Complex Harmonic Progression

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Abstract

In *Generalized Harmonic Progression*, we showed how to create formulae for the sum of the terms of a harmonic progression of order k with integer parameters, that is, $\sum_j 1/(aj+b)^k$. Those formulae were more general than the ones we created in *Generalized Harmonic Numbers Revisited*. In this new paper we make those formulae even more general by removing the restriction that a and b be integers, in other words, here we address $\sum_j 1/(a\boldsymbol{i}j+b)^k$, where a and b are complex numbers and $\boldsymbol{i}$ is the imaginary unity. These new relatively simple formulae always hold, except when $\boldsymbol{i}b/a \in \mathbb{Z}$. This paper employs a slightly modified version of the reasoning used previously. Nonetheless, we make another brief exposition of the principle used to derive such formulae.

1 Introduction

The motivation to create a formula that could work for $\sum_j 1/(a\boldsymbol{i}j+b)^k$ came from the realization that, unlike $\sum_j 1/(j^2-1)$, a sum such as $\sum_j 1/(j^2+1)$ can't be derived from the formulae we created in *Generalized Harmonic Progression*.[4]

If we can produce a formula for $\sum_j 1/(a\boldsymbol{i}j+b)$, with integer a and complex b, we may be able to produce formulae for most, if not all, sums of the type $\sum_j 1/p(j)$, where $p(j)$ is any polynomial with complex coefficients.

As we will see, the choice of integer a and complex b is good enough in most of the cases, and when it's not, we can fall back on the formulae from the previous paper,[4] which assume a and b to be integers.

Before we begin, let's recap some results from [4] for a bit of context. We've seen that the sum of the terms of a generalized harmonic progression with integer parameters, a and b, is given by:

$$\sum_{j=1}^{n}\frac{1}{(aj+b)^{2k}}=-\frac{1}{2b^{2k}}+\frac{1}{2(an+b)^{2k}}$$
$$-(-1)^k(2\pi)^{2k}\int_0^1\sum_{j=0}^{k}\frac{B_{2j}\left(2-2^{2j}\right)(1-u)^{2k-2j}}{(2j)!(2k-2j)!}\cos\pi(an+2b)u\sin\pi anu\cot\pi au\,du,$$

for the even powers, and

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(an+b)^{2k+1}}$$
$$+ (-1)^k (2\pi)^{2k+1} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)(1-u)^{2k+1-2j}}{(2j)!(2k+1-2j)!} \sin \pi(an+2b)u \sin \pi anu \cot \pi au \, du,$$

for the odd powers.

Notice that there can be singularities on both sides of these equations if, for example, $aj+b=0$ for some $j \leq n$ (that is, a divides b and they have different signs). In those cases, if we ignore the singularities wherever they occur the formula still holds. That means that, for instance, if $b=0$ then the above formulae reduce to the formulae for the generalized harmonic numbers from [3].

To see why singularities can be removed, let's transform the formula for odd powers by turning the product of sines into a sum of cosines. That way, the formula becomes:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(an+b)^{2k+1}}$$
$$- \frac{(-1)^k (2\pi)^{2k+1}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)(1-u)^{2k+1-2j}}{(2j)!(2k+1-2j)!} \left(\cos 2\pi(an+b)u - \cos 2\pi bu\right) \cot \pi au \, du$$

From the above formula, it's easier to see why it still holds when we ignore singularities. For example, let's take the forward difference of $\sum_{j=1}^{n} 1/(aj+b)$:

$$2\pi \int_0^1 (1-u)\left[\cos 2\pi(an+b)u - \cos 2\pi(a(n-1)+b)u\right] \cot \pi a(1-u) \, du = -\frac{1}{an+b} - \frac{1}{a(n-1)+b}$$

Now, if for instance $an+b=0$, the left-hand side of the above equation yields $1/a$, even though $1/(an+b)$ is a singularity on the right-hand side. Therefore, by ignoring the singularity, the equation still holds.

