DISPROOF OF THE RIEMANN HYPOTHESIS

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Abstract. In this manuscript we use the Perron formula to connect zeta evaluated on the root free halfplane to zeta evaluated on the critical strip. This is possible since the Perron formula is of the form f(s)=O f(s+w) with O being an integral operator. The variable s+w is on the root free halfplane, and yet s can be on the critical strip. Hence, the Perron formula serves as a form of a functional equation that connects the critical strip with the root free halfplane. Then, one simply notices that in the Perron formula, the left hand side converges only conditionally, whilst the right hand side converges absolutely. This, of course, cannot be, since the left side of an equation is always equal to the right side. This contradiction when examined in detail disproves the Riemann hypothesis. This method is employed on an arbitrary distribution of zeta roots as well, concluding that zeta has a root arbitrarily close to the vertical line passing through unity.

1 Introduction

Two and a half millennia ago or so, ancient Greeks knew about the fundamental theorem of arithmetics: *every integer can be written in a unique way as a product of primes only*. For instance, a version of the fundamental theorem of arithmetics is mentioned in Euclid's books *Elements* [1, Book VII, Propositions 30, 31 and 32, and Book IX, Proposition 14]. The fundamental theorem of arithmetics immediately suggests that, if one knows everything to be known about primes, one can discard integers completely and work with primes alone. This in turn means that there is a deeper level of mathematics that only deals with primes. Even though integers are the first numbers we learn as children, and even though learning integers comes naturally to us, there is a more fundamental level of numbers, a more fundamental set of numbers, namely the set of prime numbers, that supports integers.

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So ancient Greeks asked questions about primes. For instance, the first question about primes that springs in mind is "*how many primes are there?*" Ancient Greeks answered this question too [1, Book IX, Proposition 20].

So the next question that naturally springs in mind is "*how many primes not larger than some x are there?*" We do not know the answer to this question, not even today, two and a half millennia afterwards. It is the second most simple question about primes one may think of.

The first one to break through was Riemann in year 1859. He was asked to write a short article to introduce himself to the mathematical community when he became a member of the Berlin academy. He had to write the article quickly, because the printing was to start in only few days. He was just expected to introduce himself to the academic community, say a little something about himself and about what he does.

So he wrote the six pages long manuscript with no introduction, no sectioning, and without a conclusion, about the subject he never published about before nor after: about counting primes [2].

In his article [3], Riemann introduced his zeta function $\zeta(s)$, a complex function of a complex variable s. Zeta is analytic everywhere except at its only singularity, a simple pole at $s = 1$. With this function he was able to count primes, at least in principle. In order to count primes, one needs to know the exact locations of all of the roots of the Riemann zeta function. If one knew the exact locations of all the roots of zeta, then one could count primes exactly.

The problem is: we don't know where all the roots of zeta are. It is known that trivial roots are all simple and located at negative even integers. There are, however, infinitely many nontrivial roots, all of which are located on the critical strip $0 \leqslant \operatorname{Re}(s) \leqslant 1$ at generally unknown locations.

The Riemann hypothesis is given by the following theorem.

Theorem 1. *All of the nontrivial roots of zeta are located on the critical line* $\operatorname{Re}(s) = 1/2$.

In the year 1896, Hadamard and de la Valée Poussin proved independently the following theorem, known as the prime number theorem [4, p.45]. One equivalent version of the prime number theorem is as follows.

Theorem 2. *Zeta has no roots on the line* $\operatorname{Re}(s) = 1$.

The only functional equations known about zeta are the ones that relate zeta evaluated at complex numbers from both regions outside the critical strip. No functional equation is known about zeta that would relate zeta evaluated at the numbers outside the critical strip to zeta evaluated at the numbers inside the critical strip. This is problematic, because even though we do know much about the behaviour of zeta outside the critical strip, we cannot use this knowledge to inspect the behaviour of zeta inside the critical strip.

Furthermore, all the representations of zeta on the critical strip are either too complicated, or do not converge absolutely on the critical strip, making the manipulation and extracting information from them seem impossible.

Thus, the behaviour of zeta inside the critical strip is a mystery for the last 160 years.

Thus, the first step towards partially answering the question about how many primes there are not larger than some x, is to answer the Riemann hypothesis. Ideally, one would preferably have all the non-trivial roots situated on the critical line, instead of all over the critical strip, because then one would seek the knowledge about zeta on the critical line alone, and the number of primes $\pi(x)$ would oscillate about the mean $\mathrm{Li}(x)$ with the least amplitude possible.

The Riemann zeta function [4, p.1] is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{1}$$

with the sum running over all natural numbers n. This series converges absolutely on the halfplane $\mathrm{Re}(s) > 1$.

One can introduce the Möbius function, and Möbius invert the series (1). The Möbius function $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 \text{ if } n \text{ has an even number of distinct} \\ \quad \text{prime factors or if } n = 1 \\ -1 \text{ if } n \text{ has an odd number of distinct} \\ \quad \text{prime factors} \\ 0 \text{ if } n \text{ is not squarefree} \end{cases} \tag{2}$$

The Möbius inverse [4, p.3] of the series (1) defining zeta function is then

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{3}$$

The series in Eq. (3) converges absolutely on the halfplane $1 < \mathrm{Re}(s)$. This series also follows [4, p.1, Eq.(1.1.2)] from the reciprocal Euler product over all primes p,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p=2}^{\infty} \left(1 - \frac{1}{p^s}\right) \tag{4}$$

In his book *The Theory of the Riemann Zeta-Function*, Titchmarsh proved the following proposition [4, p.62, ch.3.13].

Proposition 1.

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{5}$$

on the entire line $\mathrm{Re}(s) = 1$.

Another result of use is Eq. (2.12.2) from Titchmarsh's book [4, p.29], given in the following proposition.

Proposition 2. *For every* $\sigma \geqslant \frac{1}{2}$, *one finds unconditionally for all sufficiently large* $\mathrm{Im}(s)$

$$\zeta(s) = O\left(|s|\right)$$

Yet another result of use is given in the next proposition [4, p.336].

Proposition 3. *The following estimates hold on region* $1/2 < \sigma$ *for every strictly positive real* ε *on the Riemann Hypothesis.*

$$\begin{aligned} \zeta(s) &= O\left(t^{\varepsilon}\right) \\ \frac{1}{\zeta(s)} &= O\left(t^{\varepsilon}\right) \end{aligned} \tag{6}$$

We abbreviate *the Riemann hypothesis* by *RH*.

2 The idea of this disproof

The basic idea of this disproof of RH is to use the Perron formula. One of its variants is as follows.

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \lim_{x\to\infty} \int_{c-i\infty}^{c+i\infty} \sum_{n=1^{\infty}} \frac{a_n}{n^{s+w}} \frac{x^w}{w} dw \tag{7}$$

There are two infinite series involved here. The series $\sum a_n n^{-s}$ on the left hand side, and the series $\sum a_n n^{-s-w}$ on the right hand side.

The variable $s + w$ belongs to the root free halfplane $\mathrm{Re}(s + w) > 1$ by assumption. Hence, the right hand side series converges absolutely, and its magnitude is invariant to rearranging its terms.

On the other hand, the constant c is restricted by the requirement $c > 0$. Hence, $\mathrm{Re}(s) > 1-c$. Since c is positive and otherwise arbitrary, s may be on the critical strip. Hence, the series on the left hand side converges only conditionally for any such s, changing its magnitude under rearrangements of its terms.

