1}{(\boldsymbol{i}j+b)^k}\right) = -\frac{1}{2b^k} + \frac{(2\pi)^k}{4} \frac{\phi\left(\mathrm{e}^{-2\pi b}, -k+1, 0\right)}{(k-1)!} + \frac{(2\pi)^k \mathrm{e}^{-2\pi b}}{4} \sum_{j=1}^{k} \frac{\phi\left(\mathrm{e}^{-2\pi b}, -j+1, 0\right)}{(j-1)!(k-j)!}$$

Unfortunately, doing the same for the imaginary part is not as easy, and hence we need to resort to the generating function of the generalized harmonic progression.

# 3 Generating Function

In this section we will try to come up with a generating function for the generalized harmonic progression and determine its limit when $n$ goes to infinity:

$$\sum_{k=1}^{\infty} x^k \sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k} = \sum_{k=1}^{\infty} -\frac{x^k}{2b^k} + \frac{x^k}{2(\boldsymbol{i}n+b)^k} +$$

$$\frac{1}{2}\sum_{k=1}^{\infty} \int_0^1 \sum_{j=1}^{k} \frac{\mathrm{e}^{-2\pi b}\phi\left(\mathrm{e}^{-2\pi b}, -j+1, 0\right)(2\pi x)^j}{(j-1)!} \frac{(2\pi x(1-u))^{k-j}}{(k-j)!} \mathrm{e}^{2\pi bu} \left(\sin 2\pi nu + \boldsymbol{i}(1-\cos 2\pi nu)\right) \cot \pi u \, du$$

The sum within the integral is the sum of the product of the general terms of two power series whose generating functions we know:

$$p(x) = \frac{-2\pi x}{(\mathrm{e}^{2\pi x} - \mathrm{e}^{2\pi b})} \cdot \mathrm{e}^{2\pi x(1-u)}$$

Therefore, the integrand is the power series of the product of these two functions, and we can rewrite the formula as:

$$\sum_{k=1}^{\infty} x^k \sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k} = \sum_{k=1}^{\infty} x^k \left(-\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n+b)^k}\right)$$
$$- \frac{\pi x}{\mathrm{e}^{2\pi x} - \mathrm{e}^{2\pi b}} \int_0^1 \mathrm{e}^{2\pi x(1-u)} \mathrm{e}^{2\pi bu} \left(\sin 2\pi nu + \boldsymbol{i}(1-\cos 2\pi nu)\right) \cot \pi u \, du$$

Since we want to take the limit of that expression as $n$ goes to infinity, we need to subtract the harmonic progression of order 1, which diverges (our method will show us why later down the line). Thus we have:

$$\sum_{k=2}^{\infty} x^k \sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k} = \sum_{k=2}^{\infty} x^k \left(-\frac{1}{2b^k} + \frac{1}{2(\boldsymbol{i}n+b)^k}\right)$$
$$- \pi x \int_0^1 \left(\frac{1}{\mathrm{e}^{2\pi b} - 1} + \frac{\mathrm{e}^{2\pi x(1-u)}}{\mathrm{e}^{2\pi x} - \mathrm{e}^{2\pi b}}\right) \mathrm{e}^{2\pi bu} \left(\sin 2\pi nu + \boldsymbol{i}(1-\cos 2\pi nu)\right) \cot \pi u \, du$$

## 3.1 Real Part

Let's tackle the limit of the real part first. By following the same thought process that's been laid out in section (2), we conclude that the real part is given by:

$$\Re\left(\sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty} \frac{1}{(\boldsymbol{i}j+b)^k}\right) = -\frac{x^2}{2b(b-x)} + \frac{\pi x(\mathrm{e}^{2\pi x} - 1)}{(\mathrm{e}^{-2\pi b} - 1)(\mathrm{e}^{2\pi x} - \mathrm{e}^{2\pi b})}$$

