Generalized Harmonic Progression

Jose Risomar Sousa

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Abstract

This paper presents formulae for the sum of the terms of a harmonic progression of order k with integer parameters, more precisely, $\sum_{j=1}^{n} 1/(aj+b)^k$, and for the partial sums of two Fourier series associated with them, denoted here by $C_k^m(a,b,n)$ and $S_k^m(a,b,n)$ (here, the term "harmonic progression" is used loosely, as for some parameter choices, a and b, the result may not be a harmonic progression). We provide a generalization of the formulae we created in *Generalized Harmonic Numbers Revisited*, which was achieved by using an extension of the reasoning employed before.

1 Introduction

Building upon the results of [3], *Generalized Harmonic Numbers Revisited*, this new paper demonstrates how to obtain exact formulae for the sum of the first n terms of a harmonic progression of order k with integer parameters, a and b:

$$HP_k(n) = \sum_{j=1}^{n} \frac{1}{(aj+b)^k}$$

Even though formulae for $HP_k(n)$ can probably be created using the digamma function, ψ, the ones we derive here are arguably more interesting, and they must be the first such formulae other than ψ.

We also create formulae for the partial sums of two Fourier series associated with $HP_k(n)$:

$$C_k^m(a,b,n) = \sum_{j=1}^{n} \frac{1}{(aj+b)^k} \cos\frac{2\pi(aj+b)}{m}, \text{ and } S_k^m(a,b,n) = \sum_{j=1}^{n} \frac{1}{(aj+b)^k} \sin\frac{2\pi(aj+b)}{m}$$

My first manuscript? received some criticism for including too many details, so in this one we omit unnecessary details, which can be easily understood from a reading of [3], for the interested reader. It also received criticism for being long and repetitive. In all honesty, that's because it was my very first paper, and I'm neither an academic insider nor a professor (having an advisor must make a big difference when it comes to formatting a paper). So, it turns out that the first paper wasn't suitable for an academic publication, but I hope these findings may be useful and find an application in some important problem someday. Although not excellent in exposition, paper [3] was about novelty, not about presentation.

This new manuscript, however, is concise and succinct, it focuses solely on closed-forms for $HP_k(n)$, $C_k^m(a,b,n)$ and $S_k^m(a,b,n)$. The question on the limits of these expressions as n approaches infinity will be addressed in a subsequent paper.

We make use again of Faulhaber's formula? for the sum of the i-th powers of the first n positive integers:

$$\sum_{k=1}^{n} k^i = \sum_{j=0}^{i} \frac{(-1)^j i! B_j n^{i+1-j}}{(i+1-j)!j!} \ ,$$

where B_j are the Bernoulli numbers.

Since odd Bernoulli numbers are always 0, except for B_1, we can simplify the above formula for even and odd powers as follows:

$$\sum_{k=1}^{n} k^{2i} = \frac{n^{2i}}{2} + \sum_{j=0}^{i} \frac{(2i)! B_{2j} n^{2i+1-2j}}{(2j)!(2i+1-2j)!} \tag{1}$$

$$\sum_{k=1}^{n} k^{2i+1} = \frac{n^{2i+1}}{2} + \sum_{j=0}^{i} \frac{(2i+1)! B_{2j} n^{2i+2-2j}}{(2j)!(2i+2-2j)!} \tag{2}$$

2 Lagrange's Identities

In [3] we introduced an indicator function, k divides n ($\mathbb{1}_{k|n}$), and its analog as key components of the method used to solve the harmonic numbers. For the harmonic progression, we need to modify those functions and obtain their Taylor series, though unlike in the first paper here we focus on only one of them, as the other one has an analogous behavior.

