

# Corrected weak duality theorem by way of refutation of the strong duality theorem

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**Abstract:** The equation of the weak duality theorem,  $(Ax \leq b, x \geq 0) \leq (A^T y \geq c, y \geq 0)$ , is confirmed as tautologous. Three proofs of it in the literature are *not* tautologous. The equation of the strong duality theorem,  $(Ax \leq b, x \geq 0) = (A^T y \geq c, y \geq 0)$ , is refuted as *not* tautologous. These form a *non* tautologous fragment of the universal logic VL4. What follows is the weak duality theorem could just as easily exclude the “or equal to” relation to read  $(Ax \leq b, x \geq 0) < (A^T y \geq c, y \geq 0)$  as the corrected weak duality theorem.

We assume the method and apparatus of Meth8/VL4 with Tautology as the designated proof value, **F** as contradiction, **N** as truthity (non-contingency), and **C** as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET  $\sim$  Not,  $\neg$ ; + Or,  $\vee$ ,  $\cup$ ,  $\sqcup$ ; - Not Or; & And,  $\wedge$ ,  $\cap$ ,  $\sqcap$ ,  $\cdot$ ; \ Not And;  
 $>$  Imply, greater than,  $\rightarrow$ ,  $\Rightarrow$ ,  $\mapsto$ ,  $>$ ,  $\supset$ ,  $\Rightarrow$ ;  $<$  Not Imply, less than,  $\in$ ,  $<$ , **C**,  $\neq$ ,  $\neq$ ,  $\ll$ ,  $\lesssim$ ;  
 $=$  Equivalent,  $\equiv$ ,  $:=$ ,  $\Leftrightarrow$ ,  $\leftrightarrow$ ,  $\hat{=}$ ,  $\approx$ ,  $\cong$ ; @ Not Equivalent,  $\neq$ ;  
 $\%$  possibility, for one or some,  $\exists$ ,  $\diamond$ , **M**; # necessity, for every or all,  $\forall$ ,  $\square$ , **L**;  
 $(z=z)$  **T** as tautology,  $\top$ , ordinal 3;  $(z@z)$  **F** as contradiction,  $\emptyset$ , Null,  $\perp$ , zero;  
 $(\%z\>\#z)$  **N** as non-contingency,  $\Delta$ , ordinal 1;  $(\%z\<\#z)$  **C** as contingency,  $\nabla$ , ordinal 2;  
 $\sim(y < x)$  ( $x \leq y$ ), ( $x \subseteq y$ ), ( $x \sqsubseteq y$ );  $(A=B)$   $(A\sim B)$ .  
 Note for clarity, we usually distribute quantifiers onto each designated variable.

From: [math.ubc.ca/~ansteemath340/340weakduality.pdf](http://math.ubc.ca/~ansteemath340/340weakduality.pdf); also, [www.coursera.org/lecture/approximation-algorithms-part-2/proof-of-weak-duality-theorem-eAkFN](http://www.coursera.org/lecture/approximation-algorithms-part-2/proof-of-weak-duality-theorem-eAkFN)

Theorem (Weak Duality) Let  $x^*$  be a feasible solution to the primal and let  $y^*$  be a feasible solution to the dual where

$$\begin{aligned} \text{primal max } c \cdot x: & (Ax \leq b, x \geq 0) \\ \text{dual min } b \cdot y: & (A^T y \geq c, y \geq 0). \\ \text{Then } c \cdot x^* & \leq b \cdot y^*. \end{aligned} \tag{1.1}$$

$$\begin{aligned} \text{LET } p, q, r, s, t, x, y: & \quad A, b, c, s, \top, x, y \\ \sim(((r > ((p \& t) \& y)) \& \sim((s @ s) > y)) < ((q < (p \& x)) \& ((s @ s) > x))) = (s = s); \end{aligned} \tag{1.2}$$

TTTT TTTT TTTT TTTT

Proof: ... We obtain

$$c \cdot x = x^T c \leq x^T A^T y = y^T A x \leq y^T b = b \cdot y \tag{2.1}$$

$$\begin{aligned} ((r \& x) = \sim(((x \& t) \& p) \& (t \& y)) < ((x \& t) \& r))) = (\sim(((y \& t) \& q) < ((y \& t) \& (p \& x)))) = (q \& y); \end{aligned}$$

TTTT TTTT TTTT TTTT (16)  
 TTTT **FFFF** TTTT **FFFF** (16)  
 TT**FF** TT**FF** TT**FF** TT**FF** ( 1 ) x8  
 TTTT TTTT TTTT TTTT ( 1 )  
 T**FFT** T**FFT** T**FFT** T**FFT** ( 1 ) x8

(2.2)

We read off

$$c \cdot x \leq b \cdot y. \tag{3.1}$$

$$\sim((q \& y) < (r \& x)) = (s = s); \quad \begin{array}{l} \text{TTTT TTTT TTTT TTTT (32)} \\ \text{TTFF TTFF TTFF TTFF (16)} \\ \text{TTFF TTTT TTFF TTTT (16)} \end{array} \tag{3.2}$$

The case of equality is of course of great interest and strong duality and complementary slackness deal with equality. Nonetheless, weak duality is of independent interest and is a model for other optimization problems for which we have no strong duality.