Hence, if the left hand side converges, it converges to many complex numbers under rearranging. On the other hand, if the right hand side converges, then it converges to a holomorphic single-valued function, absolutely.

Thus, we have arrived at a contradiction. The contradiction appears because one of the underlying assumptions of the Perron formula are not satisfied. It is fairly obvious that the single-valued holomorphic function represented by the right hand side cannot converge outside the region of absolute convergence of the series on the left hand side, since the underlying assumptions of the Perron formula all involve convergence.

We inspect this in detail in this manuscript. This inspection disproves RH in the end, and it also proves that the Riemann zeta function has a root arbitrarily close to the line $\sigma = 1$.

3 The Perron formula variant

Let s be a complex variable and let $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$.

We start with details of Titchmarsh's Lemma 3.12 [4, p.60, Lemma3.12], which is a variant of the Perron formula. The following lemma is the almost exact copy of Titchmarsh's Lemma 3.12. We prove it here because we need some of the results of its proof for this disproof of RH.

Lemma 1 (Perron formula variant). *Let, for any $\sigma > 1$,*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{8}$$

where $a_n = O\left\{g(n)\right\}$ with $g(n)$ being non-decreasing, and

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} = O\left\{\frac{1}{(\sigma - 1)^a}\right\} \tag{9}$$

as $\sigma \to 1$. *Then if* $c > 0$, $\sigma + c > 1$, x *is not an integer, and* $N = \lfloor x \rfloor$,

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} dw + O\left\{ \frac{x^c}{T(\sigma + c - 1)^a} \right\} + O\left\{ \frac{g(2x)x^{1-\sigma} \log x}{T} \right\} + O\left\{ \frac{g(N)x^{1-\sigma}}{T|x-N|} \right\} \tag{10}$$

Proof. We just use the Titchmarsh's results here [4, pp.61-62], since they're valid in general. We do highlight the details we'll need later on.

The starting result [4, p.61], valid in general, is

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} = \begin{cases} 1 + O\left\{ \frac{(x/n)^c}{T \log\left(x/n\right)} \right\}, \text{ if } n < x \\ O\left\{ \frac{(x/n)^c}{T \log\left(x/n\right)} \right\}, \text{ if } n > x \end{cases} \tag{11}$$

Multiply by $a_n n^{-s}$ and sum over all natural numbers n.

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}} \frac{x^w}{w} dw + O\left\{ \frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma+c} \left|\log\left(x/n\right)\right|} \right\} \tag{12}$$

Since $\sigma + c > 1$, the series in the integrand converges absolutely to $f(s+w)$, and one just estimates the error term the same way Titchmarsh did in his proof of his Lemma 3.12, and the lemma follows. □

This was nothing else but Titchmarsh's Lemma 3.12.

Of interest is next the application of Lemma 1, in the form of the following theorem, which is the almost exact copy of Titchmarsh's Theorem 14.25(A) [4, p.369, Theorem14.25(A)]. We prove it here because we need some of the results of its proof further in this manuscript.

Theorem 3 (Titchmarsh's Theorem 14.25(A)). *On RH, the series*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{13}$$

is convergent, and its sum is $1/\zeta(s)$ *for every* s *such that* $\sigma > 1/2$.

Proof. In the previous Lemma, take $a_n = \mu(n)$, $f(s) = 1/\zeta(s)$, $c = 2$, and x half an odd integer. We obtain

$$\sum_{n<x} \frac{\mu(n)}{n^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)} \frac{x^w}{w} dw + O\left(\frac{x^2}{T}\right) \tag{14}$$

Now complete the contour so that the closed contour of integration contains the point $w = 0$ and so that the point $w = 0$ is the only singularity of the integrand enclosed by the contour. For instance, with some δ such that $0 < \delta < \sigma - 1/2$, the last equation becomes by the use of the Cauchy residue theorem,

$$\begin{aligned}\sum_{n<x} \frac{\mu(n)}{n^s} &= \frac{1}{\zeta(s)} + O\left(\frac{x^2}{T}\right) + \\ &\frac{1}{2\pi i}\left(\int_{2-iT}^{1/2-\sigma+\delta-iT} + \int_{1/2-\sigma+\delta-iT}^{1/2-\sigma+\delta+iT} + \int_{1/2-\sigma+\delta+iT}^{2+iT}\right) \frac{x^w dw}{w\zeta(s+w)}\end{aligned} \tag{15}$$

Following Titchmarsh, after calculating all the error terms, and with ε being arbitrarily small strictly positive real number, coming from the estimate (6), the result reads

$$\sum_{n<x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O\left(\frac{x^2}{T^{1-\varepsilon}}\right) + O\left(\frac{T^\varepsilon}{x^{\sigma-1/2-\delta}}\right) \tag{16}$$

Taking, say, $T = x^3$ and letting $x \to \infty$, the theorem follows. □

So all of these results so far are the same as the ones in Titchmarsh's Theorem 14.25(A).

Since $\sigma + c > 1$, we rewrite Eq. (14) now in the form of Eq. (12) as follows.

$$\sum_{n<x} \frac{\mu(n)}{n^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+w}} \frac{x^w}{w} dw + O\left(\frac{x^2}{T}\right) \tag{17}$$

Consider this now: the left hand side of Eq. (17) converges only conditionally on the critical strip as $x \to \infty$. By the Riemann series rearrangement theorem, this means that the left hand side may change upon rearranging the terms. Whilst the right hand side is analytic and single valued. Namely, the above result reads, as deduced from

Eq. (14) with $T = x^3$ and as $x \to \infty$:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \lim_{x\to\infty} \frac{1}{2\pi i} \int_{2-ix^3}^{2+ix^3} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+w}} \frac{x^w}{w} dw \tag{18}$$

Do observe: the left hand side may change in the critical strip under rearranging, but the right hand side may not, since the series in the integrand converges absolutely whenever $\sigma + 2 > 1$ as required by Lemma 1. Thus, if the integral converges in the limit, it converges to a unique holomorphic function which is in this particular case single-valued, regardless of rearranging. We explore this in detail next.

Introduce a permutation $\psi(n) : \mathbb{N} \to \mathbb{N}$. Permutation $\psi(n)$ is a bijection between naturals. This way, any series $\sum_n b_n$ is rearranged by substituting $\psi(n)$ for n, resulting in $\sum_n b_{\psi(n)}$. We keep the permutation $\psi(n)$ arbitrary but fixed. This is the standard way to rearrange terms of any infinite series.

With such arbitrary permutation $\psi(n)$, we prove the following lemma.

Lemma 2 (Perron formula for rearranged series). *Let, for any* $\sigma > 1$,

$$f(s) = \sum_{n=1}^{\infty} \frac{a_{\psi(n)}}{\psi(n)^s} \tag{19}$$

where $a_n = O\left\{g(n)\right\}$ *with* $g(n)$ *being non-decreasing, and*

$$\sum_{n=1}^{\infty} \frac{\left|a_{\psi(n)}\right|}{\psi(n)^{\sigma}} = O\left\{\frac{1}{(\sigma-1)^a}\right\} \tag{20}$$

as $\sigma \to 1$. *Then if* $c > 0$, $\sigma + c > 1$, x *is not an integer, and* $N = \lfloor x \rfloor$,

$$\begin{gathered} \sum_{\substack{n\in\mathbb{N} \\ \psi(n)<x}} \frac{a_{\psi(n)}}{\psi(n)^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} dw + \\ O\left\{\frac{x^c}{T(\sigma+c-1)^a}\right\} + O\left\{\frac{g(2x)x^{1-\sigma}\log x}{T}\right\} + \\ O\left\{\frac{g(N)x^{1-\sigma}}{T|x-N|}\right\} \end{gathered} \tag{21}$$

Proof. As in the previous lemma, the starting equation is Eq. (11), valid for all n, with n replaced by $\psi(n)$.