The below is the function we're interested in, for the odd case. We can write this sum as a closed-form by using Lagrange's trigonometric identities:

$$\sum_{j=1}^{k} \sin \frac{2\pi n(aj+b)}{k} = -\frac{1}{2} \sin \frac{2\pi bn}{k} + \frac{1}{2} \sin 2\pi n \left(a + \frac{b}{k}\right) + \sin \pi n \left(a + \frac{2b}{k}\right) \sin \pi an \cot \frac{\pi an}{k}$$

Now, we can derive a power series for the left-hand side of the above equation with the employment of (??), and through comparison we can deduce the following power series for each function on the right-hand side (they hold for all real n, a and b and for all integer $k > 0$):

$$\sum_{i=0}^{\infty}(-1)^i\left(\frac{2\pi bn}{k}\right)^{2i+1}\sum_{j=0}^{i}\left(\frac{(ak/b)^{2j}}{(2j)!(2i+1-2j)!}+\frac{(ak/b)^{2j+1}}{(2j+1)!(2i-2j)!}\right)$$
$$=\sin 2\pi n\left(a+\frac{b}{k}\right) \quad (3)$$

$$\sum_{i=0}^{\infty}(-1)^i\left(\frac{2\pi bn}{k}\right)^{2i+1}\sum_{j=0}^{i}\left(\frac{(a/b)^{2j}}{(2i+1-2j)!}\sum_{p=0}^{j}\frac{B_{2p}k^{2j+1-2p}}{(2j+1-2p)!(2p)!}+\frac{(a/b)^{2j+1}}{(2i-2j)!}\sum_{p=0}^{j}\frac{B_{2p}k^{2j+2-2p}}{(2j+2-2p)!(2p)!}\right)$$
$$=\sin\pi n\left(a+\frac{2b}{k}\right)\sin\pi an\cot\frac{\pi an}{k} \quad (4)$$

For the harmonic progressions of even order, the analogous function is:

$$\sum_{j=1}^{k}\cos\frac{2\pi n(aj+b)}{k}=-\frac{1}{2}\cos\frac{2\pi bn}{k}+\frac{1}{2}\cos 2\pi n\left(a+\frac{b}{k}\right)+\cos\pi n\left(a+\frac{2b}{k}\right)\sin\pi an\cot\frac{\pi an}{k}$$

3 Formula Rationale

The rationale to build a formula for $HP_k(n)$ is to use the Taylor series expansion of $\sin 2\pi(ak+b)$,[?] and seize upon the fact that it's 0 for all integer $ak+b$, hence the need for a and b to be integers (note the k in the sum is not the same k used as subscript on $HP_k(n)$):

$$\sin 2\pi(ak+b)=0\Rightarrow 2\pi(ak+b)=-\sum_{i=1}^{\infty}\frac{(-1)^i}{(2i+1)!}(2\pi(ak+b))^{2i+1} \quad (5)$$

When we divide both sides of (**??**) by $2\pi(ak+b)^2$ we end up with a power series for $1/(ak+b)$ that only holds for integer $ak+b$.

Besides, on the right-hand side of the resulting equation the exponents of $ak+b$ are positive integers, allowing us to apply Faulhaber's formula, mentioned in the introduction. By doing that we end up with very convoluted power series that can be turned into integrals by means of equations (**??**) and (**??**), which were derived using Lagrange's identities. That sums up the rationale.

4 Harmonic Progression

We start by dividing both sides of (**??**) by $2\pi(ak+b)^2$:

$$\frac{1}{ak+b}=\sum_{i=0}^{\infty}\frac{(-1)^i(2\pi)^{2i+2}(ak+b)^{2i+1}}{(2i+3)!} \quad (6)$$

