The abc Conjecture as expansion of powers of binomials

Abstract: In this note I will show how Beal's conjecture can be used to study abc conjecture. I will first show how Beal's conjecture was proved and derive the necessary steps that will lead to further understand the abc conjecture hoping this will aid in proving it. In short, Beal's conjecture was identified as a univariate Diophantine polynomial identity derived from the binomial identity by expansion of powers of binomials, e.g. the binomial $(\lambda x^l + \gamma y^l)^n$; λ, γ, l, n are positive integers. The idea is that upon expansion and reduction to two terms we can cancel the gcd from the identity equation which leaves the coefficient terms coprime and effectively describes the abc conjecture. To further study the abc terms we need to specifically look for criterion upon which the general property of abc conjecture that states that if the two numbers a and b of the conjecture are divisible by large powers of small primes, $a + b$ tends to be divisible by small powers of large primes which leads to $a + b$ be divisible by large powers of small primes. In this note I only open the door to investigate related possible criterions that may lead to further understand the abc conjecture by expressing it in terms of binomial expansions as Beal's conjecture was handled.

Introduction and results:

The abc conjecture concerns the sum c of two relatively prime positive integers a and b; $a < b$ and the radical of the triple (a, b, c); $rad(abc)$, defined as the product of the primes dividing them. An abc-triple is a triple of relatively prime positive integers with $a + b = c$ and $rad(abc) < c$. For the majority of $a + b = c$ triples seems to follow the rule defined by $rad(abc) > c$ but few follow the exception to the rule of abc conjecture; $rad(abc) < c$. Examples of abc triples are,

$$3 + 5^3 = 2^7$$

Where $rad(abc) = 30 < (the\ size = 25)$

$$11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23$$

Where $rad(abc) = 53130 < (the\ size = 48234496)$

The abc conjecture contains some fundamental features of various problems in number theory and a number of famous conjectures and theorems would follow immediately from the abc conjecture.

In an earlier publication [1], the author proved Beal's conjecture by identifying it as a binomial identity. To start with, Beal's conjecture states that if $x^a + y^b = z^c$, where a, b, c, x, y, z are positive integers and a, b, c > 2, then x, y, z have a common prime factor. The following was the main theme of the proof: The origin of Beal's equation is a binomial identity which upon expansion and by Pascal's rule produces terms of the same power if we convert the binomial identity into univariate equation by equating the two variables as a special case. We can proceed to prove constructively by elementary algebra that Beal's equation is in fact algebraic identity. Let's recall that a binomial identity describes the expansion of powers of a binomial to produce terms of the same power if we replace y with x as a special case. If the terms on both sides of the expanded binomial identity,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\left(\lambda x^l + \gamma y^l\right)^n = \lambda^n x^{nl} + \cdots\cdots + \gamma^n y^{nl} \tag{1}$$

where λ, γ, l, n are positive integers, are converted into monomials with the same variable, the terms can be added algebraically to produce a univariate polynomial identity with one term on the LHS and two terms on the RHS, one of which must be a perfect power. This process describes Beal's equation and produces an identity on its own. This is clear since after we replace y with x we can expand the polynomial $(x+x)^n$ into a sum of terms of the form $\alpha x^b x^c$, where the coefficient α is governed by Pascal's rule and the exponents b and c are positive integers with $b + c = n$. For all the terms in the expansion expression to have the same power, the two variables x, y must be raised to the same power l. The binomial identity produces Beal's solutions of powers $(a, b, c) > 2$ when $n \geq 3, l > 1$ with signature $(a, b, c) = (nl, nl+1, n)$. A simple example is the following; for $n = 3$, the binomial $(\lambda x + \gamma y)^3$ produces Beal's solutions of the form $a, b, c = (3, 4, 3)$, e.g. the equation,

$$(x+y)^3 = x^3 + 3x^2y + 3y^2x + y^3$$

simplifies to two terms on the RHS after replacing y with x, and by taking the first term as the leading term with perfect power we get,

$$(2x)^3 = x^3 + 7x^3$$

Here we must choose $x = 7$ to get the solution,

$$7^3 + 7^4 = 14^3$$

For $l > 1$, we clearly find the single term in the Diophantine equation will always attain the power n of the binomial $(\lambda x^l + \gamma y^l)^n$ while the LHS terms attain higher powers as clear from the signature $(a, b, c) = (nl, nl+1, n)$. The trick here is to choose the coefficients λ, γ such that they produce perfect power terms when cancel the GCD in the equation to produce *abc* triples. With this restriction we see that the coefficient of the sum term (7 in the previous example) may not be a perfect power term with single prime base. But we notice that the coefficient factors in big primes compared to the coefficient itself. This leads to a large radical to the equation and a no-hit as an *abc* triple. An example is the following;

For $l = 4$, the binomial $(2^4x^4 + y^4)^3$ produces,

$$(2^4x^4 + y^4)^3 = 2^{12}x^{12} + 3 \cdot 2^8 \cdot x^8y^4 + 3 \cdot y^8 \cdot 2^4x^4 + y^{12}$$

As we can see, the coefficients are specifically chosen to produce c as a single prime base as well as the leading term (17,2), while the sum term produces a composite number.