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{x}{\psi(n)}\right)^{w}\frac{dw}{w}=\begin{cases}1+O\left\{\frac{(x/\psi(n))^{c}}{T\log\left(x/\psi(n)\right)}\right\}, \text{ if } \psi(n)<x\\ O\left\{\frac{(x/\psi(n))^{c}}{T\log\left(x/\psi(n)\right)}\right\}, \text{ if } \psi(n)>x\end{cases} \tag{22}$$

This is valid for any natural number $\psi(n)$.

Multiply by $a_{\psi(n)}\psi(n)^{-s}$ and sum over all n.

$$\begin{aligned}\sum_{\substack{n\in\mathbb{N}\\ \psi(n)<x}}\frac{a_{\psi(n)}}{\psi(n)^{s}}=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\sum_{n=1}^{\infty}\frac{a_n}{n^{s+w}}\frac{x^{w}}{w}dw+\\ O\left\{\frac{x^{c}}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}\left|\log\left(x/n\right)\right|}\right\}\end{aligned} \tag{23}$$

Since the error term, if convergent, converges absolutely, we rearranged the terms of its infinite series appropriately without affecting the result. We rearranged the terms of the series in the integrand appropriately too, since $\sigma+c>1$, and thus the series in the integrand converges absolutely to $f(s+w)$, and one just estimates the error term the same way Titchmarsh did in his Lemma 3.12, and Lemma 2 follows. □

We next prove a variant of Theorem 3, suitable for rearranged L-series.

Theorem 4 (Titchmarsh's Theorem 14.25(A) for rearranged series). *Let $\psi(n)$ be an arbitrary permutation of natural numbers n. On RH, the series*

$$\sum_{n=1}^{\infty}\frac{\mu(\psi(n))}{\psi(n)^{s}} \tag{24}$$

is convergent, and its sum is $1/\zeta(s)$ for every s such that $\sigma>1/2$.

Proof. In Lemma 2, take $a_n=\mu(\psi(n)), f(s)=1/\zeta(s)$, $c=2$, and x half an odd integer. We obtain

$$\sum_{\substack{n\in\mathbb{N}\\ \psi(n)<x}}\frac{\mu(\psi(n))}{\psi(n)^{s}}=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{1}{\zeta(s+w)}\frac{x^{w}}{w}dw+O\left(\frac{x^{2}}{T}\right) \tag{25}$$

Now complete the contour so that the closed contour of integration contains the point $w = 0$ and so that the point $w = 0$ is the only singularity of the integrand enclosed by the contour. For instance, with some δ such that $0 < \delta < \sigma - 1/2$, the last equation becomes by the use of the Cauchy residue theorem,

$$\sum_{\substack{n\in\mathbb{N}\\ \psi(n)<x}} \frac{\mu(\psi(n))}{\psi(n)^s} = \frac{1}{\zeta(s)} + O\left(\frac{x^2}{T}\right) + \frac{1}{2\pi i}\left(\int_{2-iT}^{1/2-\sigma+\delta-iT} + \int_{1/2-\sigma+\delta-iT}^{1/2-\sigma+\delta+iT} + \int_{1/2-\sigma+\delta+iT}^{2+iT}\right)\frac{x^w dw}{w\zeta(s+w)} \tag{26}$$

Following Titchmarsh's proof of his Theorem 14.25(A), after calculating all the error terms, and with ε being arbitrarily small strictly positive real number, the result reads

$$\sum_{\substack{n\in\mathbb{N}\\ \psi(n)<x}} \frac{\mu(\psi(n))}{\psi(n)^s} = \frac{1}{\zeta(s)} + O\left(\frac{x^2}{T^{1-\varepsilon}}\right) + O\left(\frac{T^\varepsilon}{x^{\sigma-1/2-\delta}}\right) \tag{27}$$

Taking, say, $T = x^3$ and letting $x \to \infty$, the theorem follows. □

The next corollary follows from Theorem 4.

Corollary 1. *On RH, the series* $\sum_n \mu(n)n^{-s}$ *converges unconditionally on the halfplane* $\sigma > 1/2$.

Proof. On RH, $1/\zeta(s)$ is holomorphic on the halfplane $\sigma > 1/2$, and so all the rearrangements of $\sum_n \mu(n)n^{-s}$ are of the same magnitude by Theorem 4. □

4 Disproof of RH

By Corollary 1, the series $\sum_n \mu(n)n^{-s}$ should converge unconditionally to $1/\zeta(s)$ for any s such that $\sigma > 1/2$. This is obviously false, since $\sum_n |\mu(n)|n^{-r}$ diverges to infinity for any r such that $r \leqslant 1$, since $\sum_n |\mu(n)|n^{-s} = \zeta(s)/\zeta(2s)$ whenever $\sigma > 1$ [4, p.5, Eq.(1.2.7)]. For instance, one immediately concludes from this that $\sum_n |\mu(n)|/n$ diverges to infinity. Hence, $\sum_n \mu(n)n^{-r}$ does not converge absolutely when $r \leqslant 1$. This disproves RH.

5 Root arbitrarily close to $\sigma = 1$

We can push the methods of previous sections further. So far we have assumed RH simply because of the estimate (6) being readily at disposal. An analogue of the estimate (6) is valid on the root free region too, without assuming RH. We prove this next.

We need the following simple definition first.

Definition 1. *Define real positive number m by*

$$m = \max \left\{ \operatorname{Re}(\rho), \zeta(\rho) = 0 \right\}$$

The following lemma is a variant of Titchmarsh's Theorem 14.2 [4, p.336, Theorem14.2]. It uses the Borel-Carathéodory theorem [5, §5.5, p.174] as well as the Hadamard three circles theorem [5, §5.3, p.172]. These two very well known theorems state the following.

Theorem 5 (Borel-Carathéodory theorem)**.** *Let a function $f(z)$ be analytic on a closed disc of radius R centred at the origin. Suppose that r is such that $r < R$. Then, the following inequality holds.*

$$\max_{|z|=r} \left| f(z) \right| \leqslant \frac{2r}{R-r} \sup_{|z| \leqslant R} \operatorname{Re}\left(f(z) \right) + \left| f(0) \right| \frac{R+r}{R-r} \tag{28}$$

Theorem 6 (Hadamard three circles theorem)**.** *Let $f(z)$ be a holomorphic function on the annulus $r_1 \leqslant |z| \leqslant r_3$. Let M_i be the maximum of $|z|$ on the circle $|z| = r_i$. Then for any three concentric circles of radii $r_1 < r_2 < r_3$ one finds*

$$\log \left(\frac{r_3}{r_1} \right) \log M_2 \leq \log \left(\frac{r_3}{r_2} \right) \log M_1 + \log \left(\frac{r_2}{r_1} \right) \log M_3 \tag{29}$$

With these two theorems we next prove the following Lemma, which is a variant of Titchmarsh's Theorem 14.2 [4, p.336, Theorem14.2].