Here we make use again of Faulhaber's formula[2] for the sum of powers of the first n positive integers, which are given by:

$$\sum_{k=1}^{n} k^{2i} = \frac{n^{2i}}{2} + \sum_{j=0}^{i} \frac{(2i)! B_{2j} n^{2i+1-2j}}{(2j)!(2i+1-2j)!} \tag{1}$$

$$\sum_{k=1}^{n} k^{2i+1} = \frac{n^{2i+1}}{2} + \sum_{j=0}^{i} \frac{(2i+1)! B_{2j} n^{2i+2-2j}}{(2j)!(2i+2-2j)!} \tag{2}$$

where B_{2j} are the non-null Bernoulli numbers.

2 Formula Rationale

The reasoning to build a formula for $\sum_j 1/(a\boldsymbol{i}j+b)$ is to use the Taylor series expansion of $\mathrm{e}^{2\pi(a\boldsymbol{i}k+b)}$, and seize upon the fact that it's constant for any integer a and complex b:

$$\mathrm{e}^{2\pi(a\boldsymbol{i}k+b)} = \mathrm{e}^{2\pi b} \Rightarrow \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}(2\pi(a\boldsymbol{i}k+b))^i = \mathrm{e}^{2\pi b}, \text{ if } a \text{ is integer}$$

Starting from the above initial equation, we end up with the below after a few calculations:

$$\frac{1}{a\boldsymbol{i}k+b}\left(\mathrm{e}^{2\pi b}-1\right) = 2\pi \sum_{i=0}^{\infty}\left(\frac{(2\pi(a\boldsymbol{i}k+b))^{2i}}{(2i+1)!} + \frac{(2\pi(a\boldsymbol{i}k+b))^{2i+1}}{(2i+2)!}\right) \tag{3}$$

Note we now have two power series, as opposed to one previously.

At this point we notice that this formula won't work if $\mathrm{e}^{2\pi b}=1$, which happens when b is a pure complex number with integer imaginary part ($\Re(b)=0$ and $\Im(b)\in\mathbb{Z}$). In most situations, we can work around this issue by making b non-integer, by dividing b by a. By the same token, the formula won't apply if a is not integer ($a \notin \mathbb{Z}$). In that case we can also make a integer by dividing both parameters by a:

$$\sum_{k=1}^{n}\frac{1}{a\boldsymbol{i}k+b} = \frac{1}{a}\sum_{k=1}^{n}\frac{1}{\boldsymbol{i}k+b/a}$$

However, in both cases this work-around won't work if $\boldsymbol{i}b/a$ is integer, including 0. In those cases we can resort to the previous formulae.

3 Baseline Functions

The baseline functions are holomorphic functions whose power series take similar forms as the power series we obtain when we expand the initial equation from the previous section.

For the complex harmonic progression, the baseline functions are entirely analogous to the ones used in [4] (we just need to replace a with $\boldsymbol{i}a$).

Again, we can obtain them using the so-called Lagrange's trigonometric identities (which stem from the sum of the terms of a geometric progression with complex terms), along with the identities $\cos(x+y)=\cos x\cos y-\sin x\sin y$ or $\sin(x+y)=\sin x\cos y+\cos x\sin y$:

$$\sum_{j=1}^{k}\cos\frac{2\pi n(a\boldsymbol{i}j+b)}{k} = -\frac{1}{2}\cos\frac{2\pi bn}{k}+\frac{1}{2}\cos 2\pi n\left(a\boldsymbol{i}+\frac{b}{k}\right)+\cos\pi n\left(a\boldsymbol{i}+\frac{2b}{k}\right)\sin\pi a\boldsymbol{i}n\cot\frac{\pi a\boldsymbol{i}n}{k}$$

$$\sum_{j=1}^{k}\sin\frac{2\pi n(a\boldsymbol{i}j+b)}{k} = -\frac{1}{2}\sin\frac{2\pi bn}{k}+\frac{1}{2}\sin 2\pi n\left(a\boldsymbol{i}+\frac{b}{k}\right)+\sin\pi n\left(a\boldsymbol{i}+\frac{2b}{k}\right)\sin\pi a\boldsymbol{i}n\cot\frac{\pi a\boldsymbol{i}n}{k}$$

