An Exact Solution of the Einstein Equations for Cylindrically Symmetric Empty Space

Xuan Zhong Ni, Campbell, CA, USA

(June 2019)

Abstract

In this article, we derived an exact solution of the Einstein equations for cylindrically symmetric empty space .

Here we derive an exact solution of the Einstein equations for cylindrically symmetric empty space.

The static condition[1] means that, with a static coordinate system, the fundamental tensors, the $g_{\mu\nu}$ are independent of the time x^0 or t and also $g_{0m} = 0$. The spatial coordinates may be taken to be cylindrical coordinates $x^1 = r, x^2 = \varphi, x^3 = z$. The general form for the square of invariant distance, the ds^2 compatible with cylindrical symmetry is

$$ds^2 = e^{2v}dt^2 - e^{2h}dr^2 - r^2d\varphi^2 - e^{2u}dz^2, \tag{1}$$

where v, h, and u are functions of r and z only.

We can read off the value of $g_{\mu\nu}$ from Eq.(1), namely,

$$g_{00} = e^{2v}, g_{11} = -e^{2h}, g_{22} = -r^2, g_{33} = -e^{2u},$$

and

$$g_{\mu\nu} = 0 for \mu \neq \nu.$$

We find

$$g^{00} = e^{-2v}, g^{11} = -e^{-2h}, g^{22} = -r^{-2}, g^{33} = -e^{-2u},$$

and

$$g^{\mu\nu} = 0 for \mu \neq \nu.$$

The Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$ can be calculated by

$$\Gamma^{\mu}_{\nu\sigma} = g^{\mu\lambda}\Gamma_{\lambda\nu\sigma}, \tag{2}$$

and

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}). \tag{3}$$

Many of them vanish.

Then we calculate the Ricci tensors by

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}. \tag{4}$$

The non vanishing components of $R_{\mu\nu}$ are

$$R_{00} = e^{2(v-h)} \times \{\frac{\partial v}{\partial r}[-\frac{\partial(v+u-h)}{\partial r} - \frac{1}{r}] - \frac{\partial^2 v}{\partial r^2}\} +$$

$$+ e^{2(v-u)} \times \{\frac{\partial v}{\partial z}[-\frac{\partial(v-u+h)}{\partial z}] - \frac{\partial^2 v}{\partial z^2}\},$$

$$R_{11} = e^{2(h-u)} \times \{\frac{\partial h}{\partial z}[\frac{\partial(v-u+h)}{\partial z}] + \frac{\partial^2 h}{\partial z^2}\} - \frac{\partial h}{\partial r}[\frac{\partial(v+u)}{\partial r} + \frac{1}{r}] +$$

$$+ \frac{\partial^2(v+u)}{\partial r^2} + (\frac{\partial v}{\partial r})^2 + (\frac{\partial u}{\partial r})^2,$$

$$R_{22} = e^{-2h} \times r \times \frac{\partial(v+u-h)}{\partial r},$$

and

$$R_{33} = e^{2(u-h)}\{\frac{\partial u}{\partial r}[\frac{\partial(v+u-h)}{\partial r} + \frac{1}{r}] + \frac{\partial^2 u}{\partial r^2}\} + (\frac{\partial v}{\partial z})^2 + (\frac{\partial h}{\partial z})^2 +$$
$$+ \frac{\partial^2(v+h)}{\partial z^2} - \frac{\partial u}{\partial z}\frac{\partial(v+h)}{\partial z}.$$

Einstein's law requires all $R_{\mu\nu}$ vanish. Thus

$$R_{\mu\nu} = 0 \tag{5}$$

for all μ, ν.

From the $R_{22} = 0$, we can expect that $u + v - h = -f(z)$ or $h = u + v + f(z)$.

The equations $R_{00} = 0, R_{11} = 0, and R_{33} = 0$ will be,

$$e^{2(v+f)} \times \ (2(\frac{\partial v}{\partial z})^2 + \frac{\partial^2 v}{\partial z^2} \ + \frac{\partial f}{\partial z}\frac{\partial v}{\partial z}) + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2} = 0, \tag{6}$$

$$e^{2(v+f)}\times\{\frac{\partial(v+u+f)}{\partial z}\frac{\partial(2v+f)}{\partial z}+\frac{\partial^2(v+u+f)}{\partial z^2}\}-\frac{1}{r}\frac{\partial(v+u)}{\partial r}-2\frac{\partial v}{\partial r}\frac{\partial u}{\partial r}+\frac{\partial^2(v+u)}{\partial r^2} = 0, \tag{7}$$

and

$$e^{2(v+f)}\{2(\frac{\partial v}{\partial z})^2+\frac{\partial(2v+u+f)}{\partial z}\frac{\partial f}{\partial z}+\frac{\partial^2(2v+u+f)}{\partial z^2}\}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2} = 0. \tag{8}$$

It should be able to solve Eq.(6) for v and then solve the other equations for u and h.