If we take the term $2^{12}x^{12}$ as the leading term because it is a single power term we get the equation,

$$(17x^4)^3 = 2^{12}x^{12} + 817x^{12}$$

Substituting $x = 817$ we get Beal's solution,

$$1634^{12} + 817^{13} = 7574206607057^3$$

If we cancel the GCD of 817^{12} to obtain a relative prime terms as abc conjecture requires, we get an abc triple $(817, 2^{12}, 17^3)$ with a solution,

$$2^{12} + 817 = 17^3$$

Since 817 factors to 43·19, it follows that the solution is not an abc-hit since rad (a, b, c) is 27778 while c is 4913.

We notice that the abc conjecture is insensitive to l since we are cancelling the gcd to get the abc equation from Beal's binomial $\left(\lambda x^l + \gamma y^l\right)^n$. This reduces abc binomial identity to $(\lambda x + \gamma y)^n$. Here I am only introducing the abc binomial identity with some examples of hit and non-hit to abc conjecture in the hope that it introduces a step forward to fully understand the abc conjecture. Basically then, to produce abc triples, we choose any positive integer for λ, γ, n in the binomial $(\lambda x + \gamma y)^n$, replace y with x, expand it, reduce it to two terms on the RHS and one term on the LHS, and finally cancel the common factor x^n.

The following are *abc* binomials that are *abc*-hits,

Example 1: For the binomial,

$$(x + 47y)^2$$

Expanding, replacing y with x, and reducing the terms we get,

$$(48x)^2 = x^2 + 2303x^2$$

By cancelling the gcd $x^2$and expressing the terms in their base primes we get the ***abc*-hit**,

$$1 + 7^2 \cdot 47 = 2^8 \cdot 3^2$$

Example 2: For the binomial,

$$(7x + 243y)^2$$

Expanding, replacing y with x, and reducing the terms we get,

$$(250x)^2 = 7^2x^2 + 62451x^2$$

By cancelling the gcd x^2 and expressing the terms in their base primes we get the ***abc*-hit**,

$$7^2 + 3^5 \cdot 257 = 2^2 \cdot 125^2$$

Example 3: For the binomial

$$\left(3^2x + 32y\right)^2$$

Expanding, replacing y with x, and reducing the terms we get,

$$(41x)^2 = 3^4x^2 + 1600x^2$$

By cancelling the gcd x^2 and expressing the terms in their base primes we get the ***abc*-hit**,

$$3^4 + 2^6 \cdot 5^2 = 41^2$$

The following are *abc* binomials that are not-*abc*-hits,

Example 4: For the binomial,

$$(3x + 2y)^3$$

Expanding, replacing y with x, and reducing the terms we get,

$$(5x)^3 = 3^3x^3 + 206x^3$$

By cancelling the gcd $x^3$and expressing the terms in their base primes we get the **not-*abc*-hit**,

$$5^3 + 3^3 = 2 \cdot 7^2$$

Example 5: For the binomial,

$$(5x + 8y)^2$$

Expanding, replacing y with x, and reducing the terms we get,

$$(13x)^2 = 5^2x^2 + 144x^2$$

By cancelling the gcd x^2 and expressing the terms in their base primes we get the **not-*abc*-hit**,

$$5^2 + 2^4 \cdot 3^2 = 13^2$$

Example 6: For the binomial

$$(2x + y)^4$$

Expanding, replacing y with x, and reducing the terms we get,

$$(3x)^4 = 2^4x^4 + 65x^4$$

By cancelling the gcd x^4 and expressing the terms in their base primes we get the **not-*abc*-hit**,

$$3^4 = 2^4 + 5 \cdot 13$$

Conclusions

I introduced an equation that produces a triple for the abc conjecture as the coefficients of the terms of the expansion of powers of binomials, therefore defining it as a binomial identity of the form $(\lambda x + \gamma y)^n$, opening the door to further investigate possible criterion that helps understand why there are exceptions to the general trend of the triples that the abc conjecture produces and obeys the rule $d = rad(abc) > c$, whether there are infinitely many of them and, when there is exception to the general rule, why d is usually not much smaller than c. It is obvious that we can easily produce infinitely many abc triples from expansion of powers of binomials by the infinitely many integer choices to the constants $\lambda, \gamma \ and \ n$, but it is not so obvious whether the triples obey the general rule or not.

References

[1] Kamal Barghout, "Beal's Conjecture as Univariate Polynomial Identity Derived from Algebraic Expansion of Powers of Binomials: Analysis and Proof" http://vixra.org/abs/1712.0135