We can then derive power series for the left-hand side of the above equations with the employment of (1) and (2), and come up with the following power series for each function on the right-hand side (these equations aren't numbered as it's plain to see which is which):

$$\sum_{i=0}^{\infty}(-1)^i\left(\frac{2\pi bn}{k}\right)^{2i}\left(\sum_{j=0}^{i}\frac{(a\boldsymbol{i}k/b)^{2j}}{(2j)!(2i-2j)!}+\sum_{j=0}^{i-1}\frac{(a\boldsymbol{i}k/b)^{2j+1}}{(2j+1)!(2i-2j-1)!}\right)=\cos 2\pi n\left(a\boldsymbol{i}+\frac{b}{k}\right)$$

$$\sum_{i=0}^{\infty}(-1)^i\left(\frac{2\pi bn}{k}\right)^{2i+1}\left(\sum_{j=0}^{i}\frac{(a\boldsymbol{i}k/b)^{2j}}{(2j)!(2i+1-2j)!}+\sum_{j=0}^{i}\frac{(a\boldsymbol{i}k/b)^{2j+1}}{(2j+1)!(2i-2j)!}\right)=\sin 2\pi n\left(a\boldsymbol{i}+\frac{b}{k}\right)$$

$$\sum_{i=0}^{\infty}(-1)^i\left(\frac{2\pi bn}{k}\right)^{2i}\left(\sum_{j=0}^{i}\frac{(a\boldsymbol{i}/b)^{2j}}{(2i-2j)!}\sum_{p=0}^{j}\frac{B_{2p}k^{2j+1-2p}}{(2j+1-2p)!(2p)!}+\sum_{j=0}^{i-1}\frac{(a\boldsymbol{i}/b)^{2j+1}}{(2i-2j-1)!}\sum_{p=0}^{j}\frac{B_{2p}k^{2j+2-2p}}{(2j+2-2p)!(2p)!}\right)$$
$$=\cos\pi n\left(a\boldsymbol{i}+\frac{2b}{k}\right)\sin\pi a\boldsymbol{i}n\cot\frac{\pi a\boldsymbol{i}n}{k}$$

$$\sum_{i=0}^{\infty}(-1)^i\left(\frac{2\pi bn}{k}\right)^{2i+1}\left(\sum_{j=0}^{i}\frac{(a\boldsymbol{i}/b)^{2j}}{(2i+1-2j)!}\sum_{p=0}^{j}\frac{B_{2p}k^{2j+1-2p}}{(2j+1-2p)!(2p)!}+\sum_{j=0}^{i}\frac{(a\boldsymbol{i}/b)^{2j+1}}{(2i-2j)!}\sum_{p=0}^{j}\frac{B_{2p}k^{2j+2-2p}}{(2j+2-2p)!(2p)!}\right)$$
$$=\sin\pi n\left(a\boldsymbol{i}+\frac{2b}{k}\right)\sin\pi a\boldsymbol{i}n\cot\frac{\pi a\boldsymbol{i}n}{k}$$

4 Complex Harmonic Progression

Going back to equation (3), we need to sum it over k, expand $(a\boldsymbol{i}k+b)^{2i}$ and $(a\boldsymbol{i}k+b)^{2i+1}$ with Newton's binomial theorem, and replace $\sum_j k^{2j}$ and $\sum_j k^{2j+1}$ with their respective Faulhaber's formulae. After we make all possible simplifications, we end up with a bunch of power series.