Now, we sum $1/(ak+b)$ over k and expand $(ak+b)^{2i+1}$ according to the binomial theorem and apply the respective Faulhaber's formulae, ending up with the following power series after all the calculations are performed:

$$\sum_{k=1}^{n}\frac{1}{ak+b} = -\frac{1}{2b}+\frac{1}{2}\sum_{i=0}^{\infty}(-1)^i\frac{(2\pi b)^{2i+2}(2i+1)!}{b(2i+3)!}\sum_{j=0}^{i}\left(\frac{(an/b)^{2j}}{(2j)!(2i+1-2j)!}+\frac{(an/b)^{2j+1}}{(2j+1)!(2i-2j)!}\right)+$$
$$+\sum_{i=0}^{\infty}(-1)^i\frac{(2\pi b)^{2i+2}(2i+1)!}{b(2i+3)!}\sum_{j=0}^{i}\left(\frac{(a/b)^{2j}}{(2i+1-2j)!}\sum_{p=0}^{j}\frac{B_{2p}n^{2j+1-2p}}{(2j+1-2p)!(2p)!}+\frac{(a/b)^{2j+1}}{(2i-2j)!}\sum_{p=0}^{j}\frac{B_{2p}n^{2j+2-2p}}{(2j+2-2p)!(2p)!}\right)$$

The above sums can be manipulated conveniently and then obtained from (**??**) and (**??**), the two equations we derived previously, by means of decompositions into linear combinations followed by integrations. Though we have omitted the details here, the reader can refer to the precursor paper[?] for a more detailed description of the steps involved.

After all the appropriate calculations are performed, we end up with:

$$\frac{1}{2b}\sum_{i=0}^{\infty}(-1)^i\frac{(2\pi b)^{2i+2}(2i+1)!}{(2i+3)!}\sum_{j=0}^{i}\left(\frac{(an/b)^{2j}}{(2j)!(2i+1-2j)!}+\frac{(an/b)^{2j+1}}{(2j+1)!(2i-2j)!}\right)=\frac{2\pi(an+b)-\sin 2\pi(an+b)}{4\pi(an+b)^2}$$

$$\sum_{i=0}^{\infty}(-1)^i\frac{(2\pi b)^{2i+2}(2i+1)!}{b(2i+3)!}\sum_{j=0}^{i}\left(\frac{(a/b)^{2j}}{(2i+1-2j)!}\sum_{p=0}^{j}\frac{B_{2p}n^{2j+1-2p}}{(2j+1-2p)!(2p)!}+\frac{(a/b)^{2j+1}}{(2i-2j)!}\sum_{p=0}^{j}\frac{B_{2p}n^{2j+2-2p}}{(2j+2-2p)!(2p)!}\right)$$
$$= 2\pi\int_0^1(1-u)\sin\pi anu\sin\pi(an+2b)u\cot\pi au\,du$$

Now, by summing up all the results (disregarding the sine of multiples of π), we arrive at a formula for $HP(n)$:

$$\sum_{k=1}^{n}\frac{1}{ak+b} = -\frac{1}{2b}+\frac{1}{2(an+b)}+2\pi\int_0^1(1-u)\sin\pi(an+2b)u\sin\pi anu\cot\pi au\,du$$

In the next sections we state a generalization of this result without getting into too much detail (again, the reader may refer to [3] if needed).

4.1 General Formula

If we keep dividing (**??**) by $ak+b$, we get similar recursions to the ones we got for the generalized harmonic numbers. All the results presented next follow from analogous reasonings to those from reference [3].

4.2 Harmonic Progression of Order $2k$

Let $HP_{2j}(n)$ be the harmonic progression of order $(2j)$-th:

$$HP_{2j}(n) = \sum_{j=1}^{n} \frac{1}{(an+b)^{2j}}$$

Then we have the following recursion for $HP_{2k}(n)$:

$$HP_{2k}(n) = -\frac{1}{2b^{2k}} \sum_{j=0}^{k} \frac{(-1)^j (2\pi b)^{2j}}{(2j+1)!} + \frac{1}{2(an+b)^{2k}} \sum_{j=0}^{k} \frac{(-1)^j (2\pi(an+b))^{2j}}{(2j+1)!}$$
$$-\sum_{j=0}^{k-1} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!} HP_{2j}(n) - \frac{(-1)^k (2\pi)^{2k}}{(2k)!} \int_0^1 (1-u)^{2k} \cos \pi(an+2b)u \sin \pi anu \cot \pi au \, du$$