Lemma 3. *The following result holds uniformly on region $m < \sigma \leqslant 1$ for every strictly positive real ε.*

$$\log \zeta(s) = O\left\{ (\log t)^{\frac{1-\sigma}{m} + \varepsilon} \right\} \tag{30}$$

Consequently, the following estimates hold on region $m < \sigma$ for every strictly positive real ε.

$$\begin{aligned} \zeta(s) &= O\left(t^{\varepsilon}\right) \\ \frac{1}{\zeta(s)} &= O\left(t^{\varepsilon}\right) \end{aligned} \tag{31}$$

Proof. Apply the Borel-Carathéodory theorem to the function $\log\zeta(s)$ and the circles with centre $2+it$ and radii $2-m-\frac{1}{2}\delta$ and $2-m-\delta$, with δ such that $0<\delta<1-m$. On the larger circle

$$\operatorname{Re}\left\{\log\zeta(z)\right\}=\log\left|\zeta(z)\right|<A\log t$$

by Proposition 2, with some strictly positive real A.

Hence, on the smaller circle

$$\begin{aligned}\left|\zeta(z)\right|&\leqslant\frac{2(2-m-\delta)}{\frac{1}{2}\delta}A\log t+\frac{2(2-m)-\frac{3}{2}\delta}{\frac{1}{2}\delta}\left|\log\left|\zeta(2+it)\right|\right|\\&<A\frac{\log t}{\delta}\end{aligned}\tag{32}$$

with some new strictly positive real A.

Now apply Hadamard's three circle theorem to circles C_1, C_2 and C_3 with centre σ_1+it with $1<\sigma_1\leqslant t$, passing through the points $1+\eta+it$, $\sigma+it$ and $m+\delta+it$, with some strictly positive η. The radii are thus

$$r_1=\sigma_1-1-\eta$$
$$r_2=\sigma_1-\sigma$$
$$r_3=\sigma_1-m-\delta$$

If the maxima of $\left|\log\zeta(z)\right|$ on the circles are M_1, M_2 and M_3, we obtain

$$M_2\leqslant M_1^{1-a}M_2^a$$

with

$$a=\frac{\log\frac{r_2}{r_1}}{\log\frac{r_3}{r_1}}=\frac{\log\left(1+\frac{1+\eta-\sigma}{\sigma_1-1-\eta}\right)}{\log\left(1+\frac{m+\eta-\delta}{\sigma_1-1-\eta}\right)}$$
$$=\frac{1+\eta-\sigma}{m+\eta-\delta}+O\left(\frac{1}{\sigma_1}\right)=\frac{1-\sigma}{m}+O(\delta)+O(\eta)+O\left(\frac{1}{\sigma_1}\right)$$

for all sufficiently small δ and η.

By (32), $M_3<A\delta^{-1}\log t$, and, since [4, p.4, Eq.(1.1.9)],

$$\log\zeta(z)=\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^z}$$

with $\Lambda_1(n)\leqslant 1$ and with $\operatorname{Re}(z)>1$, one finds

$$M_1\leqslant\max_{\operatorname{Re}(z)\geqslant 1+\eta}\left|\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^z}\right|\leqslant\sum_{n=2}^{\infty}\frac{1}{n^{1+\eta}}<\zeta(1+\eta)<\frac{A}{\eta}$$

for some A, since zeta has a simple pole at $s=1$, and since circle C_1 lies entirely on the half-plane $\operatorname{Re}(z)>1$.

Hence

$$\begin{aligned}\left|\log\zeta(\sigma+it)\right| &< \left(\frac{A}{\eta}\right)^{1-a}\left(\frac{A\log t}{\delta}\right)^{a}\\ &< \frac{A}{\eta^{1-a}\delta^{a}}\,(\log t)^{\frac{1-\sigma}{m}+O(\delta)+O(\eta)+O\left(\frac{1}{\sigma_1}\right)}\end{aligned}$$

The first part of the lemma follows on taking δ and η small enough, and σ_1 large enough. More precisely, we can take

$$\sigma_1 = \frac{1}{\eta} = \frac{1}{\delta} = \log\log t$$

Since for any sufficiently small φ,

$$(\log t)^{O(\varphi)} = e^{O(\varphi\log\log t)} = e^{O(1)} = O(1)$$

we obtain on the region $m + \frac{1}{\log\log t} \leqslant \sigma \leqslant 1$

$$\log\zeta(s) = O\left\{\log\log t(\log t)^{\frac{1-\sigma}{m}}\right\}$$

Since the term $\frac{1-\sigma}{m}$ in the exponent of $\log t$ in (30) is less than unity, it follows that, with new ε, Eq. (30) leads to

$$-\varepsilon\log t < \log\left|\zeta(s)\right| < \varepsilon\log t$$

and the lemma follows. □

The next lemma is a variant of Titchmarsh's Theorem 14.25(A) [4, p.369, Theorem 14.25(A)].

Lemma 4. *The series*

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{s}} \tag{33}$$

is convergent, and its sum is $1/\zeta(s)$ for every s with $m < \sigma$.

Proof. In Lemma 1 take $a_n = \mu(n)$, $f(s) = 1/\zeta(s)$, $c = 2$, and x half an odd integer. We obtain

$$\sum_{n<x} \frac{\mu(n)}{n^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)} \frac{x^w}{w} dw + O\left(\frac{x^2}{T}\right)$$

$$= \frac{1}{2\pi i} \left(\int_{2-iT}^{m-\sigma+\delta-iT} + \int_{m-\sigma+\delta-iT}^{m-\sigma+\delta+iT} + \int_{m-\sigma+\delta+iT}^{2+iT} \right) \frac{1}{\zeta(s+w)} \frac{x^w}{w} dw \tag{34}$$

$$+ \frac{1}{\zeta(s)} + O\left(\frac{x^2}{T}\right)$$

with

$$0 < \delta < \sigma - m$$

By Eq. (31) of Lemma 3, the first and the third integrals are

$$O\left(T^{-1+\varepsilon} \int_{m-\sigma+\delta}^{2} x^u du \right) = O\left(x^2 T^{-1+\varepsilon} \right)$$

and the second integral is

$$O\left(x^{m-\sigma+\delta} \int_{-T}^{T} \left(1 + |v|\right)^{-1+\varepsilon} dv \right) = O\left(x^{m-\sigma+\delta} T^{\varepsilon} \right)$$

Hence

$$\sum_{n<x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O\left(x^2 T^{-1+\varepsilon} \right) + O\left(x^{m-\sigma+\delta} T^{\varepsilon} \right)$$

Taking, for instance, $T = x^3$, the O-terms tend to zero as $x \to \infty$ and the result follows.

Conversely, if (33) is convergent for $\sigma > m$, it is uniformly convergent, and so in this region represents an analytic function, which is $1/\zeta(s)$ for $\sigma > 1$, and so throughout the entire region by analytic continuation. □

We have in fact proven the following lemma, which is a variant of Titchmarsh's Theorem 14.25(B) [4, p.370, Theorem 14.25(B)].

Lemma 5. *The convergence of (33) for $\sigma > m$ is a necessary and sufficient condition for $\zeta(s)$ being root-free on the half-plane $\sigma > m$.*

It's a trivial exercise now to demonstrate that Theorem 4 becomes the following theorem when one replaces $m = 1/2$ with some general m, since the estimates remain of the same magnitude, as demonstrated in Eq. (31).

