An Exact Solution of the Einstein Equations for Cylindrically Symmetric Empty Space

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Abstract

In this article, we derived an exact solution of the Einstein equations for cylindrically symmetric empty space .

Here we derive an exact solution of the Einstein equations for cylindrically symmetric empty space.

The static condition[1] means that, with a static coordinate system, the fundamental tensors, the $g_{\mu\nu}$ are independent of the time x^0 or t and also $g_{0m} = 0$. The spatial coordinates may be taken to be cylindrical coordinates $x^1 = r, x^2 = \varphi, x^3 = z$. The general form for the square of invariant distance, the ds^2 compatible with cylindrical symmetry is

$$ds^2 = e^{2v}dt^2 - e^{2h}dr^2 - r^2d\varphi^2 - e^{2u}dz^2, \tag{1}$$

where v, h, and u are functions of r and z only.

We can read off the value of $g_{\mu\nu}$ from Eq.(1), namely,

$$g_{00} = e^{2v}, g_{11} = -e^{2h}, g_{22} = -r^2, g_{33} = -e^{2u},$$

and

$$g_{\mu\nu} = 0 for \mu \neq \nu.$$

We find

$$g^{00}=e^{-2v},g^{11}=-e^{-2h},g^{22}=-r^{-2},g^{33}=-e^{-2u},$$

and

$$g^{\mu\nu}=0 for \mu\neq\nu.$$

The Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$ can be calculated by

$$\Gamma^{\mu}_{\nu\sigma}=g^{\mu\lambda}\Gamma_{\lambda\nu\sigma}, \tag{2}$$

and

$$\Gamma_{\mu\nu\sigma}=\frac{1}{2}(g_{\mu\nu,\sigma}+g_{\mu\sigma,\nu}-g_{\nu\sigma,\mu}). \tag{3}$$

Many of them vanish.

Then we calculate the Ricci tensors by

$$R_{\mu\nu}=\Gamma^{\alpha}_{\mu\alpha,\nu}-\Gamma^{\alpha}_{\mu\nu,\alpha}-\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}+\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}. \tag{4}$$

The non vanishing components of $R_{\mu\nu}$ are

$$R_{00}=e^{2(v-h)}\times\{\frac{\partial v}{\partial r}[-\frac{\partial(v+u-h)}{\partial r}-\frac{1}{r}]-\frac{\partial^2 v}{\partial r^2}\}+$$

$$+e^{2(v-u)}\times\{\frac{\partial v}{\partial z}[-\frac{\partial(v-u+h)}{\partial z}]-\frac{\partial^2 v}{\partial z^2}\},$$

$$R_{11}=e^{2(h-u)}\times\{\frac{\partial h}{\partial z}[\frac{\partial(v-u+h)}{\partial z}]+\frac{\partial^2 h}{\partial z^2}\}-\frac{\partial h}{\partial r}[\frac{\partial(v+u)}{\partial r}+\frac{1}{r}]+$$

$$+\frac{\partial^2(v+u)}{\partial r^2}+(\frac{\partial v}{\partial r})^2+(\frac{\partial u}{\partial r})^2,$$

$$R_{22}=e^{-2h}\times r\times\frac{\partial(v+u-h)}{\partial r},$$

and

$$R_{33} = e^{2(u-h)}\{\frac{\partial u}{\partial r}[\frac{\partial(v+u-h)}{\partial r} + \frac{1}{r}] + \frac{\partial^2 u}{\partial r^2}\} + (\frac{\partial v}{\partial z})^2 + (\frac{\partial h}{\partial z})^2 +$$

$$+ \frac{\partial^2(v+h)}{\partial z^2} - \frac{\partial u}{\partial z}\frac{\partial(v+h)}{\partial z}.$$

Einstein's law requires all $R_{\mu\nu}$ vanish. Thus

$$R_{\mu\nu} = 0 \tag{5}$$

for all μ, ν.

From the $R_{22} = 0$, we can expect that $u + v - h = -f(z)$ or $h = u + v + f(z)$.

The equations $R_{00} = 0, R_{11} = 0, and R_{33} = 0$ will be,

$$e^{2(v+f)} \times \ (2(\frac{\partial v}{\partial z})^2 + \frac{\partial^2 v}{\partial z^2} \ + \frac{\partial f}{\partial z}\frac{\partial v}{\partial z}) + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2} = 0, \tag{6}$$

$$e^{2(v+f)} \times \{\frac{\partial(v+u+f)}{\partial z}\frac{\partial(2v+f)}{\partial z} + \frac{\partial^2(v+u+f)}{\partial z^2}\} - \frac{1}{r}\frac{\partial(v+u)}{\partial r} - 2\frac{\partial v}{\partial r}\frac{\partial u}{\partial r} + \frac{\partial^2(v+u)}{\partial r^2} = 0, \tag{7}$$

and

$$e^{2(v+f)}\{2(\frac{\partial v}{\partial z})^2 + \frac{\partial(2v+u+f)}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial^2(2v+u+f)}{\partial z^2}\} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = 0. \tag{8}$$

It should be able to solve Eq.(6) for v and then solve the other equations for u and h.