Here we limit our solution when all the functions are variables separable. $v(r,z) = v_1(r)+v_2(z)$, $u(r,z) = u_1(r)+u_2(z)$, and $h(r,z) = h_1(r)+h_2(z)$,

$$e^{2(v_2+f)} \times (2(\frac{dv_2}{dz})^2 + \frac{d^2 v_2}{dz^2} \ + \frac{df}{dz}\frac{dv_2}{dz}) = -e^{-2v_1} \times (\frac{1}{r}\frac{dv_1}{dr} + \frac{d^2 v_1}{dr^2}) = k_1, \tag{9}$$

$$e^{2(v_2+f)} \times \{ \frac{d(v_2+u_2+f)}{dz} \frac{d(2v_2+f)}{dz} + \frac{d^2(v_2+u_2+f)}{dz^2} \} = \tag{10}$$
$$= -e^{-2v_1} \times (-\tfrac{1}{r}\tfrac{d(v_1+u_1)}{dr} - 2\tfrac{dv_1}{dr}\tfrac{du_1}{dr} + \tfrac{d^2(v_1+u_1)}{dr^2}) = k_2,$$

and

$$e^{2(v_2+f)} \{ 2(\frac{dv_2}{dz})^2 + \frac{d(2v_2+u_2+f)}{dz} \frac{df}{dz} + \frac{d^2(2v_2+u_2+f)}{dz^2} \} = \tag{11}$$
$$= -e^{-2v_1} (\tfrac{1}{r}\tfrac{du_1}{dr} + \tfrac{d^2u_1}{dr^2}) = k_3.$$

Solve the equations of eq(9),eq(10) and eq(11) for varibals z and r separately.

We get the exact solution,

$$e^{2v_1} = \frac{r^{3\kappa}}{(r^\kappa + \frac{1}{c_0\kappa})^2} (\frac{\kappa}{c_0 k_1}), \tag{12}$$

when $c_0 k_1 > 0$,

or

$$e^{2v_1} = \frac{1}{r^\kappa (r^\kappa - c_0\kappa)^2} (\frac{-c_0\kappa^3}{k_1}), \tag{13}$$

when $c_0 k_1 < 0$,

here $\kappa = 2\sqrt{\frac{k_1}{k_3}}$, and c_0 is a constant of integration.

Also we will have $k_3/k_1 = \frac{4}{\kappa^2}$ and $k_2/k_1 = 1 - \frac{4}{\kappa^2}$.

We get the rest functions as following,

$$e^{2u_1} = (e^{2v_1})^{\frac{k_3}{k_1}} \times (\frac{e^{2u_0}}{r^{\frac{2k_2}{k_1}}}), \tag{14}$$

$$e^{2h_1} = e^{2v_1+2u_1}, \tag{15}$$

For functions with variable z we have the following solutions,

$$e^{2v_2} = 2k_1 \int e^{-2f} F dz, \tag{16}$$

here $F = e^f \int e^{-f} dz$,

$$e^{2u_2} = \frac{\exp(-2f + \frac{2k_2}{k_1} \int \frac{dz}{F})}{2k_1 \int e^{-2f} F dz} \tag{17}$$

$$e^{2h_2} = \exp(-2f + \frac{2k_2}{k_1} \int \frac{dz}{F}), \tag{18}$$

here $k_2 = k_1 - k_3$, and we have,

$$e^{2v} = e^{2v_1+2v_2}, \tag{19}$$

$$e^{2u} = e^{2u_1+2u_2}, \tag{20}$$

$$e^{2h} = e^{2h_1+2h_2}, \tag{21}$$

For any given function of e^f, we will have a solution. For example, if we have the following function,

$$e^f = \frac{1}{\cosh z} \tag{22}$$

Then we will have,

$$e^{2v_2} = k_1 (\frac{\cosh(2z)}{2} + 2\sqrt{2} \sinh z), \tag{23}$$

$$e^{2u_2} = \frac{2}{k_1} \frac{(\sinh z + 1/\sqrt{2})^{2k_2/k_1} \times (coshz)^2}{(\cosh(2z) + 4\sqrt{2}\sinh z)}, \tag{24}$$

$$e^{2h_2} = (\sinh z + \frac{1}{\sqrt{2}})^{\frac{2k_2}{k_1}}, \tag{25}$$

The interesting solution here is when $c_0 k_1 > 0$, and $c_0 < 0$, or when $c_0 k_1 < 0$, and $c_0 > 0$, the e^{2v_1} has a singular point for $r > 0$, or the space are separated into two regions.

References

[1] P.A.M. Dirac,General Theory of Relativity, Wiley, 1975.