## 3.2 Imaginary Part

When it comes to the imaginary part, we can obviously discard the two terms outside of the integral, as one is real and the other one goes to zero as $n$ goes to infinity, leaving us with:

$$\Im\left(\sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty} \frac{1}{(\boldsymbol{i}j+b)^k}\right) = \lim_{n\to\infty} -\pi x \int_0^1 \left(\frac{1}{e^{2\pi b}-1} + \frac{e^{2\pi x(1-u)}}{e^{2\pi x}-e^{2\pi b}}\right) e^{2\pi bu}(1-\cos 2\pi nu)\cot \pi u\, du$$

Here we realize that in order to figure out the above limits, we need to solve the below problem. To solve the two limits at once, let's use $c = c(x)$ as the coefficient of $u$:

$$\lim_{n\to\infty} \int_0^1 e^{-c\,u}(1-\cos 2\pi nu)\cot \pi u\, du$$

Now we need to go back to one of the results from [2]. It provides us with an equivalence between certain integrals and generalized harmonic numbers. More specifically we've seen that:

$$\int_0^1 u^{2k+1}\,(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du = \frac{2(-1)^k(2k+1)!}{(2\pi)^{2k+1}} \left(\sum_{j=0}^{k} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!} H_{2j+1}(n) - \frac{1}{2n^{2k+1}} \sum_{j=0}^{k} \frac{(-1)^j(2\pi n)^{2j}}{(2j+1)!}\right)$$

$$\int_0^1 u^{2k+2}\,(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du = \frac{2(-1)^k(2k+2)!}{(2\pi)^{2k+1}} \left(\sum_{j=0}^{k} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+2-2j)!} H_{2j+1}(n) - \frac{1}{2n^{2k+1}} \sum_{j=0}^{k} \frac{(-1)^j(2\pi n)^{2j}}{(2j+2)!}\right)$$

Let's consider their limit as $n$ approaches infinity. If $n$ is sufficiently large:

$$\int_0^1 u^{2k+1}\,(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du \sim \frac{2(-1)^k(2k+1)!}{(2\pi)^{2k+1}} \left(\frac{(-1)^k(2\pi)^{2k}}{(2k+1)!}(\gamma+\log n) + \sum_{j=1}^{k} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!}\zeta(2j+1)\right)$$

$$\int_0^1 u^{2k+2}\,(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du \sim \frac{2(-1)^k(2k+2)!}{(2\pi)^{2k+1}} \left(\frac{(-1)^k(2\pi)^{2k}}{(2k+2)!}(\gamma+\log n) + \sum_{j=1}^{k} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+2-2j)!}\zeta(2j+1)\right)$$

Just to be sure we're not missing anything, let's make our initial formula more in sync with the above formulae by changing $u$ for $1-u$:

$$\Im\left(\sum_{k=2}^{\infty} x^k \sum_{j=1}^{n} \frac{1}{(\boldsymbol{i}j+b)^k}\right) = \lim_{n\to\infty} -\pi x e^{2\pi b} \int_0^1 \left(\frac{e^{-2\pi bu}}{e^{2\pi b}-1} + \frac{e^{-2\pi(b-x)u}}{e^{2\pi x}-e^{2\pi b}}\right)(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du$$

Also, let's introduce a variable $y$ in the integral we need to evaluate, to help us with the power series manipulations that we need to perform. In the end we just need to remember to set $y$ to 1 (and $c$ to $2\pi b$ or $2\pi(b-x)$):

$$\int_0^1 e^{-cy\,u}(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du =$$
$$\int_0^1 \left(1+\sum_{k=0}^{\infty} \frac{(-cy)^{2k+1}}{(2k+1)!}u^{2k+1} + \frac{(-cy)^{2k+2}}{(2k+2)!}u^{2k+2}\right)(1-\cos 2\pi n(1-u))\cot \pi(1-u)\, du$$

Note we can ignore the constant 1, since that integral is 0 for all integer $n$. After we plug the approximation formulae for the integrals into the sum, part of the above expression reduces to:

$$\frac{\gamma+\log n}{\pi}\sum_{k=0}^{\infty}\frac{-c^{2k+1}}{(2k+1)!}+\frac{c^{2k+2}}{(2k+2)!}=\frac{\gamma+\log n}{\pi}\left(-\sinh c-1+\cosh c\right)$$

This part explodes to infinity, and it's due to the harmonic progression of order 1 mentioned in the introduction, which we subtracted from the generating function. Therefore, as expected, these infinities cancel out when we add up the terms that have $c = 2\pi b$ and $c = 2\pi(b-x)$ in our initial formula.

