Refutation of the algebra of binary relations as basis of free Kleene algebras with domain

Abstract: The definitions for composition of relations and set-theoretic union are *not* tautologous. This refutes the algebra of binary relations on which is based the free Kleene algebras with domain, to form a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **`F`** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓ , · , ⊗ ; \ Not And;
> Imply, greater than, →, ⇒ , ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠, ⊕;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **`F`** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1; (%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: McLean, B. (2019). Free Kleene algebras with domain. arxiv.org/pdf/1907.10386.pdf

Abstract: First we identify the free algebras of the class of algebras of binary relations equipped with the composition and domain operations.

2. Algebras of binary relations

We begin by making precise what is meant by an algebra of binary relations.

Definition 2.1. An algebra of binary relations of the signature {;, +, *, 0, 1} is a universal algebra A = (A, ;, +, *, 0, 1) where the elements of the universe A are all binary relations on some (common) set X, the base, and the interpretations of the symbols are given as follows:

• the binary operation ; is interpreted as composition of relations:
R ; S ≔ {(x, y) ∈ X^2 | ∃z ∈ X : (x,z) ∈ R ∧ (z, y) ∈ S }, (2.1.1.1)

Remark 2.1.1.1: We map only the consequent in Eq. 2.1.1.1 because the symbol " ; " is a not the symbol of a connective in classical logic.

LET p, q, r, s, t, x, y, z: p, q, R, S, X, x, y, z.

((x&y)<(t&t)) > ((%z<t)>(((x&%z)<r)&((%z&y)<s))) ;

```
TTTT TTTT TTTT TTTT( 48)
TTTT NNNN NNNN NNNN(  1)}x8
TTTT TTTT TTTT TTTT(  1)}
TTTT TTTT TTTT TTTT( 48)
TTTT FFFF FFFF FFFF(  1)}x8
TTTT TTTT TTTT TTTT(  1)}
```
(2.1.1.2)

• the binary operation + is interpreted as set-theoretic union:
R + S ≔ {(x, y) ∈ X^2 | (x, y) ∈ R ∨ (x, y) ∈ S }, … (2.1.2.1)

(((x&y)<(t&t)) > (((x&y)<r)+((x&y)<s))) ;

FFFF TTTT TTTT TTTT(48)
FFFF TTTT TTTT **FFFF**(1)}x8
FFFF TTTT TTTT TTTT(1)} (2.1.2.2)

The definitions for composition of relations and set-theoretic union as rendered in Eqs. 2.1.1.2 and 2.1.2.2 are *not* tautologous. This refutes the algebra of binary relations on which is based the free Kleene algebras with domain.