**Level Set Theory with an Explicit Mapping to the Natural Numbers**

*by Jim Rock*

**Abstract.** We show that Cantor’s diagonal argument starts with an invalid premise. Rather than unending decimal expansions, we represent the real numbers as the limit of their partial decimal sums. By working with the completed set of natural numbers, we show that the power set of the natural numbers has the same cardinality as the natural numbers. We create an explicit mapping of the real numbers to the set of natural numbers.

The natural numbers can be put in one to one correspondence with the terminating decimal fractions in the closed interval \([0, 1]\). \(0 \rightarrow .0, 1 \rightarrow .1, 2 \rightarrow .2, \ldots, 10 \rightarrow .01, \ldots\) Each terminating decimal is the mirror image reflection through the decimal point of a natural number. The mapping does not include any repeating decimals. From this mapping the set of all rational numbers would appear to be uncountable. This shows that attempting to map the real numbers in the closed interval \([0, 1]\) to the natural numbers, by listing them as infinite decimal fractions is futile. Cantor’s diagonal proof that the real numbers are an uncountable set can never even get started.

The problem is in defining the real numbers as the set of all infinite decimal expansions. That’s vague. Most non-algebraic real numbers cannot be explicitly referenced. We need a new definition. Let \(p\) be an integer. Each real number \(S_n\) is the limit of its partial decimal sums:

\[
\text{the limit } m \to \infty \quad j = 1 \text{ to } m, \quad 0 \leq a_j \leq 9 \quad \sum p + a_j/10^j = S_n
\]

For any \(n\) the subsets of all the natural numbers between \(0\) and \(n\) can be mapped to \(0, 1, 2, 3, \ldots, 2^n - 1\). By setting \(1/\mathbb{N} = 0\), this process can be extended to the entire set of natural numbers. \(|\mathbb{N}| = |2^{\mathbb{N}}|\).

Let \(1/\mathbb{N} = 0\). Then the limit \(x \to \infty f(x) = \mathbb{N} \leftrightarrow \lim x \to \infty 1/f(x) = 0\).

The limit \(n \to \infty \quad n = \mathbb{N}\). and the limit \(n \to \infty \quad 2^n = \mathbb{N}\). Thus, \(|\mathbb{N}| = |2^{\mathbb{N}}|\).

**There is no hierarchy of infinites.** The limit \(n \to \infty \quad 2^n = 2^{\mathbb{N}} = \mathbb{N}\). \(|2^{\mathbb{N}}| = |\mathbb{N}|\). Since the power set of the natural numbers has the same cardinality as the set of natural numbers and the set of real numbers has the same cardinality as the power set of the natural numbers, the set of real numbers has the same cardinality as the set of natural numbers.

In a terminating decimal the \(S_n\) is the negative integer form of the decimals.

\(0.14533778 \rightarrow S_{14533778} \quad 0.1453 \rightarrow S_{1453}\) For zero \(0 \rightarrow S_0\)

An unending decimal expansion \(S_n\) is from the part of the decimal that can be listed.

\(0.14533778… \rightarrow S_{14533778} \quad 0.45226… \rightarrow S_{45226}\)

**Each \(S_n\) represents an unique decimal expansion.** For non-repeating unending decimal expansions only the listed part, which generates the \(n\) of \(S_n\) is known. The decimal \(0.1453…\) is unending. It may or may not be non-repeating. \(S_{1453}\) is the convergent limit of the partial decimal sums of \(0.1453…\) \(0.1453\) does not generate \(0.1453…\) it is just the first 4 decimal places of \(0.1453…\), while \(0.1453\) is a terminating decimal whose value \(0.1453\) is represented by \(S_{1453}\).

\(0.1453…\) and \(0.14533778…\) could be the same unending decimal, which would make \(S_{1453}\) and \(S_{14533778}\) the limit of the same unending decimal expansion. In order to keep that from happening we devise a rule
which generates a difference in the two decimal expansions. The extra known decimals between 
.1453... and .14533778... there are 8 known decimal places in .14533778... hence we assign to .1453... and .14533778... a different decimal in the \(3778 + 8 = 3786\)th decimal position in the two unending decimals.

\[
\begin{align*}
.14533778... & \text{ differs from } .1453... \text{ in the } 3778 + 8 \text{ decimal places } = 3786\text{th} \text{ decimal position.} \\
.335443... & \text{ differs from } .335... \text{ in the } 443 + 6 \text{ decimal places } = 449\text{th} \text{ decimal position.}
\end{align*}
\]

and so forth.