Note that $HP_0(n) = 0$ for all positive integer n (just like $H_0(n) = 0$, previously). Therefore, from the recursion, we conclude that for all integer $a \neq 0$, $b \neq 0$ and $k \geq 0$:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(an+b)^{2k}} +$$
$$-(-1)^k (2\pi)^{2k} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)(1-u)^{2k-2j}}{(2j)!(2k-2j)!} \cos \pi(an+2b)u \sin \pi anu \cot \pi au \, du$$

And if we turn the product of cosine and sine into a sum of sines, we derive another, perhaps more useful, way to express this formula:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} = -\frac{1}{2b^{2k}} + \frac{1}{2(an+b)^{2k}} +$$
$$-\frac{(-1)^k (2\pi)^{2k}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)(1-u)^{2k-2j}}{(2j)!(2k-2j)!} \left(\sin 2\pi(an+b)u - \sin 2\pi bu\right) \cot \pi au \, du$$

This formula also works for the generalized harmonic numbers ($a = 1, b = 0$), if the term $-1/(2b^{2k})$ is disregarded. In fact, if we disregard any term of the equation that has a null denominator, the equation still holds.

As an example, $HP_2(n)$ is given by:

$$\sum_{k=1}^{n} \frac{1}{(ak+b)^2} = -\frac{1}{2b^2} + \frac{1}{2(an+b)^2} +$$
$$+ 4\pi^2 \int_0^1 \left(\frac{1}{3} - u + \frac{u^2}{2}\right) \cos \pi(an+2b)u \sin \pi anu \cot \pi au \, du$$

4.3 Harmonic Progression of Order $2k+1$

We have the following recursion for $HP_{2k+1}(n)$:

$$HP_{2k+1}(n) = -\frac{1}{2b^{2k+1}} \sum_{j=0}^{k} \frac{(-1)^j (2\pi b)^{2j}}{(2j+1)!} + \frac{1}{2(an+b)^{2k+1}} \sum_{j=0}^{k} \frac{(-1)^j (2\pi(an+b))^{2j}}{(2j+1)!}$$

$$-\sum_{j=0}^{k-1} \frac{(-1)^{k-j}(2\pi)^{2k-2j}}{(2k+1-2j)!} HP_{2j+1}(n) + \frac{(-1)^k (2\pi)^{2k+1}}{(2k+1)!} \int_0^1 (1-u)^{2k+1} \sin \pi(an+2b)u \sin \pi anu \cot \pi au \, du$$

Therefore, for all integer $a \neq 0$, $b \neq 0$ and $k \geq 0$:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(an+b)^{2k+1}} +$$

$$+ (-1)^k (2\pi)^{2k+1} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)(1-u)^{2k+1-2j}}{(2j)!(2k+1-2j)!} \sin \pi(an+2b)u \sin \pi anu \cot \pi au \, du$$

If we turn the product of sines into a sum of cosines, we get:

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} = -\frac{1}{2b^{2k+1}} + \frac{1}{2(an+b)^{2k+1}} +$$

$$-\frac{(-1)^k (2\pi)^{2k+1}}{2} \int_0^1 \sum_{j=0}^{k} \frac{B_{2j}\left(2-2^{2j}\right)(1-u)^{2k+1-2j}}{(2j)!(2k+1-2j)!} (\cos 2\pi(an+b)u - \cos 2\pi bu) \cot \pi au \, du$$

As an example, $HP_3(n)$ is given by:

$$\sum_{k=1}^{n} \frac{1}{(ak+b)^3} = -\frac{1}{2b^3} + \frac{1}{2(an+b)^3} +$$

$$-\frac{4\pi^3}{3} \int_0^1 \left(-2u + 3u^2 - u^3\right) \sin \pi anu \sin \pi(an+2b)u \cot \pi au \, du$$

5 Associated Fourier Series

Let's briefly recall the formulae we found for $C_k^m(n)$ and $S_k^m(n)$, the partial sums of the Fourier series associated with the generalized harmonic numbers, $H_k(n)$, from reference [3].

