On : $z^3 - 7z^2 + 3z - 1 = 0$

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Abstract

We give a formula for Pi.

1. Introduction

Entry 1.

$$z \mapsto f(z) = z^3 - 7z^2 + 3z - 1 \tag{1}$$

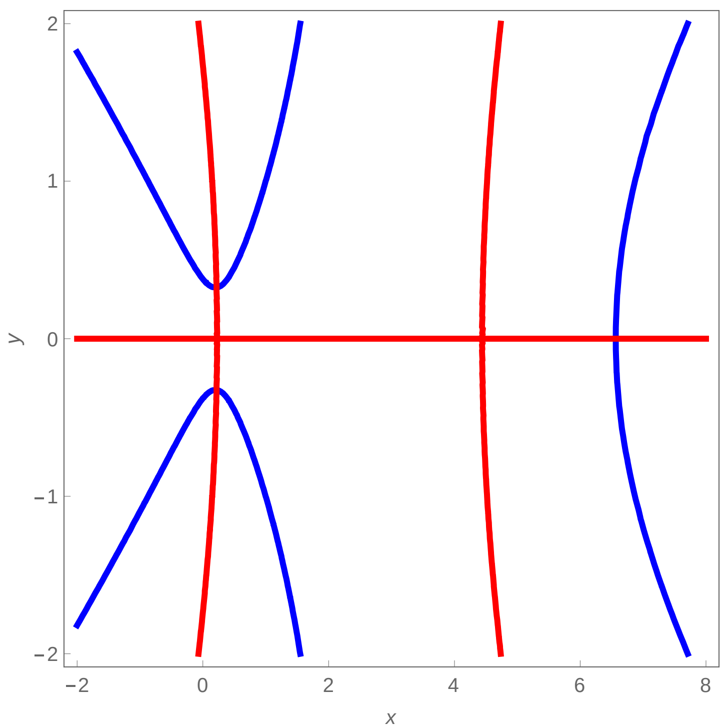

Figure 1. • $\mathrm{Re}(f(x+iy)) = 0$, • $\mathrm{Im}(f(x+iy)) = 0$

Entry 2. If $i=\sqrt{-1}$ then

$$\begin{aligned} z^3-7z^2+3z-1&=(z-r)\left(z^2-(7-r)z+r^2-7r+3\right)= \\ &=(z-r)(z-a-bi)(z-a+bi) \end{aligned} \tag{2}$$

where

$$r=\frac{7}{3}+\sqrt[3]{\frac{262}{27}+\sqrt{\frac{172}{27}}}+\sqrt[3]{\frac{262}{27}-\sqrt{\frac{172}{27}}} \tag{3}$$

$$a=\frac{7-r}{2} \quad , \quad b=\frac{1}{2}\sqrt{3r^2-14r-37} \tag{4}$$

Entry 3.

$$r=\frac{7}{3}+\frac{1}{3}\left(262+6\sqrt{129}\right)^{1/3}+\frac{40}{3}\left(262+6\sqrt{129}\right)^{-1/3} \tag{5}$$

$$a=\frac{7}{3}-\frac{1}{6}\left(262+6\sqrt{129}\right)^{1/3}-\frac{20}{3}\left(262+6\sqrt{129}\right)^{-1/3} \tag{6}$$

$$b=\frac{\sqrt{3}}{2}\left(\frac{1}{3}\left(262+6\sqrt{129}\right)^{1/3}-\frac{40}{3}\left(262+6\sqrt{129}\right)^{-1/3}\right) \tag{7}$$

Entry 4. Define $c(n,k)$ and u_n by

$$c(n,k)=\begin{cases} 1 & n\ge k \\ 0 & n<k \end{cases} \tag{8}$$

$$u_n=\sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{n-k-m}{k}\binom{k}{m}7^{n-2k-m}(-3)^{k-m}c(n-k-m,k)c(n-k-m,m) \tag{9}$$

then

$$u_n=\{1,7,46,302,1983,13021,\ldots\} \tag{10}$$

$$\frac{u_{n+1}}{u_n}\to r \tag{11}$$

2. Pi formulas

Recall that $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.1415926535...$

Entry 5.

$$\pi = \frac{2}{\sqrt{r}}\sum_{n=0}^{\infty} r^{-n} A_n \tag{12}$$

where

$$r = \frac{7}{3} + \sqrt[3]{\frac{262}{27} + \sqrt{\frac{172}{27}}} + \sqrt[3]{\frac{262}{27} - \sqrt{\frac{172}{27}}} \tag{13}$$

$$A_n = \binom{2n}{n}\frac{2^{-2n}}{2n+1} + 2\sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{n-k}\frac{1}{2k+1} \tag{14}$$

Entry 6. If $i = \sqrt{-1}, z = a + bi, w = \frac{(1-z)^2}{4z}$, then

$$\pi = 4\tan^{-1}\left(\frac{b-a}{b+a}\right) + 4\tan^{-1}\left(\frac{b}{1-a}\right) + 2\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{n}\left(\mathrm{Im}\left(z^n\right) - \mathrm{Im}\left(w^n\right)\right) \tag{15}$$

Remark: $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the real and imaginary part of the complex z .

References

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