Proof of \[ \sum_{n=1}^{\infty} (-1)^n = -\frac{1}{2} \]

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First, \( \pm \infty \) is constant at any observation point (position). If a set of real numbers is \( \mathbb{R} \), then,
\[
\begin{align*}
R \times (\pm \infty) &= \pm \infty \\
R + (\pm \infty) &= \pm \infty \\
(-1) \times (\pm \infty) &= \mp \infty
\end{align*}
\]
On the other hand, when \( x (\in \mathbb{R}) \) is taken on a number line, the absolute value \( X \) becomes larger toward \( \pm \infty \) as the absolute value \( X \) is expanded. Similarly, as the size decreases, the absolute value \( X \) decreases toward 0. Furthermore, \( \times (-1) \) represents the reversal of the direction of the axis.

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Second, we consider the figure below.

From the figure above, I got the following equation.

\[
\theta = \arcsin \left( \frac{1}{2} \right)
\]

Here, when take \( \pm \infty \) to the consideration,
\[
\tan \theta = \frac{x+1}{2x(x+1)} = i \left( (-1) \cdot (\pm \infty) = \frac{1}{\pm \infty} \right)
\]
\[
\therefore x = \frac{1}{2} (-1 + i)
\]

Here, when we consider the figure above,
\[
\frac{x^2}{(x+1)^2} = -1
\]

So, when we put \( x = (-1+i)/2 \), \( 1/(x^2) = 2i = -2i \).

Here, from Fig.1, \( i + (\pm \infty) = 0 \).

\[
\begin{align*}
-2i + 2 \cdot (-1) + (-2i) + 2 \cdot (-1) + (-2i) + \cdots &= i + (\pm \infty) = 0 \\
-2i \cdot (1 - 2 + 2 - 2 + 2 - 2 + \cdots) &= i + (\pm \infty) = 0 \\
1 - 2 + 2 - 2 + 2 - 2 + \cdots &= -\frac{1}{2} \left( 1 + (\pm \infty) \right) = -\frac{1}{2} \left( -1 - (-1) \right) = 0 \\
1 + 2((-1) + 1 + (-1) + 1 + (-1) + \cdots) &= 0 \\
\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n &= 0
\end{align*}
\]
\[
\therefore \sum_{n=1}^{\infty} (-1)^n = -\frac{1}{2}
\]