First we have the independent term, given by:

$$-\pi\sum_{i=0}^{\infty}\left(\frac{(2\pi b)^{2i}}{(2i+1)!}+\frac{(2\pi b)^{2i+1}}{(2i+2)!}\right)=-\frac{\mathrm{e}^{2\pi b}-1}{2b}$$

For the other, more complicated, power series, we can obtain closed-forms with the aid of their baseline functions:

$$\pi\sum_{i=0}^{\infty}\frac{(2\pi b)^{2i}}{2i+1}\left(\sum_{j=0}^{i}\frac{(a\boldsymbol{i}n/b)^{2j}}{(2j)!(2i-2j)!}+\sum_{j=0}^{i-1}\frac{(a\boldsymbol{i}n/b)^{2j+1}}{(2j+1)!(2i-2j-1)!}\right)=-\frac{\boldsymbol{i}\pi}{n}\int_0^{\boldsymbol{i}n}\cos 2\pi x\left(a\boldsymbol{i}+\frac{b}{n}\right)dx$$

$$\pi\sum_{i=0}^{\infty}\frac{(2\pi b)^{2i+1}}{2i+2}\left(\sum_{j=0}^{i}\frac{(a\boldsymbol{i}n/b)^{2j}}{(2j)!(2i+1-2j)!}+\sum_{j=0}^{i}\frac{(a\boldsymbol{i}n/b)^{2j+1}}{(2j+1)!(2i-2j)!}\right)=-\frac{\pi}{n}\int_{0}^{\boldsymbol{i}n}\sin 2\pi x\left(a\boldsymbol{i}+\frac{b}{n}\right)dx$$

$$2\pi\sum_{i=0}^{\infty}\frac{(2\pi b)^{2i}}{2i+1}\left(\sum_{j=0}^{i}\frac{(a\boldsymbol{i}/b)^{2j}}{(2i-2j)!}\sum_{p=0}^{j}\frac{B_{2p}n^{2j+1-2p}}{(2j+1-2p)!(2p)!}+\sum_{j=0}^{i-1}\frac{(a\boldsymbol{i}/b)^{2j+1}}{(2i-2j-1)!}\sum_{p=0}^{j}\frac{B_{2p}n^{2j+2-2p}}{(2j+2-2p)!(2p)!}\right)$$
$$=-\frac{2\pi\boldsymbol{i}}{n}\int_{0}^{\boldsymbol{i}n}\cos\pi x\left(a\boldsymbol{i}+\frac{2b}{n}\right)\sin\left(\pi a\boldsymbol{i}x\right)\cot\frac{\pi a\boldsymbol{i}x}{n}\,dx$$

$$2\pi\sum_{i=0}^{\infty}\frac{(2\pi b)^{2i+1}}{2i+2}\left(\sum_{j=0}^{i}\frac{(a\boldsymbol{i}/b)^{2j}}{(2i+1-2j)!}\sum_{p=0}^{j}\frac{B_{2p}n^{2j+1-2p}}{(2j+1-2p)!(2p)!}+\sum_{j=0}^{i}\frac{(a\boldsymbol{i}/b)^{2j+1}}{(2i-2j)!}\sum_{p=0}^{j}\frac{B_{2p}n^{2j+2-2p}}{(2j+2-2p)!(2p)!}\right)$$
$$=-\frac{2\pi}{n}\int_{0}^{\boldsymbol{i}n}\sin\pi x\left(a\boldsymbol{i}+\frac{2b}{n}\right)\sin\left(\pi a\boldsymbol{i}x\right)\cot\frac{\pi a\boldsymbol{i}x}{n}\,dx$$

So, by putting it all together and making a change of variables on the integral, we get the below closed-form, which works as long as a is integer and $\boldsymbol{i}b$ is not integer:

$$\sum_{k=1}^{n}\frac{1}{a\boldsymbol{i}k+b}=-\frac{1}{2b}+\frac{1}{2(a\boldsymbol{i}n+b)}+\frac{2\pi}{\mathrm{e}^{2\pi b}-1}\int_{0}^{1}\mathrm{e}^{\pi(a\boldsymbol{i}n+2b)u}\sin\pi anu\cot\pi au\,du$$

In this formula, the independent term came from the first power series, the second term from the next two power series and the third term from the last two power series, which is somewhat evident.