Now, let's see the part that doesn't explode:

$$\frac{1}{\pi}\sum_{k=0}^{\infty}-(cy)^{2k+1}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+1-2j)!}\zeta(2j+1)+(cy)^{2k+2}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+2-2j)!}\zeta(2j+1)$$

It's not a very simple sum, and it's not very easy to know how to proceed from here. But, let's recall the integral representation we derived for $\zeta(2j+1)$ in [2], only here it's been slightly modified:

$$\zeta(2j+1)=-\frac{(-1)^j(2\pi)^{2j+1}}{2}\int_0^1\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}\cot\pi u\,du$$

Let's start with the first part of the previous sum:

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(cy)^{2k+1}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+1-2j)!}\left(-\frac{(-1)^j(2\pi)^{2j+1}}{2}\int_0^1\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}\cot\pi u\,du\right)\Rightarrow$$

$$\sum_{k=0}^{\infty}(cy)^{2k+1}\sum_{j=1}^{k}\frac{1}{(2k+1-2j)!}\int_0^1\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}\cot\pi u\,du=$$
$$\frac{1}{cy}\int_0^1\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{(cy)^{2k+1-2j}}{(2k+1-2j)!}(cy)^{2j+1}\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}-\frac{(cy)^{2k+2}u}{(2k+1)!}\right)\cot\pi u\,du$$

Again, the above power series is the product of two somewhat familiar functions. One of them is obviously $\sinh cy$, and the other one (which appeared in [2] in non-hyperbolic form) is:

$$\sum_{j=0}^{\infty}(cy)^{2j+1}\sum_{p=0}^{j}\frac{B_{2p}\left(2-2^{2p}\right)u^{2j-2p+1}}{(2p)!(2j-2p+1)!}=cy\operatorname{csch}cy\sinh cuy$$

Therefore, the final conclusion is:

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(cy)^{2k+1}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+1-2j)!}\zeta(2j+1)=\sinh cy\int_0^1\left(\operatorname{csch}cy\sinh cuy-u\right)\cot\pi u\,du$$

And since the second part follows an analogous thought process, its development is omitted, but the end result is below:

$$\frac{1}{\pi}\sum_{k=0}^{\infty}(cy)^{2k+2}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+2-2j)!}\zeta(2j+1) = (1-\cosh cy)\int_0^1 (\operatorname{csch} cy \sinh cuy - u)\cot \pi u \, du$$

Now, we can sum it all up by making $y = 1$:

$$\frac{1}{\pi}\sum_{k=0}^{\infty} -c^{2k+1}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+1-2j)!}\zeta(2j+1) + c^{2k+2}\sum_{j=1}^{k}\frac{(-1)^j(2\pi)^{-2j}}{(2k+2-2j)!}\zeta(2j+1) = \\ -(\mathrm{e}^{-c}-1)\int_0^1 (\operatorname{csch} c \sinh cu - u)\cot \pi u \, du$$

To wrap it up, we just need to evaluate the initial expression with the above identity, and when we do so we find that:

$$\Im\left(\sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty}\frac{1}{(\boldsymbol{i}j+b)^k}\right) = \pi x \int_0^1 (\operatorname{csch} 2\pi(b-x)\sinh 2\pi(b-x)u - \sinh 2\pi b \operatorname{csch} 2\pi b u)\cot \pi u \, du$$

