Two cubic equations and Pi

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abstract
This note presents some formulas for Pi

1. Introduction

The number Pi is defined by

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.1415926535... \tag{1}$$

In this note we give some series for π .

2. Two cubic equations

Entry 1. The roots $\{a, z, \bar{z}\}$ of the equation: $x^3 + x - 1 = 0$ are

$$a = \frac{1}{6}\alpha^{1/3} - 2\alpha^{-1/3} \tag{2}$$

$$z = -\frac{1}{12}\alpha^{1/3} + \alpha^{-1/3} + \frac{i\sqrt{3}}{2}\left(\frac{1}{6}\alpha^{1/3} + 2\alpha^{-1/3}\right) \tag{3}$$

$$\bar{z} = -\frac{1}{12}\alpha^{1/3} + \alpha^{-1/3} - \frac{i\sqrt{3}}{2}\left(\frac{1}{6}\alpha^{1/3} + 2\alpha^{-1/3}\right) \tag{4}$$

where

$$\alpha = 108 + 12\sqrt{93} \tag{5}$$

Entry 2. The roots $\{b, w, \bar{w}\}$ of the equation: $x^3 + x^2 - 1 = 0$ are

$$b = \frac{1}{6}\beta^{1/3} + \frac{2}{3}\beta^{-1/3} - \frac{1}{3} \tag{6}$$

$$w = -\frac{1}{12}\beta^{1/3} - \frac{1}{3}\beta^{-1/3} - \frac{1}{3} + \frac{i\sqrt{3}}{2}\left(\frac{1}{6}\beta^{1/3} - \frac{2}{3}\beta^{-1/3}\right) \tag{7}$$

$$\bar{w} = -\frac{1}{12}\beta^{1/3} - \frac{1}{3}\beta^{-1/3} - \frac{1}{3} - \frac{i\sqrt{3}}{2}\left(\frac{1}{6}\beta^{1/3} - \frac{2}{3}\beta^{-1/3}\right) \tag{8}$$

where

$$\beta = 100 + 12\sqrt{69} \tag{9}$$

Entry 3.

$$a = \left\{1 + \left(1 + \left(1 + \left(1 + \ldots\right)^{2/3}\right)^{2/3}\right)^{2/3}\right\}^{-1/3} \tag{10}$$

Entry 4.

$$b = \frac{1}{\sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \ldots}}}} = \frac{1}{\sqrt{1 + \frac{1}{\sqrt{1 + \frac{1}{\sqrt{1 + \ldots}}}}}} \tag{11}$$

3. Pi Series

Entry 5.

$$\pi = -4\tan^{-1}(ab) - 8\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}a^{2n+1}\operatorname{Re}\left(w^{2n+1}\right) \tag{12}$$

$$\pi = -4\tan^{-1}(ab) - 8\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}b^{2n+1}\operatorname{Re}\left(z^{2n+1}\right) \tag{13}$$

Entry 6.

$$\begin{aligned}\pi &= 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left\{\left(\left(\frac{1}{z}\right)^{2n+1} + \left(\frac{1}{\bar{z}}\right)^{2n+1}\right)w^{-2n-1} - (aw)^{2n+1}\right\} = \\ &= 4\sum_{n=0}^{\infty}A_n w^{-2n-1} - 4\sum_{n=0}^{\infty}B_n w^{2n+1}\end{aligned} \tag{14}$$

where

$$A_n = \frac{(-1)^n}{2n+1}\left(\left(\frac{1}{z}\right)^{2n+1} + \left(\frac{1}{\bar{z}}\right)^{2n+1}\right) \quad , \quad B_n = \frac{(-1)^n}{2n+1}a^{2n+1} \quad , n = 0,1,2,3,\ldots \tag{15}$$

Entry 7.

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left\{\left(\left(\frac{1}{w}\right)^{2n+1}+\left(\frac{1}{\overline{w}}\right)^{2n+1}\right)z^{-2n-1}-(bz)^{2n+1}\right\}=$$

$$= 4\sum_{n=0}^{\infty}A_n z^{-2n-1} - 4\sum_{n=0}^{\infty}B_n z^{2n+1} \qquad (16)$$

where

$$A_n = \frac{(-1)^n}{2n+1}\left(\left(\frac{1}{w}\right)^{2n+1}+\left(\frac{1}{\overline{w}}\right)^{2n+1}\right) \quad , \quad B_n = \frac{(-1)^n}{2n+1}b^{2n+1} \quad , n = 0,1,2,3,... \qquad (17)$$

Remarks:

- For $z \in \mathbb{C}$ (complex numbers) , $\overline{z}$ is the conjugate of z .
- For $z \in \mathbb{C}$ (complex numbers) , $\mathrm{Re}(z)$ is the real part of z .

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