Theorem 7 (Titchmarsh's Theorem 14.25(A) for rearranged series with general m). *Let $\psi(n)$ be an arbitrary permutation of natural numbers n. The series*

$$\sum_{n=1}^{\infty} \frac{\mu(\psi(n))}{\psi(n)^s} \tag{35}$$

is convergent, and its sum is $1/\zeta(s)$ for every s such that $\sigma > m$.

Proof. In Lemma 2, take $a_n = \mu(\psi(n))$, $f(s) = 1/\zeta(s)$, $c = 2$, and x half an odd integer. We obtain

$$\sum_{\substack{n\in\mathbb{N} \\ \psi(n)<x}} \frac{\mu(\psi(n))}{\psi(n)^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)} \frac{x^w}{w} dw + O\left(\frac{x^2}{T}\right) \tag{36}$$

Now complete the contour so that the closed contour of integration contains the point $w = 0$ and so that the point $w = 0$ is the only singularity of the integrand enclosed by the contour. For instance, with some δ such that $0 < \delta < \sigma - m$, the last equation becomes by the use of the Cauchy residue theorem,

$$\begin{aligned} \sum_{\substack{n\in\mathbb{N} \\ \psi(n)<x}} \frac{\mu(\psi(n))}{\psi(n)^s} &= \frac{1}{\zeta(s)} + O\left(\frac{x^2}{T}\right) + \\ \frac{1}{2\pi i} &\left(\int_{2-iT}^{m-\sigma+\delta-iT} + \int_{m-\sigma+\delta-iT}^{m-\sigma+\delta+iT} + \int_{m-\sigma+\delta+iT}^{2+iT} \right) \frac{x^w dw}{w\zeta(s+w)} \end{aligned} \tag{37}$$

The right hand side of Eq. (37) is identical to the right hand side of Eq. (34). Hence the estimates of Lemma 4 following Eq. (34) apply, and thus Theorem 7 follows. □

We have arrived at the following theorem.

Theorem 8. *Zeta has a root arbitrarily close to the line $\sigma = 1$.*

Proof. This conclusion follows immediately, since the series (35) in Theorem 7 is not convergent for some permutations $\psi(n)$ for every s such that $\sigma \leq 1$.

Since $1/\zeta(s)$ converges on the halfplane $\sigma \geqslant 1$, one concludes that in the proof of Theorem 7 one cannot close the contour of integration in such a way that the only singularity enclosed is $w = 0$ whenever $\sigma = 1$, since the condition $\sigma - \varepsilon > 1$ is never true for any positive real ε when $\sigma = 1$.

To demonstrate this more precisely, rewrite δ from Theorem 7, which is defined by the condition $0 < \delta < \sigma - m$, in the form $\delta = \sigma - m - \varepsilon$, with some arbitrary but fixed strictly positive real ε. Then the integrals in Eq. (37) can be rewritten in the form

$$\int_{2-iT}^{-\varepsilon-iT} + \int_{\varepsilon-iT}^{-\varepsilon+iT} + \int_{-\varepsilon+iT}^{2+iT} \tag{38}$$

Hence, the integration contour in Theorem 7 is the rectangle with the left-most vertical side passing through the point $-\varepsilon$ on the real axis. Thus, one can write $\mathrm{Re}(w)$ inside the integration contour in the form $\mathrm{Re}(w) = -\varepsilon + \xi$, with a strictly positive real ξ. When $\sigma = 1$, the variable $s + w$ inside the integration contour has the real part equal to $1 - \varepsilon + \xi$. Hence, whenever $\xi < \varepsilon$, the variable $s + w$ is inside the critical strip, and hence $\zeta(s + w)$ could have roots inside the integration contour. If zeta had no roots there, then $\sum_{\psi} \mu(n) n^{-1-it}$ should converge unconditionally to $1/\zeta(1+it)$ by Theorem 7. However, $\sum_{\psi} \mu(n) n^{-1-it}$ does not converge unconditionally. Hence, the assumption that m exists such that the integration contour can be defined so that zeta has no roots in some δ-neighbourhood of the line $\sigma = 1$ was wrong. Since δ is arbitrary, one concludes that zeta has a root with $1 - \varepsilon < m < 1$ for all strictly positive real numbers ε. □

6 On the Riemann series theorem

One may argue that all the versions of the Perron formula above with a permutation $\psi(n)$ are identical to the original Titchmarsh's version of the Perron formula, whenever $\psi(n)$ is finite, and hence that there may not be any new information given by them. For instance, in Titchmarsh's Lemma 3.12 in Eq. (10) the left hand side is

$$\sum_{n<x} a_n n^{-s} \tag{39}$$

whilst in a variant of Titchmarsh's Lemma 3.12 for rearranged series in Eq. (21) the left hand side is

$$\sum_{\substack{n\in\mathbb{N}\\ \psi(n)<x}} \frac{a_{\psi(n)}}{\psi(n)^s} \tag{40}$$

Do notice that Eqs. (39) and (40) are identical for every finite x. Also, right hand sides of Eqs. (10) and (21) are identical as well. Furthermore, both series (39) and (40) are invariant to rearranging the terms for every finite x. Thus, if one argues that the limiting process $x \to \infty$ simply means that x grows large without bounds remaining finite, then one may argue that there is no permutation acting on the terms of the series at all, and hence that there are no rearrangements in any of the equations above at all. This would invalidate the main argument of this manuscript, rendering it false.

We address this argument next.

This manuscript uses the Weierstrass prescription, which is the standard prescription:

Weierstrass prescription: *If you want to know what happens at the point at infinity, find the finite result, and then take the limit.*

Consider the infinite series

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots \tag{41}$$

This infinite series converges to zero, since after sufficiently many terms, the series comes arbitrarily close to zero, and the pairwise sum of consecutive terms is zero.

Consider now the rearrangement of series (41), done by taking the first two positive terms, then adding the first negative term, then adding the next two positive terms, then adding the next negative term from (41), and so on:

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \dots \tag{42}$$

This series is the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \tag{43}$$

and it's a very well known fact that it converges to $\ln 2$.

Thus, by rearranging the terms of the original series (41), which converges to 0, we created the new series (42), which converges to

$\ln 2$. This is the consequence of the Riemann series theorem [6, p.76, Theorem3.54], which states that the terms of any conditionally convergent series can be rearranged to form the new series, which can converge to any number, or even diverge.

But consider the following now. We rearranged the original series (41) by applying this algorithm: take two positive terms and one negative term. And so on, take two positive terms and one negative term. So, we added twice as many positive terms than the negative ones in the end. However, the original series (41), which we now rewrite in the form $\lim_{N\to\infty}\sum_1^N(1/n-1/n)$, has equally many positive and negative terms for any N. While rearranging the terms with this algorithm, we discarded half of the negative terms. More precisely, we discarded the smallest negative terms. We should add those negative terms back in, if we want to believe that the rearranged series (42) represents the original series (41). So, we have to add

$$-\lim_{N\to\infty}\sum_{N/2}^{N}\frac{1}{n}=-\lim_{N\to\infty}\Big(\ln N-\ln\big(N/2\big)\Big)=-\ln 2 \tag{44}$$

to the rearranged series (42) to ensure that it truly represents the rearrangement of the original series (41), and that all the terms of the original series are counted in.

But now, after adding all the missing negative terms back to the rearranged series, the rearranged series (42) converges to $\ln 2-\ln 2=0$. So, the rearrangement changed the series by omitting infinitely many infinitely small terms given in Eq. (44).

