Corrected weak duality theorem by way of refutation of the strong duality theorem

Abstract: The equation of the weak duality theorem, (Ax≤b, x≥0) ≤ (A^Ty≥c, y≥0), is confirmed as tautologous. Three proofs of it in the literature are *not* tautologous. The equation of the strong duality theorem, (Ax≤b, x≥0) = (A^Ty≥c, y≥0), is refuted as *no*t tautologous. These form a *non* tautologous fragment of the universal logic VŁ4. What follows is the weak duality theorem could just as easily exclude the "or equal to" relation to read (Ax≤b, x≥0) < (A^Ty≥c, y≥0) as the corrected weak duality theorem.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **`F`** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **`F`** as contradiction, Ø, Null, ⊥, zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1; (%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: math.ubc.ca/~anstee/math340/340weakduality.pdf; also,
www.coursera.org/lecture/approximation-algorithms-part-2/proof-of-weak-duality-theorem-eAkFN

Theorem (Weak Duality) Let x* be a feasible solution to the primal and let y* be a feasible solution to the dual where

primal max c·x: (Ax ≤ b, x ≥ 0)
dual min b·y: (A^Ty ≥ c, y ≥ 0).
Then c·x* ≤ b·y* . (0.1.1)

LET p, q, r, s, t, x, y: A, b, c, s, ^{T}, x, y

~(((r>((p&t)&y))&~((s@s)>y))<((q<(p&x))&((s@s)>x)))=(s=s) ;

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TTTT TTTT TTTT TTTT
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(0.1.2)

Proof: … We obtain

$$c \cdot x = x^T c \le x^T A^T y = y^T A x \le y^T b = b \cdot y \quad (0.2.1)$$

((r&x)=~((((x&t)&p)&(t&y))<((x&t)&r))) =(~(((y&t)&q)<((y&t)&(p&x)))=(q&y)) ;

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TTTT TTTT TTTT TTTT(16)
TTTT FFFF TTTT FFFF(16)
TTFF TTFF TTFF TTFF( 1)}x8
TTTT TTTT TTTT TTTT( 1)}
TFTT FFFT TFTT FFFT( 1)}x8
```
(0.2.2)

We read off

$c \cdot x \leq b \cdot y.$ (0.3.1)

~((q&y)<(r&x))=(s=s) ;

TTTT TTTT TTTT TTTT(32)
TT**FF** TT**FF** TT**FF** TT**FF**(16)
TT**FF** TTTT TT**F**F TTTT(16) (0.3.2)

The case of equality is of course of great interest and strong duality and complementary slackness deal with equality. Nonetheless, weak duality is of independent interest and is a model for other optimization problems for which we have no strong duality.

From: en.wikipedia.org/wiki/Weak_duality

In applied mathematics, weak duality is a concept in optimization which states that the duality gap is always greater than or equal to 0. That means the solution to the primal (minimization) problem is always greater than or equal to the solution to an associated dual problem. This is opposed to strong duality which only holds in certain cases.

The primal problem: Maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$; (1.1)
The dual problem: Minimize $b^T y$ subject to $A^T y \geq c$, $y \geq 0$. (2.1)
The weak duality theorem: $c^T x \leq b^T y$. (3.1)

Remark 3.1: We write the weak duality theorem Eq. 3.1 as 1.1 ≤ 2.1:

$(Ax \leq b,\ x \geq 0) \leq (A^T y \geq c,\ y \geq 0)$ (4.1)

LET p, q, r, s, t, x, y: A , b , c , s, ^{T},x, y.

~(((r>((p&t)&y))&~((s@s)>y))<((q<(p&x))&((s@s)>x)))=(s=s) ;

TTTT TTTT TTTT TTTT (4.2)

Proof: $c^T x = x^T c \leq x^T A^T y \leq b^T y$ (5.1)

((r&t)&x)=~(~(((q&t)&y)<((x&t)&((p&t)&y)))<((x&t)&r)) ;

TTTT TTTT TTTT TTTT(32)
TTTT TTTT TTTT TTTT(1)}x8
TT**FF** TT**FF** TT**FF** TT**FF**(1)}
TTTT TTTT TTTT TTTT(1)}x8
TT**F**T TTTT TT**F**T TTTT(1)} (5.2)

Eqs. 1.2 and 4.2 as rendered are tautologous and are equivalents, hence confirming the weak duality theorem. However, Eqs. 2.2, 3.2, and 5.2 are not tautologous and not equivalents, hence refuting three proofs of the theorem in the literature.

We turn to strong duality.

From: www.coursera.org/lecture/approximation-algorithms-part-2/proof-of-weak-duality-theorem-eAkFN

Strong duality theorem in general (10.1)
(P) primal max $c \cdot x$: $(Ax \leq b,\ x \geq 0)$
(D) dual min $b \cdot y$: $(A^T y \geq c,\ y \geq 0)$.
[empty value is 0]

Four possible cases:

(P) is empty, (D) has value plus infinity (11.1)

(s@s)&((q<(p&x))&((s@s)>x)) ;
FFFF FFFF FFFF FFFF (11.2)

(D) is empty, (P) has value minus infinity (12.1)

((r>((p&t)&y))&~((s@s)>y))=(s=s) ;
FFFF FFFF FFFF FFFF (12.2)

value(P)=value(D) (13.1)

((q<(p&x))&((s@s)>x))=((r>((p&t)&y))&~((s@s)>y)) ;
TT**FF** TT**FF** TT**FF** TT**FF**(16)
TT**F**T TT**F**T TT**F**T TT**F**T(16) (13.2)

[(P) and (D) empty] (14.1)

(s@s)&(s@s) ;
FFFF FFFF FFFF FFFF (14.2)

Remark 10.0: We write strong duality as the four possible states of Eqs. 11.1 or 12.1 or 13.1 or 14.1 (15.1)

((((s@s)&((q<(p&x))&((s@s)>x)))+((s@s)&((r>((p&t)&y))&~((s@s)>y))))+
(((((q<(p&x))&((s@s)>x))=((r>((p&t)&y))&~((s@s)>y)))+((s@s)&(s@s))))=(s=s) ;
TT**FF** TT**FF** TT**FF** TT**FF**(16)
TT**F**T TT**F**T TT**F**T TT**F**T(16) (15.2)

Eq. 15.2 is equivalent to 13.2, as expected, and *not* tautologous, hence refuting the strong duality theorem.

What follows is that the weak duality theorem of Eq. 4.2 as (Ax≤b, x≥0) ≤ (A^Ty≥c, y≥0) could just as easily exclude the "or equal to" relation to read (Ax≤b, x≥0) $<$ (A^Ty≥c, y≥0) as the corrected weak duality theorem.