From: [en.wikipedia.org/wiki/Weak\\_duality](http://en.wikipedia.org/wiki/Weak_duality)

In applied mathematics, weak duality is a concept in optimization which states that the duality gap is always greater than or equal to 0. That means the solution to the primal (minimization) problem is always greater than or equal to the solution to an associated dual problem. This is opposed to strong duality which only holds in certain cases.

$$\text{The primal problem: Maximize } c^T x \text{ subject to } Ax \leq b, x \geq 0; \tag{1.1}$$

$$\text{The dual problem: Minimize } b^T y \text{ subject to } A^T y \geq c, y \geq 0. \tag{2.1}$$

$$\text{The weak duality theorem: } c^T x \leq b^T y. \tag{3.1}$$

**Remark 3.1:** We write the weak duality theorem Eq. 3.1 as  $1.1 \leq 2.1$ :

$$(Ax \leq b, x \geq 0) \leq (A^T y \geq c, y \geq 0) \tag{4.1}$$

$$\begin{array}{l} \text{LET } p, q, r, s, t, x, y: \quad A, b, c, s, ^T, x, y. \\ \sim(((r > ((p \& t) \& y)) \& \sim((s @ s) > y)) < ((q < (p \& x)) \& ((s @ s) > x))) = (s = s); \\ \text{TTTT TTTT TTTT TTTT} \end{array} \tag{4.2}$$

$$\text{Proof: } c^T x = x^T c \leq x^T A^T y \leq b^T y \tag{5.1}$$

$$\begin{array}{l} ((r \& t) \& x) = \sim(\sim(((q \& t) \& y) < ((x \& t) \& ((p \& t) \& y))) < ((x \& t) \& r)); \\ \text{TTTT TTTT TTTT TTTT (32)} \\ \text{TTTT TTTT TTTT TTTT ( 1) } \times 8 \\ \text{TTFF TTFF TTFF TTFF ( 1) } \\ \text{TTTT TTTT TTTT TTTT ( 1) } \times 8 \\ \text{TTFT TTTT TTFT TTTT ( 1) } \end{array} \tag{5.2}$$

Eqs. 1.2 and 4.2 as rendered are tautologous and are equivalents, hence confirming the weak duality theorem. However, Eqs. 2.2, 3.2, and 5.2 are not tautologous and not equivalents, hence refuting three proofs of the theorem in the literature.

We turn to strong duality.

From: [www.coursera.org/lecture/approximation-algorithms-part-2/proof-of-weak-duality-theorem-eAkFN](http://www.coursera.org/lecture/approximation-algorithms-part-2/proof-of-weak-duality-theorem-eAkFN)

$$\text{Strong duality theorem in general} \tag{10.1}$$

$$(P) \text{ primal max } c \cdot x: (Ax \leq b, x \geq 0)$$

$$(D) \text{ dual min } b \cdot y: (A^T y \geq c, y \geq 0).$$

[empty value is 0]

Four possible cases:

$$(P) \text{ is empty, } (D) \text{ has value plus infinity} \quad (11.1)$$

$$(s@s) \& ((q < (p \& x)) \& ((s@s) > x)) ;$$

$$\mathbf{FFFF \ FFFF \ FFFF \ FFFF} \quad (11.2)$$

$$(D) \text{ is empty, } (P) \text{ has value minus infinity} \quad (12.1)$$

$$((r > ((p \& t) \& y)) \& \sim((s@s) > y)) = (s=s) ;$$

$$\mathbf{FFFF \ FFFF \ FFFF \ FFFF} \quad (12.2)$$

$$\text{value}(P) = \text{value}(D) \quad (13.1)$$

$$((q < (p \& x)) \& ((s@s) > x)) = ((r > ((p \& t) \& y)) \& \sim((s@s) > y)) ;$$

$$\mathbf{TTF\!F \ TTF\!F \ TTF\!F \ TTF\!F} \quad (16)$$

$$\mathbf{TTF\!T \ TTF\!T \ TTF\!T \ TTF\!T} \quad (16) \quad (13.2)$$

$$[(P) \text{ and } (D) \text{ empty}] \quad (14.1)$$

$$(s@s) \& (s@s) ;$$

$$\mathbf{FFFF \ FFFF \ FFFF \ FFFF} \quad (14.2)$$

**Remark 10.0:** We write strong duality as the four possible states of Eqs. 11.1 or 12.1 or 13.1 or 14.1 (15.1)

$$(((s@s) \& ((q < (p \& x)) \& ((s@s) > x))) + ((s@s) \& ((r > ((p \& t) \& y)) \& \sim((s@s) > y)))) +$$

$$(((q < (p \& x)) \& ((s@s) > x)) = ((r > ((p \& t) \& y)) \& \sim((s@s) > y))) + ((s@s) \& (s@s))) = (s=s) ;$$

$$\mathbf{TTF\!F \ TTF\!F \ TTF\!F \ TTF\!F} \quad (16)$$

$$\mathbf{TTF\!T \ TTF\!T \ TTF\!T \ TTF\!T} \quad (16) \quad (15.2)$$

Eq. 15.2 is equivalent to 13.2, as expected, and *not* tautologous, hence refuting the strong duality theorem.

What follows is that the weak duality theorem of Eq. 4.2 as  $(Ax \leq b, x \geq 0) \leq (A^T y \geq c, y \geq 0)$  could just as easily exclude the “or equal to” relation to read  $(Ax \leq b, x \geq 0) < (A^T y \geq c, y \geq 0)$  as the corrected weak duality theorem.