This is true for any convergent series. No rearrangement of terms can change the series, unless infinitely many infinitely small terms are being omitted or added. For – if one does not add or subtract anything from the original series, where does the difference in magnitudes come from then?

In order to demonstrate this more precisely, let us consider any finite sum $\sum_1^N a_n$. One can rearrange terms a_n at will, the result will stay the same, since the sum is finite. If the original sum $\sum_1^N a_n$ converges to some a, so does the rearranged one. This is so because the series is finite.

But, what do we mean exactly when we write an infinite series in the form $\sum_1^\infty a_n$? We abuse the notation slightly by writing an infinite series this way, because we really mean to write $\lim_{N\to\infty} \sum_1^N a_n$. So, when we rearrange the terms of an infinite series, what we really do is we rearrange the terms of finite series, and then take the limit. In other words, we employ the Weierstrass prescription. This way, rearrangement of terms of infinite series does not change series at all, because this is true in the finite case, unless we add or subtract some terms from the original series.

Thus, in this particular sense, the Riemann series theorem is wrong. No series is conditionally convergent. All convergent series converge unconditionally. The series becomes conditionally convergent only if one adds or subtracts infinitely many infinitely small terms while performing a rearrangement.

The same holds for divergent series as well. If a finite series grows without bounds, then the rearranged series grows with no bounds as well.

In short, a rearrangement of terms of finite series does not change the series. Therefore, by the Weierstrass prescription, the same holds for infinite series as well.

However, the Riemann series rearrangement theorem is a very important tool of course: *There is a procedure that can change the magnitude of a series to any value if the series does not converge absolutely, whilst the same procedure leaves the magnitude of any absolutely convergent series invariant.* This procedure is called *rearranging the terms of a series.*

The point of this section so far is to notice that series rearrangements can be done by adding or subtracting infinitely many infinitely small terms in the limit of large N to the original series, loosely speaking. We collect this result in the following preliminary definition.

Definition 2. *A rearrangement of terms of an infinite series $\sum_n a_n$ can be done by adding or subtracting infinitely many infinitely small terms a_n to and from the original series $\sum_n a_n$ in the limit $n \to \infty$.*

The precise definition of such rearranging requires the introduction of indicator functions, and shall be given in one of the following sections.

We provide some examples of interest next, in order to introduce the necessary concepts.

7 The set $\mathbb{N}_P$

Infinite series are generally understood as limits of the upper limit of summation. For instance, let $\mathbb{N}$ be the set of all natural numbers. Then Eq. (1) is understood in the sense of a limit of the finite sum:

$$\zeta(s) = \lim_{N\to\infty} \sum_{n=1}^{N} n^{-s} \tag{45}$$

Similarly, in the Euler product

$$\zeta(s) = \prod_{p=2}^{\infty} \left(1 - p^{-s}\right)^{-1} \tag{46}$$

the infinite product is understood as a limit of a finite product as an arbitrary prime number P grows large without bounds:

$$\zeta(s) = \lim_{P\to\infty} \prod_{p=2}^{P} \left(1 - p^{-s}\right)^{-1} \tag{47}$$

The natural numbers n created by expanding this product and multiplying all the primes in the product out are all such that their largest prime factor does not exceed P. So, if $\mathbb{N}_P$ denotes the set of all natural numbers n such that their largest prime factor does not exceed P, one finds by expanding this product

$$\zeta(s) = \lim_{P\to\infty} \sum_{n\in\mathbb{N}_P} n^{-s} \tag{48}$$

This is so because one can rewrite each factor $\left(1 - p^{-s}\right)^{-1}$ in the form of an infinite series:

$$\frac{1}{1-p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots = \sum_{n=1}^{\infty} \frac{1}{p^{ns}}$$

The two series in Eqs. (45) and (48) converge to the same function $\zeta(s)$ on the halfplane $\mathrm{Re}(s) > 1$ because they both converge absolutely and uniformly on that region. Do notice that one series is the rearrangement of the other.

We pay attention to the set $\mathbb{N}_P$. Its elements are natural numbers n with each prime factor not exceeding the prime P.

The first thing to notice is that, among other elements, $\mathbb{N}_P$ contains all the natural numbers up to P. This is so because any number up to P cannot have a prime factor larger than P.

Another thing to notice is that $\mathbb{N}_P$ has infinitely many elements. For instance the product $P \times P \times P \times \cdots = P^N$ is an element of $\mathbb{N}_P$ for any natural N.

One can rewrite the definition of zeta (45) in the form

$$\zeta(s) = \lim_{P\to\infty} \sum_{n=1}^{P} n^{-s} \tag{49}$$

with P being a prime number. This way, the three definitions of zeta (45), (48) and (49) differ one from another by infinitely many summands after taking the limit $P \to \infty$ or $N \to \infty$. The difference tends to zero on the halfplane $\mathrm{Re}(s) > 1$ of course because on that region all three series converge absolutely and uniformly to $\zeta(s)$.

We gather these results in the following definition and proposition.

Definition 3. *The set $\mathbb{N}_P$ is the set of all natural numbers n whose largest prime factor does not exceed the fixed prime number P.*

Proposition 4. *Among other elements, the set $\mathbb{N}_P$ contains all the natural numbers up to the prime P.*

8 The set $\mathbb{S}_P$

Rewrite Möbius inverse (3) in the form

$$\frac{1}{\zeta(s)} = \lim_{P\to\infty} \sum_{n=1}^{P} \frac{\mu(n)}{n^s} \tag{50}$$

One can derive a similar expression for $1/\zeta(s)$ by expanding the reciprocal of the Euler product:

$$\frac{1}{\zeta(s)} = \lim_{P\to\infty} \prod_{p\le P} \left(1 - \frac{1}{p^s}\right) = \lim_{P\to\infty} \sum_{\mathbb{S}_P} \frac{\mu(n)}{n^s} \tag{51}$$

Here natural numbers n are all squarefree and each prime factor does not exceed P. So define the elements of the set $\mathbb{S}_P$ to be squarefree natural numbers n such that their largest prime factor of n does not exceed P.

The largest element of $\mathbb{S}_P$ is the primorial $P\#$ since it is the largest squarefree number whose largest prime factor does not exceed P. Hence, $\mathbb{S}_P$ has only finitely many elements for any finite P.

Furthermore, among other elements, $\mathbb{S}_P$ contains all the squarefree numbers not exceeding P. Hence, the two definitions (50) and (51) differ by infinitely many terms in the limit $P \to \infty$. The difference tends to zero however on the region of absolute convergence $\mathrm{Re}(s) > 1$.

We collect these results in the following definition and proposition.

Definition 4. *The set $\mathbb{S}_P$ is the set of all square-free natural numbers n whose largest prime factor does not exceed the fixed prime number P.*

Proposition 5. *Among other elements, the set $\mathbb{S}_P$ contains all the square-free natural numbers up to the prime P.*

9 Indicator function

Introduce the indicator function $\delta_P(n)$ that indicates whether n is an element of $\mathbb{N}_P$ or not by

$$\delta_P(n) = \begin{cases} 1 \text{ if } n \in \mathbb{N}_P \\ 0 \text{ if } n \notin \mathbb{N}_P \end{cases} \tag{52}$$

Hence one finds generally for any b_n

$$\sum_{n \in \mathbb{N}_P} b_n = \sum_{n \in \mathbb{N}} b_n \delta_P(n)$$

Introduce notation

$$\delta_\infty(n) = \lim_{P \to \infty} \delta_P(n)$$

From the Euler product one finds for s on the halfplane $\mathrm{Re}(s) > 1$

$$\lim_{P \to \infty} \prod_{p=2}^{P} \left(1 - \frac{1}{p^s}\right) = \sum_{n \in \mathbb{N}} \frac{\mu(n)\delta_\infty(n)}{n^s} = \frac{1}{\zeta(s)}$$

One generalizes all of these observations by introducing the sets $\mathbb{N}_N$ and $\mathbb{S}_N$ as follows.