Next to each one we show their harmonic progression analogs, $C_k^m(a,b,n)$ and $S_k^m(a,b,n)$, which are based entirely on analogy.

The below expressions hold for all complex m, a and b (unlike the $HP_k(n)$ formulae), and for all integer $n \geq 1$. As mentioned before, if $b = 0$, we can disregard the term that has b in the denominator, and the equation still holds. Again, by definition $H_0(n) = 0$ and $HP_0(n) = 0$ for all positive integer, so they actually have no effect in the sums.

5.1 $C_{2k}^m(n)$ and $C_{2k}^m(a,b,n)$

For all integer $k \geq 1$:

$$\sum_{j=1}^{n} \frac{1}{j^{2k}} \cos\frac{2\pi j}{m} = \frac{1}{2n^{2k}}\left(\cos\frac{2\pi n}{m} - \sum_{j=0}^{k} \frac{(-1)^j(\frac{2\pi n}{m})^{2j}}{(2j)!}\right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!} H_{2j}(n) + \frac{(-1)^k(\frac{2\pi}{m})^{2k}}{2(2k-1)!}\int_0^1 (1-u)^{2k-1} \sin\frac{2\pi n u}{m} \cot\frac{\pi u}{m}\, du$$

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k}} \cos\frac{2\pi(aj+b)}{m} = -\frac{1}{2b^{2k}}\left(\cos\frac{2\pi b}{m} - \sum_{j=0}^{k} \frac{(-1)^j(\frac{2\pi b}{m})^{2j}}{(2j)!}\right) + \frac{1}{2(an+b)^{2k}}\left(\cos\frac{2\pi(an+b)}{m} - \sum_{j=0}^{k} \frac{(-1)^j(\frac{2\pi(an+b)}{m})^{2j}}{(2j)!}\right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!} HP_{2j}(n) + \frac{(-1)^k(\frac{2\pi}{m})^{2k}}{2(2k-1)!}\int_0^1 (1-u)^{2k-1}\left(\sin\frac{2\pi(an+b)u}{m} - \sin\frac{2\pi b u}{m}\right) \cot\frac{\pi a u}{m}\, du$$

5.2 $S_{2k+1}^m(n)$ and $S_{2k+1}^m(a,b,n)$

For all integer $k \geq 0$:

$$\sum_{j=1}^{n} \frac{1}{j^{2k+1}} \sin\frac{2\pi j}{m} = \frac{1}{2n^{2k+1}}\left(\sin\frac{2\pi n}{m} - \sum_{j=0}^{k} \frac{(-1)^j(\frac{2\pi n}{m})^{2j+1}}{(2j+1)!}\right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k+1-2j}}{(2k+1-2j)!} H_{2j}(n) + \frac{(-1)^k(\frac{2\pi}{m})^{2k+1}}{2(2k)!}\int_0^1 (1-u)^{2k} \sin\frac{2\pi n u}{m} \cot\frac{\pi u}{m}\, du$$

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^{2k+1}} \sin\frac{2\pi(aj+b)}{m} = -\frac{1}{2b^{2k+1}}\left(\sin\frac{2\pi b}{m} - \sum_{j=0}^{k} \frac{(-1)^j(\frac{2\pi b}{m})^{2j+1}}{(2j+1)!}\right) + \frac{1}{2(an+b)^{2k+1}}\left(\sin\frac{2\pi(an+b)}{m} - \sum_{j=0}^{k} \frac{(-1)^j(\frac{2\pi(an+b)}{m})^{2j+1}}{(2j+1)!}\right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k+1-2j}}{(2k+1-2j)!} HP_{2j}(n) + \frac{(-1)^k(\frac{2\pi}{m})^{2k+1}}{2(2k)!}\int_0^1 (1-u)^{2k}\left(\sin\frac{2\pi(an+b)u}{m} - \sin\frac{2\pi b u}{m}\right) \cot\frac{\pi a u}{m}\, du$$