# 4 Conclusion

If we don't assume that $b$ is real then, provided that $\boldsymbol{i}b \notin \mathbb{Z}$ (and provided that the right-hand side doesn't contain singularities, such as $b = x$), we can simply state that, for $b, x \in \mathbb{C}$, when the limit exists it's given by:

$$\sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty}\frac{1}{(\boldsymbol{i}j+b)^k} = -\frac{x^2}{2b(b-x)} + \frac{\pi x(\mathrm{e}^{2\pi x}-1)}{(\mathrm{e}^{-2\pi b}-1)(\mathrm{e}^{2\pi x}-\mathrm{e}^{2\pi b})} + \\ \boldsymbol{i}\pi x\int_0^1 (\operatorname{csch} 2\pi(b-x)\sinh 2\pi(b-x)u - \operatorname{csch} 2\pi b \sinh 2\pi b u)\cot \pi u \, du$$

However, when the left-hand side diverges, the expression on the right can still converge, if it doesn't have singularities, meaning that it's probably an analytic continuation of the left-hand side.

The above can be turned into a better looking equation, which in principle holds if $b \neq 0$, $b \neq x$ and $2b$ and $2(b-x)$ are not integers:

$$f(x) = \sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty}\frac{1}{(j+b)^k} = -\frac{x^2}{2b(b-x)} + \frac{\pi x \sin \pi x}{2 \sin \pi b}\csc \pi(b-x) \\ -\pi x\int_0^1 \left(\frac{\sin 2\pi(b-x)u}{\sin 2\pi(b-x)} - \frac{\sin 2\pi b u}{\sin 2\pi b}\right)\cot \pi u \, du$$

Notice that $x = 2b$ causes the integral to vanish, which hints at a possible set of solutions for the zeros of this equation, though we won't pursue it.

To know what the formula looks like when $b$ is a positive integer, we can rely on the below identity:

$$f(x) = \sum_{k=2}^{\infty} x^k \sum_{j=1}^{\infty} \frac{1}{(j+b)^k} = \sum_{k=2}^{\infty} x^k (\zeta(k) - H_k(b))$$

In general, when $b$ is an integer, we can obtain $f(x)$ by using the generating functions we created for $H_k(n)$ and $\zeta(k)$ in [2], which, after all the necessary calculations are performed, lead us to:

$$f(x) = \begin{cases} -\frac{x^2}{2b(b-x)} + \frac{\pi x \sin \pi x}{2 \sin \pi b} \csc \pi(b-x) - \pi x \int_0^1 \left( \frac{\sin 2\pi(b-x)u}{\sin 2\pi(b-x)} - \frac{\sin 2\pi b u}{\sin 2\pi b} \right) \cot \pi u \, du & , \text{if } 2b \notin \mathbb{Z} \\ \frac{1}{2} - \frac{\pi x}{2} \cot \pi x - \pi x \int_0^1 \left( \frac{\sin 2\pi x u}{\sin 2\pi x} - u \right) \cot \pi u \, du & , \text{if } b = 0 \\ -\frac{x^2}{2b(b-x)} - \frac{\pi x}{2} \cot \pi x + \pi x \int_0^1 \left( \frac{\sin 2\pi(b-x)u}{\sin 2\pi x} + u \cos 2\pi b u \right) \cot \pi u \, du & , \text{if } b \in \mathbb{Z}_+ \\ \frac{2b^2 - 2bx - x^2}{2b(b-x)} - \frac{\pi x}{2} \cot \pi x + \pi x \int_0^1 \left( \frac{\sin 2\pi(b-x)u}{\sin 2\pi x} + u \cos 2\pi b u \right) \cot \pi u \, du & , \text{if } b \in \mathbb{Z}_- \end{cases}$$

Finally, if we want to know the limit of the harmonic progression of order $k \geq 2$:

$$\sum_{j=1}^{\infty} \frac{1}{(j+b)^k} = \frac{f^{(k)}(0)}{k!}$$

# References


[1] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (9th printing ed.), New York: Dover*, 1972.

[2] Risomar Sousa, Jose *Generalized Harmonic Numbers Revisited, eprint arXiv:1810.07877*, 2018 2018arXiv181007877R.

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[4] Risomar Sousa, Jose *Complex Harmonic Progression, eprint arXiv:1902.01008*, 2019 2019arXiv190201008R.