Definition 5. *The set $\mathbb{N}_N$ contains all the natural numbers n such that $n \leqslant N$.*

Definition 6. *The set $\mathbb{S}_N$ contains all the natural numbers n such that some rule $\mathcal{R}(M, N, n)$ is satisfied.*

Definition 7. *The indicator function $\delta_N(n)$ is defined for all $n \in \mathbb{S}_N$ as follows.*

$$\delta_N(n) = \begin{cases} 1, \textit{ if } n \in \mathbb{S}_N \\ 0, \textit{ if } n \notin \mathbb{S}_N \end{cases} \tag{53}$$

Thus, for instance, if the rule $\mathcal{R}(M, N, n)$ is given by:

n does not have a prime factor larger than prime P, n is squarefree, and P is the largest prime not exceeding N,
then one can rewrite the sum $\sum_{n\in\mathbb{S}_N} a_n$ in the form

$$\sum_{n\in\mathbb{S}_P} a_n = \sum_{n\leqslant P\#} a_n\delta_P(n) = \sum_{n\in\mathbb{N}_{P\#}} a_n\delta_P(n)$$

We can now reformulate Definition 2 accordingly, by introducing the notion of a conditional alteration of an infinite series.

Definition 8. *A conditional alteration of an infinite series*

$$\lim_{N\to\infty}\sum_{n\leqslant N} a_n$$

is done by introducing an indicator function $\delta_N(n)$ into the series as follows.

$$\lim_{N\to\infty}\sum_{n\leqslant N} a_n\delta_N(n)$$

The indicator function $\delta_N(n)$ is such that

$$\delta_N(n) = 1,\ n < M(N)$$
$$\delta_N(n) = 0,\ n > N$$

for some function $M(N)$, such that

$$M(N) < N$$

for all sufficiently large N, and such that

$$\lim_{N\to\infty} M(N) = \infty$$

Whenever n is such that $M(N) \leqslant n \leqslant N$, the action of $\delta_N(n)$ on n is dictated by some rule $\mathcal{R}(M, N, n)$. The rule $\mathcal{R}(M, N, n)$ also defines $M(N)$, for all sufficiently large N.

We next claim in the following theorem that the conditional alteration of a series is equivalent to permuting the indices of the terms of a series, in the sense that they both can alter the magnitude of a conditionally convergent series to any number, but leave the magnitude of an absolutely convergent series invariant.

Theorem 9. *The conditional alteration can change the magnitude of an conditionally convergent series to any number, but cannot alter the magnitude of an absolutely convergent series.*

Proof. Let the series $\sum_{n=1}^{\infty} a_n$ converge. In other words,

$$\sum_{n=1}^{\infty} a_n < \infty \tag{54}$$

Also, let the series $\sum_{n=1}^{\infty} a_n$ diverge absolutely:

$$\sum_{n=1}^{\infty} |a_n| = \infty \tag{55}$$

Define terms a_n^+ and a_n^- as follows.

$$\begin{aligned} a_n^+ &= \frac{|a_n| + a_n}{2} = \begin{cases} a_n, \text{ if } a_n \geqslant 0 \\ 0, \text{ if } a_n < 0 \end{cases} \\ a_n^- &= \frac{|a_n| - a_n}{2} = \begin{cases} 0, \text{ if } a_n \geqslant 0 \\ |a_n|, \text{ if } a_n < 0 \end{cases} \end{aligned} \tag{56}$$

Thus, a_n^+ is positive and different from zero only when a_n is strictly positive, and a_n^- is positive and different from zero only when a_n is strictly negative.

Obviously, one finds

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- < \infty \\ \sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^- = \infty \end{aligned} \tag{57}$$

The divergence of $\sum_{n=1}^{\infty} |a_n|$ demonstrates that at least one of the series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ diverges to infinity. On the other hand, the convergence of $\sum_{n=1}^{\infty} a_n$ demands that both $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ diverge to infinity. We collect this result in the next equation.

$$\sum_{n=1}^{\infty} a_n^{\pm} = \infty \tag{58}$$

Furthermore, the convergence of $\sum_{n=1}^{\infty} a_n$ demonstrates that $a_n \to 0$ as $n \to \infty$. Consequently, $a_n^+ \to 0$ and $a_n^- \to 0$ as $n \to \infty$ as well.

Now introduce some functions $M^{\pm}(N)$ such that $M^{\pm}(N) < N$ for all sufficiently large N, and such that $M^{\pm}(N) \to \infty$ as $N \to \infty$. Define functions $f^{\pm}(N)$ by

$$\sum_{1}^{N} a_n^{\pm} = f^{\pm}(N) \tag{59}$$

Do notice that, since $a_n^{\pm} \geqslant 0$, the functions $f^{\pm}(N)$ are increasing, and without bounds.

Consider the following, not necessarily infinite, series now.

$$\lim_{N\to\infty} \sum_{n=M^{\pm}(N)}^{N} a_n^{\pm} = \lim_{N\to\infty} \left(\sum_{n=1}^{N} a_n^{\pm} - \sum_{n=1}^{M^{\pm}(N)-1} a_n^{\pm} \right) = \lim_{N\to\infty} \Big\{ f^{\pm}(N) - f^{\pm}\big(M^{\pm}(N)-1\big) \Big\} \tag{60}$$

We first notice that there exist functions $M^{\pm}(N)$ such that (60) is infinitely large, since functions $f^{\pm}(N)$ are increasing without bounds in the limit $N \to \infty$. To demonstrate this, pick numbers $M^{\pm}$. Obviously, $\sum_{1}^{M^{\pm}} a_n^{\pm} \to \infty$ as $M^{\pm} \to \infty$. Since $f^{\pm}(N)$ increases without bounds, there is N such that for any positive function $g(M^{\pm})$ one finds $f^{\pm}(N) - f^{\pm}\big(M^{\pm}(N) - 1\big) = g(M^{\pm})$. Choose $g(M^{\pm})$ such that $g(M^{\pm}) \to \infty$ as $M^{\pm} \to \infty$ and the claim follows.

Since $g(M^{\pm})$ is arbitrary and $a^{\pm} \to 0$ in the limit $M^{\pm} \to \infty$, it follows that there exist functions $M^{\pm}(N)$ such that (60) converges to any number.

Hence, as a further refinement, for any positive real R, finite or infinite, since $a^{\pm} \to 0$ in the limit $M^{\pm} \to \infty$, there exist functions $M^{\pm}(N)$ and $\delta_N(n)$ such that

$$\lim_{N\to\infty} \sum_{n=M^{\pm}(N)}^{N} a_n^{\pm} \delta_N(n) = R^{\pm} \tag{61}$$

To demonstrate this, choose $\delta(n) = 1$ for all n. Then with a suitable $M^{\pm}(N)$, the number $R^{\pm}$ can be infinitely large. Now choose $\delta(n) = 0$ whenever n is such that $M^{\pm}(N) < n \leqslant N$ for both a_n^{+} and a_n^{-}. Then $R^{\pm} = 0$. Thus, since $a^{\pm} \to 0$, by picking a suitable rule $\mathcal{R}(M, N, n)$, one can make $R^{\pm}$ any positive number.