5.3 $C_{2k+1}^m(n)$ and $C_{2k+1}^m(a,b,n)$

For all integer $k \geq 0$:

$$\sum_{j=1}^{n} \frac{1}{j^{2k+1}} \cos\frac{2\pi j}{m} = \frac{1}{2n^{2k+1}}\left(\cos\frac{2\pi n}{m} - \sum_{j=0}^{k} \frac{(-1)^j(\frac{2\pi n}{m})^{2j}}{(2j)!}\right) + \sum_{j=0}^{k} \frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!} H_{2j+1}(n) + \frac{(-1)^k(\frac{2\pi}{m})^{2k+1}}{2(2k)!}\int_0^1 (1-u)^{2k}\left(\cos\frac{2\pi n u}{m} - 1\right) \cot\frac{\pi u}{m}\, du$$

$$\sum_{j=1}^{n}\frac{1}{(aj+b)^{2k+1}}\cos\frac{2\pi(aj+b)}{m}=-\frac{1}{2b^{2k+1}}\left(\cos\frac{2\pi b}{m}-\sum_{j=0}^{k}\frac{(-1)^j(\frac{2\pi b}{m})^{2j}}{(2j)!}\right)$$
$$+\frac{1}{2(an+b)^{2k+1}}\left(\cos\frac{2\pi(an+b)}{m}-\sum_{j=0}^{k}\frac{(-1)^j(\frac{2\pi(an+b)}{m})^{2j}}{(2j)!}\right)+\sum_{j=0}^{k}\frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k-2j}}{(2k-2j)!}HP_{2j+1}(n)$$
$$+\frac{(-1)^k(\frac{2\pi}{m})^{2k+1}}{2(2k)!}\int_0^1(1-u)^{2k}\left(\cos\frac{2\pi(an+b)u}{m}-\cos\frac{2\pi bu}{m}\right)\cot\frac{\pi au}{m}\,du$$

5.4 $S_{2k}^m(n)$ and $S_{2k}^m(a,b,n)$

For all integer $k\geq 1$:

$$\sum_{j=1}^{n}\frac{1}{j^{2k}}\sin\frac{2\pi j}{m}=\frac{1}{2n^{2k}}\left(\sin\frac{2\pi n}{m}-\sum_{j=0}^{k-1}\frac{(-1)^j(\frac{2\pi n}{m})^{2j+1}}{(2j+1)!}\right)-\sum_{j=0}^{k-1}\frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k-1-2j}}{(2k-1-2j)!}H_{2j+1}(n)$$
$$-\frac{(-1)^k(\frac{2\pi}{m})^{2k}}{2(2k-1)!}\int_0^1(1-u)^{2k-1}\left(\cos\frac{2\pi nu}{m}-1\right)\cot\frac{\pi u}{m}\,du$$

$$\sum_{j=1}^{n}\frac{1}{(aj+b)^{2k}}\sin\frac{2\pi(aj+b)}{m}=-\frac{1}{2b^{2k}}\left(\sin\frac{2\pi b}{m}-\sum_{j=0}^{k-1}\frac{(-1)^j(\frac{2\pi b}{m})^{2j+1}}{(2j+1)!}\right)$$
$$+\frac{1}{2(an+b)^{2k}}\left(\sin\frac{2\pi(an+b)}{m}-\sum_{j=0}^{k-1}\frac{(-1)^j(\frac{2\pi(an+b)}{m})^{2j+1}}{(2j+1)!}\right)-\sum_{j=0}^{k-1}\frac{(-1)^{k-j}(\frac{2\pi}{m})^{2k-1-2j}}{(2k-1-2j)!}HP_{2j+1}(n)$$
$$-\frac{(-1)^k(\frac{2\pi}{m})^{2k}}{2(2k-1)!}\int_0^1(1-u)^{2k-1}\left(\cos\frac{2\pi(an+b)u}{m}-\cos\frac{2\pi bu}{m}\right)\cot\frac{\pi au}{m}\,du$$

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