Refutation of the axiom of infinity

Abstract: We evaluate the following. Two definitions of the axiom of infinity are both contradictory. Two definitions to extract natural numbers from the infinite are are *not* equivalent and *not* tautologous. Therefore the four definitions for the axiom of infinity are refuted. Therefore these form a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **`F`** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **`F`** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1; (%z<#z) `C` as contingency, ▽, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: en.wikipedia.org/wiki/Axiom_of_infinity

Formal statement
In the formal language of the Zermelo–Fraenkel axioms, the axiom reads:

$$\exists \mathbf{I} (\varnothing \in \mathbf{I} \wedge \forall x \in \mathbf{I} ((x \cup \{x\}) \in \mathbf{I})). \quad (1.1)$$

LET p, q: x, I.

((p@p)<%q)&((#p<%q)&((#p&p)<%q)) ;
`FFFF FFFF FFFF FFFF` (1.2)

In words, there is a set **I** (the set which is postulated to be infinite), such that the empty set is in **I**, and such that whenever any x is a member of **I**, the set formed by taking the union of x with its singleton {x} is also a member of **I**. Such a set is sometimes called an **inductive set**.

An apparently weaker version
Some old texts use an apparently weaker version of the axiom of infinity, to wit:

$$\exists x(\exists y(y \in x) \wedge \forall y(y \in x \rightarrow \exists z(z \in x \wedge y \subseteq z \wedge y \neq z))). \quad (2.1)$$

LET p, q, r: x, y, z.

(%q<%p)&((# q<%p)>((%r<%p)&(~(%r<#q)&~(#q=%r)))) ;
`FFFF FFFF FFFF FFFF` (2.2)

Extracting the natural numbers from the infinite set
To extract the natural numbers, we need a definition of which sets are natural numbers. ... In formal

language, the definition says:

$$\forall n(n\in N\Leftrightarrow([n=\emptyset\vee\exists k(n=k\cup\{k\})]\wedge\forall m\in n[m=\emptyset\vee\exists k\in n(m=k\cup\{k\})])). \quad (3.1)$$

LET p, q, r, s, t: k, m, n, N, j.

```
(#r<s)=(((#r=(s@s))+(#r=(%p+p)))&
((#q<#r)&((#q=(s@s))+((%p<#r)&(#q=(%p+p)))))) ;
```

TTTC CCCC TTTC TTTT (3.2)

Or, even more formally:

$$\forall n(n\in N\Leftrightarrow([\forall k(\neg k\in n)\vee\exists k\forall j(j\in n\Leftrightarrow(j\in k\vee j=k))]\wedge$$
$$\forall m(m\in N\Rightarrow[\forall k(\neg k\in m)\vee\exists k(k\in n\&\forall j(j\in m\Leftrightarrow(j\in k\vee j=k)))]))). \quad (4.1)$$

```
(#r<s)=(((~#p<#r)+((#t<#r)=((#t<%p)+(#t=%p))))&
((#q<s)>((~p<#q)+((%p<#q)&((#t<#q)=((#t<%p)+(#t=%p))))))) ;
```

FFNN **F**N**FF** **FFFF** N**F**N**F**(1)
FFNN **FFFF** **FFFF** NNNN(1) (4.2)

Two definitions of the axiom of infinity are both contradictory. Two definitions to extract natural numbers from the infinite are are *not* equivalent and *not* tautologous. Therefore the four definitions for the axiom of infinity are refuted.