As discussed previously, provided that $\boldsymbol{i}b/a\notin\mathbb{Z}$, we can remove the restriction that a be an integer by transforming the formula, for example, as follows:

$$\frac{1}{a}\sum_{k=1}^{n}\frac{1}{\boldsymbol{i}k+b/a}=-\frac{1}{2b}+\frac{1}{2(a\boldsymbol{i}n+b)}+\frac{2\pi}{a\left(\mathrm{e}^{2\pi b/a}-1\right)}\int_{0}^{1}\mathrm{e}^{\pi(\boldsymbol{i}n+2b/a)u}\sin\pi nu\cot\pi u\,du,$$

which holds for any a, $b\in\mathbb{C}$. There are endless ways of doing that though, the example is just one of the possibilities.

4.1 Application Examples

Let's see two examples on how we can apply the formula we just got to obtain sums that weren't possible with the formulae we created in the previous paper. We won't show all the plethora of calculations here, if the reader is interested they can do their own math.

The first example is $\sum_k 1/(k^2+1)$. The first step to derive its formula is to decompose $1/(k^2+1)$ into a linear combination:

$$\frac{1}{k^2+1} = \frac{1}{2(\boldsymbol{i}k+1)} - \frac{1}{2(\boldsymbol{i}k-1)}$$

Thus we get the below closed-form:

$$\sum_{k=1}^{n} \frac{1}{k^2+1} = -\frac{1}{2} + \frac{1}{2(n^2+1)} + \frac{4\pi}{\mathrm{e}^{4\pi}-1} \int_0^1 \mathrm{e}^{4\pi u} \cos 2\pi n(1-u) \sin 2\pi nu \cot 2\pi u \, du$$

Generally, if we want to know what the formula for $\sum_{j=1}^{n} 1/(j^{2k}+1)$ looks like, we need to find all the $2k$ complex roots of $x^{2k} = -1$ (say they are x_j), and a linear combination such that $\sum_{j=1}^{2k} c_j/(x-x_j) = 1/(x^{2k}+1)$. We won't go over this exercise here though, but it shouldn't be that hard, presumably.

The second example is $\sum_k 1/(k^2+2k+2)$. It can be linearly decomposed as:

$$\frac{1}{k^2+2k+2} = \frac{1}{2(\boldsymbol{i}k+1+\boldsymbol{i})} - \frac{1}{2(\boldsymbol{i}k-1+\boldsymbol{i})},$$

which implies the below closed-form:

$$\sum_{k=1}^{n} \frac{1}{k^2+2k+2} = -\frac{1}{4} + \frac{1}{2(n^2+2n+2)} + \frac{2\pi}{\mathrm{e}^{4\pi}-1} \int_0^1 \left(\mathrm{e}^{4\pi(1-u)} + \mathrm{e}^{4\pi u}\right) \cos 2\pi(n+2)u \sin 2\pi n(1-u) \cot 2\pi(1-u) \, du$$

So, it turns out the real problem here is finding out the linear combination that yields the polynomial that we're interested in.