Decimals that start with .0... are explicitly listed until non-zero decimals are reached. Then the \(S_n\) is the mirror image of the decimal reflected through the decimal point.

\[
\begin{align*}
.000715... & \rightarrow S_{517000} \\
.000715 & \rightarrow S-517000
\end{align*}
\]

Decimals that contain a series of 0’s are explicitly listed until non-zero decimals are reached.

\[
\begin{align*}
.7000157... & \rightarrow S_{7000157} \\
.7000157 & \rightarrow S-7000157
\end{align*}
\]

Based on the listed part of the decimal, which is used to generate the \(S_n\), each \(S_n\) that represents an unending decimal could potentially represent an infinite number of different decimals in [0, 1]. For each \(S_n\) (A) there are an infinite number of \(S_{n(S)}\) that contain (A) at their beginning. .1453... \(S_{1453}\) is contained in .1453xxxx... \(S_{1453xxxx}\) where xxxx is any finite number of digits, and we have a rule that assures us each \(S_n\) represents a unique decimal fraction. Every unending decimal that starts with .1453... is eventually reached.

Suppose you will live forever and each day you want to write a journal chronicling the activity in heaven for an infinite number of days, but it takes you a whole day to record a day’s activities. It would seem like you are destined to fall farther and farther behind in your chronicling as the different infinities of days build up. But even though you can only record one day’s activity each day, you can still record an infinite number of infinite days; to do so you set up an infinite grid. Put the first set of infinite days column 1, and the second set of infinite days column 2 and so forth. This infinite grid is matched to your individual days by a diagonalization technique. Let the grid element in row \(a\) and column \(b\) be \((a, b)\). That’s day \(a\) of the set \(b\) of infinite days.

Then day \(1 \rightarrow (1, 1)\) \(2 \rightarrow (2, 1)\) \(3 \rightarrow (1, 2)\) \(4 \rightarrow (3, 1)\) \(5 \rightarrow (2, 2)\) \(6 \rightarrow (1, 3)\) \(7 \rightarrow (4, 1)\) \(8 \rightarrow (3, 2)\) \(9 \rightarrow (2, 3)\) \(10 \rightarrow (1, 4)\) ...

You will eventually record every day from every different set of infinity in your journal. Similarly, the \(S_{n(S)}\) are all of finite length but they still cover all unending decimals in [0, 1]. Any unending decimal beginning with .1453 can be reached, and eventually every unending decimal beginning with .1453 will be reached. It does not matter that .1453... is potentially any one of an infinite number of unending decimal expansions. An infinite number of unending decimals expansions in [0, 1] each different from .1453..., which like .1453... can each themselves represent any one of an infinite number of unending decimal expansions is a countable infinity of countable infinities, and as such is denumerable as the daily journal example shows.

Each \(S_n\) always represents a different decimal from every other \(S_n\). The rules for generating the \(S_n\) assures us that each generated \(S_n\) is different from every other one. The definition of the \(S_n\) as the limit of the partial decimal sums assures us that each one is unique. No two unending decimals converge to the same limit. For each \(S_n\) \(n\) is a different integer and all integers are \(S_n\).

All the decimal expansions in [0, 1] can be mapped to the natural numbers.

\[
0 \rightarrow S_0 = .0, 1 \rightarrow S_1 = .1, 2 \rightarrow S_2 = .2, 3 \rightarrow S_3 = .3, 4 \rightarrow S_4 = .4, 5 \rightarrow S_5 = .5, 6 \rightarrow S_6 = .6, ...
\]
To extend the mapping to all real numbers we create columns of the decimals in \((0, 1], (1, 2], (-1, 0], \ldots\)

To represent them we make an infinite grid.

The first column is all the decimals in \((0, 1]\)
The second column is all the decimals in \((1, 2]\)
The third column is all the decimals in \((-1, 0]\)
The fourth column is all the decimals in \((2, 3]\)
The fifth column is all the decimals in \((-2, -1]\)… and so forth.

We use ordered pairs of positive integers to represent the numbers in the grid.

\[ P(a, b) \] represents the \(a\text{th}\) decimal in the \(b\text{th}\) column.

For all natural numbers \(n\) we make a diagonal mapping of \(n\) to \(P(a, b)\).

\[
\begin{align*}
0 \rightarrow P(1, 1) & \quad 1 \rightarrow P(2, 1) & \quad 2 \rightarrow P(1, 2) & \quad 3 \rightarrow P(3, 1) & \quad 4 \rightarrow P(2, 2) & \quad 5 \rightarrow P(1, 3) & \quad 6 \rightarrow P(4, 1) & \quad 7 \rightarrow P(3, 2) & \quad 8 \rightarrow P(2, 3) & \quad 9 \rightarrow P(1, 4) & \quad \ldots
\end{align*}
\]

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