Finally, we notice that this procedure does not change the magnitude of any series that converges absolutely, since then (60) and (61) both tend to zero by the Cauchy convergence criterion. □

We can also demonstrate this by the use of the standard proof of the Riemann series theorem [6, p.76, Theorem3.54]. In the standard proof one alters the magnitude of a conditionally convergent series $\sum a_n$ to any arbitrary positive number A by taking the finite sub-series of the first k_1^+ terms a^+ such that the magnitude of the sub-series $\sum_{n\leqslant k_1^+} a_n^+$ is the least of all the sub-series of consecutive terms of $\sum a_n^+$ not less than A. Then one subtracts the first k_1^- terms a^- such that the magnitude of the sub-series $\sum_{n\leqslant k_1^-} a_n^+$ is the least of all the sub-series of consecutive terms of $\sum a_n^-$ such that $\sum_{n\leqslant k_1^+} a_n^+ - \sum_{n\leqslant k_1^-} a_n^+ \leqslant A$.

Do notice that if $k_1^+ \neq k_1^-$, this way one omitted some positive or negative terms a form the first $\max\left\{k_1^+, k_1^-\right\}$ terms a_n of the series $\sum a_n$.

In the next step, one ads and subtracts the next remaining consecutive terms a^+ and a_- so that the new sub-series is as close to A as possible. Since $a \to 0$, one finds that in each step the newly formed sub-series comes ever closer to A. Do notice that at each step l, one possibly omits some positive or negative terms of the first k_l terms of the original series. In the end, one omits infinitely many infinitely small terms in the limit of growing n. The difference of the magnitudes after the re-arranging is done comes from these omitted terms, given by Eq. (60).

A further refinement can then be done by introducing the rule $\mathcal{R}(M, N, n)$, as demonstrated earlier.

So, Definition 8 is equivalent to rearranging the terms by permuting indices, in the sense that both techniques can change the magnitude of a series that converges conditionally to any number, and cannot change the magnitude of an absolutely convergent series.

10 The Perron formula with indicator functions

We next rewrite the Perron formula for rearranged series by rearranging the terms by conditional alteration as prescribed by Definition 8.

Lemma 6 (Perron formula with indicator functions). *Let, for any* $\sigma >$ 1,

$$f_N(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \delta_N(n) \tag{62}$$

where $a_n = O\left\{g(n)\right\}$ *with* $g(n)$ *being non-decreasing, and*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} = O\left\{\frac{1}{(\sigma-1)^a}\right\} \tag{63}$$

as $\sigma \to 1$. *Then if* $c > 0$, $\sigma + c > 1$, x *is not an integer, and* $N = \lfloor x \rfloor$,

$$\sum_{n<x} \frac{a_n}{n^s}\delta_N(n) =$$

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f_N(s+w)\frac{x^w}{w}dw + O\left\{\frac{x^c}{T(\sigma+c-1)^a}\right\} + \tag{64}$$

$$O\left\{\frac{g(2x)x^{1-\sigma}\log x}{T}\right\} + O\left\{\frac{g(N)x^{1-\sigma}}{T|x-N|}\right\}$$

Proof. We again use the Titchmarsh's results here [4, pp.61-62], since they're valid in general.

The starting result [4, p.61], valid in general, given earlier by Eq. (11), is

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^w \frac{dw}{w} = \begin{cases} 1 + O\left\{\frac{(x/n)^c}{T\log\left(x/n\right)}\right\}, \text{ if } n < x \\ O\left\{\frac{(x/n)^c}{T\log\left(x/n\right)}\right\}, \text{ if } n > x \end{cases} \tag{65}$$

Multiply by $a_n n^{-s}\delta_N(n)$ and sum over all natural numbers n.

$$\sum_{n<x}\frac{a_n}{n^s}\delta_N(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\sum_{n=1}^{\infty}\frac{a_n}{n^{s+w}}\delta_N(n)\frac{x^w}{w}dw + \tag{66}$$

$$O\left\{\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}\left|\log\left(x/n\right)\right|}\delta_N(n)\right\}$$

Since $\sigma + c > 1$, the series in the integrand converges absolutely to $f_N(s+w)$ for any N, as well as in the limit $N \to \infty$, and since $\delta_N(n) \in \{0, 1\}$, one just substitutes $\delta_N(n) \to 1$ and estimates the error term the same way Titchmarsh did in his Lemma 3.12, and the lemma follows. □

We notice here that in Eq. (64) neither the left hand side nor the right hand side coincide with Eq. (10) of Lemma 1, which wasn't so when rearranging the terms by permuting.

Lemma 7. *The series*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \delta_\infty(n) \tag{67}$$

is convergent, and its sum is $1/\zeta(s)$ *for every* s *with* $m < \sigma$.

Proof. Define function $\zeta_N(s)$ by

$$\frac{1}{\zeta_N(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \delta_N(n) \tag{68}$$

We notice here that $1/\zeta_N(s)$ is holomorphic and the series in Eq. (68) converges absolutely on the halfplane $\sigma > 1$ for every natural N, as well as in the limit $N \to \infty$, for every rule $\mathcal{R}(M, N, n)$.

Obviously, whenever $\sigma > 1$, one finds

$$\lim_{N\to\infty} \frac{1}{\zeta_N(s)} = \frac{1}{\zeta(s)} \tag{69}$$

In Lemma 6 take $a_n = \mu(n)$, $f_N(s) = 1/\zeta_N(s)$, $c = 2$, and x half an odd integer. We obtain

$$\begin{aligned} \sum_{n<x} \frac{\mu(n)}{n^s} \delta_N(n) &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta_N(s+w)} \frac{x^w}{w} dw + O\left(\frac{x^2}{T}\right) \\ &= \frac{1}{2\pi i} \left(\int_{2-iT}^{m-\sigma+\delta-iT} + \int_{m-\sigma+\delta-iT}^{m-\sigma+\delta+iT} + \int_{m-\sigma+\delta+iT}^{2+iT} \right) \frac{1}{\zeta_N(s+w)} \frac{x^w}{w} dw \\ &\quad + \frac{1}{\zeta_N(s)} + O\left(\frac{x^2}{T}\right) \end{aligned} \tag{70}$$

with $0 < \delta < \sigma - m$.

The rest of the proof goes exactly the same way as did the proof of Lemma 4, since $\zeta_N(s) \to \zeta(s)$ as $x \to \infty$. □

Hence, since rearranging with indicator functions is equivalent to rearranging by permuting, we conclude that all the results put forward so far are valid.

11 Conclusions

In this manuscript we use the Perron formula to connect zeta evaluated on the root free halfplane to zeta evaluated on the critical strip. This is possible since the Perron formula is of the form f(s)=O f(s+w) with O being an integral operator. The variable s+w is on the root free halfplane, and yet s can be on the critical strip. Hence, the Perron formula serves as a form of a functional equation that connects the critical strip with the root free halfplane. Then, one simply notices that in the Perron formula, the left hand side converges only conditionally, whilst the right hand side converges absolutely. This, of course, cannot be, since the left side of an equation is always equal to the right side. This contradiction when examined in detail disproves the Riemann hypothesis. This method is employed on an arbitrary distribution of zeta roots as well, concluding that zeta has a root arbitrarily close to the vertical line passing through unity.

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