5 Generalization

To generalize the previous formula we got, first we note that we get a recursion at each new step. For example, for the third power we have:

$$\left(\mathrm{e}^{2\pi b}-1\right) \sum_{k=1}^{n} \frac{1}{(a\boldsymbol{i}k+b)^3} = \sum_{k=1}^{n} \left(\frac{(2\pi)^2}{2!} \frac{1}{a\boldsymbol{i}k+b} + \frac{2\pi}{1!} \frac{1}{(a\boldsymbol{i}k+b)^2} + (2\pi)^3 \sum_{i=0}^{\infty} \frac{(2\pi(a\boldsymbol{i}k+b))^{2i}}{(2i+3)!} + \frac{(2\pi(a\boldsymbol{i}k+b))^{2i+1}}{(2i+4)!} \right)$$

That means that we only need to worry about the last sum in the above equation. As before, this recurrence is such that in the final formula the terms that go outside of the integral reduce to very simple forms, the integral being the only challenging part.

Let $H_k(u)$ be the polynomial in u that goes within the integral. Then for integer a and non-integer $\boldsymbol{i}b/a$:

$$\sum_{j=1}^{n} \frac{1}{(a\boldsymbol{i}j+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(a\boldsymbol{i}n+b)^k} + (2\pi)^k \int_0^1 H_k(u) \mathrm{e}^{\pi(a\boldsymbol{i}n+2b)u} \sin(\pi anu) \cot(\pi au)\, du,$$

where $H_k(u)$ is given by the recurrence equation:

$$\left(\mathrm{e}^{2\pi b}-1\right) H_k(u) = \begin{cases} 1, & \text{if } k=1 \\ \frac{(1-u)^{k-1}}{(k-1)!} + \sum_{j=1}^{k-1} \frac{1}{(k-j)!} H_j(u), & \text{if integer } \ k>1 \end{cases}$$

and where term $(1-u)^{k-1}/(k-1)!$ came from the observation of the patterns.

Now we only need to solve this recurrence. Let $p(x)$ be the generating function of $H_k(u)$. Then, looking at the recurrence, we conclude that:

$$\left(\mathrm{e}^{2\pi b}-1\right) p(x) - \left(-1+\mathrm{e}^{x}\right) p(x) = x\,\mathrm{e}^{(1-u)x} \Rightarrow p(x) = -\frac{x\,\mathrm{e}^{(1-u)x}}{\mathrm{e}^{x}-\mathrm{e}^{2\pi b}}$$

The solution of this equation is not trivial, that is, the general term of the power series of $p(x)$ is not simple, unfortunately. Nonetheless, it gives us the following generalization, after we drop the requirement that a be integer.

Let ϕ be the Hurwitz-Lerch transcendent function:

$$\phi(z,s,\alpha) = \sum_{k=0}^{\infty} \frac{z^k}{(k+\alpha)^s}$$

Then, if $\boldsymbol{i}b/a \notin \mathbb{Z}$:

$$\sum_{j=1}^{n} \frac{1}{(a\boldsymbol{i}j+b)^k} = -\frac{1}{2b^k} + \frac{1}{2(a\boldsymbol{i}n+b)^k} +$$

$$\mathrm{e}^{-2\pi b/a} \left(\frac{2\pi}{a}\right)^k \int_0^1 \sum_{j=1}^{k} \frac{\phi\left(\mathrm{e}^{-2\pi b/a}, -j+1, 0\right)(1-u)^{k-j}}{(j-1)!(k-j)!} \mathrm{e}^{\pi(\boldsymbol{i}n+2b/a)u} \sin \pi nu \cot \pi u\, du$$

It'd also work to use Lerch's zeta function, as $\phi\left(\mathrm{e}^{-2\pi b/a}, -j+1, 0\right) = L\left(\boldsymbol{i}b/a, 0, -j+1\right)$.

References

[1] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (9th printing ed.), New York: Dover,* 1972.

[2] John H. Conway, Richard Guy, *The Book of Numbers, Springer,* 1996 ISBN 0-387-97993-X.

[3] Risomar Sousa, Jose *Generalized Harmonic Numbers Revisited, eprint arXiv:1810.07877,* 2018 2018arXiv181007877R.

[4] Risomar Sousa, Jose *Generalized Harmonic Progression, eprint arXiv:1811.11305,* 2018 2018arXiv181111305R.