Refutation of Hilbert's first epsilon theorem in intuitionistic and intermediate logics

Abstract: From the universal quantifier shift of (∀xA(x)→B)→∃x(B→A(x)) as *not* tautologous, the intermediate logic **L** is refuted, refuting Hilbert's first epsilon theorem and intuitionistic logic, and forming a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **`F`** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **`F`** as contradiction, Ø, Null, ⊥, zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1; (%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Baaz, M.; Zach, R. (2019). The first epsilon theorem in pure intutionistic and intermediate logics. arxiv.org/pdf/1907.04477.pdf rzach@ucalgary.ca

§1. Introduction. In 1921, Hilbert introduced the ε-calculus as a formalism on which to build his proof-theoretic project. The ε-calculus was originally introduced as a formalization of classical first-order logic. It can be seen as an attempt to reduce proofs in first-order logic to proofs in propositional logic, where the role of quantifiers is taken over by certain terms. … In the presence of identity, the formalism is more complicated, as axioms for identity have to be added to propositional logic. Hilbert called the resulting system the "elementary calculus of free variables"— essentially a formalism with predicates and terms, as well as open axioms for identity, but without quantifiers.

§3. ετ-Calculi for intermediate logics. An intermediate logic **L** is a set of formulas that contains intuitionistic logic **H** and is contained in classical logic **C**, and is closed under modus ponens and substitution. For intermediate predicate logics, we also require closure under the universal and existential quantifier rules.

Definition 3.1. Suppose **L** is an intermediate logic. … Some of these are obtained from **QH** simply by adding propositional axiom schemes. Equivalently, they can be obtained by expanding a propositional intermediate logic **L** to a language with predicates and terms, the standard quantifier axioms ∀xA(x)→A(t) and A(t)→∃xA(x) and closing under substitution, modus ponens, and the quantifier rules. This results in the weakest pure intermediate predicate logic extending **L**. Not every intermediate predicate logic is obtained in this way, as it is possible to consistently add additional first-order principles to **L**. Some important first-order principles are, e.g., the constant domain principle ∀x(A(x)∨B)→(∀xA(x)∨B), (CD) the double negation shift (or Kuroda's principle), ∀x¬¬A(x)→¬¬∀xA(x) (K) and the quantifier shifts

(B→∃xA(x))→∃x(B→A(x)) (Q∃) (3.1.1.1)

LET p, q, r: A, B, x.

(q>(%r&(p&r)))>(%r&(q>(p&r))) ;
`CCTT TTTT CCTT TTTT` (3.1.1.2)

Remark 3.1.1.2: If the existential quantifier is distributed, the result is tautologous.

$(\forall xA(x)\to B)\to\exists x(B\to A(x))$ $(Q\forall)$ (3.1.2.1)

((#r&(p&r))>q)>(%r&(q>(p&r))) ;
CC**FF** TT**F**T CCC**F** TT**F**T (3.1.2.2)

Remark 3.1.2.2: If the universal quantifier is distributed, the result is *not* tautologous as ((p&#r)>q)>(q>(p&%r)) ; TT**F**C TT**F**T TT**F**C TT**F**T (3.1.3.2)

§4. Critical formulas and quantifier shifts.

We might think of ετ-terms semantically as terms for objects which serve the role of generics taking on the role of quantifiers, and indeed in classical logic this connection is very close. Because of the validity of

$\exists x(\exists yA(y)\to A(x))$ (Wel 1) (4.1.1.1)
$\exists x(A(x)\to\forall yA(y))$ (Wel 2) (4.1.2.1)

in classical logic, there always is an object x which behaves as an ε-term (A(x) holds iff $\exists x\, A(x)$ holds), and an object x which behaves as a τ-term (i.e., A(x) holds iff $\forall y\, A(y)$ holds). One might expect then that Wel1 and Wel 2, when added to **QH**, have the same effect as adding critical formulas, i.e., that all quantifier shifts become provable. Note that Wel 1 and Wel 2 are intuitionistically equivalent to

$\exists x\forall y(A(y)\to A(x))$ (Wel′ 1) (4.2.1.1)
$\exists x\forall y(A(x)\to A(y))$ (Wel′ 2) (4.2.2.1)

Remark 4: We test Eqs 4.1.1.1=4.2.1.1 (4.3.1.1); 4.1.2.1=4.2.2.1 is a trivially tautologous.

((p&%r)>(p&%q))=((p&#r)>(p&%r)) ;
`TTTT TCTT TTTT TCTT` (4.3.1.2)

From the universal quantifier shift of Eq. 3.1.2.1 as not tautologous, the intermediate logic **L** is refuted along with Hilbert's first epsilon theorem in intuitionistic logic.