Here we limit our solution when all the functions are variables separable. $v(r,z) = v_1(r) + v_2(z)$, $u(r,z) = u_1(r) + u_2(z)$, and $h(r,z) = h_1(r) + h_2(z)$,

$$e^{2(v_2+f)} \times (2(\frac{dv_2}{dz})^2 + \frac{d^2 v_2}{dz^2} \ + \frac{df}{dz}\frac{dv_2}{dz}) = -e^{-2v_1} \times (\frac{1}{r}\frac{dv_1}{dr} + \frac{d^2 v_1}{dr^2}) = k_1, \tag{9}$$

$$e^{2(v_2+f)} \times \{\frac{d(v_2+u_2+f)}{dz}\frac{d(2v_2+f)}{dz} + \frac{d^2(v_2+u_2+f)}{dz^2}\} = \tag{10}$$
$$= -e^{-2v_1} \times (-\tfrac{1}{r}\tfrac{d(v_1+u_1)}{dr} - 2\tfrac{dv_1}{dr}\tfrac{du_1}{dr} + \tfrac{d^2(v_1+u_1)}{dr^2}) = k_2,$$

and

$$e^{2(v_2+f)}\{2(\frac{dv_2}{dz})^2 + \frac{d(2v_2+u_2+f)}{dz}\frac{df}{dz} + \frac{d^2(2v_2+u_2+f)}{dz^2}\} = \tag{11}$$
$$= -e^{-2v_1}(\tfrac{1}{r}\tfrac{du_1}{dr} + \tfrac{d^2u_1}{dr^2}) = k_3.$$

Solve the equations of eq(9),eq(10) and eq(11) for varibals z and r separately.

We get the exact solution,

$$e^{2v_1} = \frac{r^{3\kappa}}{(r^\kappa + \frac{1}{c_0\kappa})^2}(\frac{\kappa}{c_0k_1}), \tag{12}$$

when $c_0k_1 > 0$,

or

$$e^{2v_1} = \frac{1}{r^\kappa(r^\kappa - c_0\kappa)^2}(\frac{-c_0\kappa^3}{k_1}), \tag{13}$$

when $c_0k_1 < 0$,

here $\kappa = 2\sqrt{\frac{k_1}{k_3}}$, and c_0 is a constant of integration.

Also we will have $k_3/k_1 = \frac{4}{\kappa^2}$ and $k_2/k_1 = 1 - \frac{4}{\kappa^2}$.

We get the rest functions as following,

$$e^{2u_1} = (e^{2v_1})^{\frac{k_3}{k_1}} \times (\frac{e^{2u_0}}{r^{\frac{2k_2}{k_1}}}), \tag{14}$$

$$e^{2h_1} = e^{2v_1+2u_1}, \tag{15}$$

For functions with variable z we have the following solutions,

$$e^{2v_2} = 2k_1 \int e^{-2f} F dz, \tag{16}$$

here $F = e^f \int e^{-f} dz$,

$$e^{2u_2} = \frac{\exp(-2f + \frac{2k_2}{k_1} \int \frac{dz}{F})}{2k_1 \int e^{-2f} F dz} \tag{17}$$

$$e^{2h_2} = \exp(\frac{2k_2}{k_1} \int \frac{dz}{F}), \tag{18}$$

here $k_2 = k_1 - k_3$, and we have,

$$e^{2v} = e^{2v_1+2v_2}, \tag{19}$$

$$e^{2u} = e^{2u_1+2u_2}, \tag{20}$$

$$e^{2h} = e^{2h_1+2h_2}, \tag{21}$$

For any given function of e^f, we will have a solution. For example, if we have the following function,

$$e^f = \frac{1}{\cosh z} \tag{22}$$

Then we will have,

$$e^{2v_2} = k_1(\frac{\cosh(2z)}{2} + 2\sqrt{2}\sinh z), \tag{23}$$

$$e^{2u_2} = \frac{2}{k_1} \frac{(\sinh z + 1/\sqrt{2})^{2k_2/k_1} \times (coshz)^2}{(\cosh(2z) + 4\sqrt{2}\sinh z)}, \tag{24}$$

$$e^{2h_2} = (\sinh z + \frac{1}{\sqrt{2}})^{\frac{2k_2}{k_1}}, \tag{25}$$

The interesting solution here is when $c_0 k_1 > 0$, and $c_0 < 0$, or when $c_0 k_1 < 0$, and $c_0 > 0$, the e^{2v_1} has a singular point for $r > 0$, or the space are separated into two regions.

References

[1] P.A.M. Dirac,General Theory of Relativity, Wiley, 1975.