Using Special Relativity (SR) as a starting point, then noting a few empirical 4-Vector facts, one can derive the Principles that are normally considered to be Axioms of Quantum Mechanics (QM).

Since many of the QM Axioms are rather obscure, this seems a more logical and understandable paradigm than QM as a separate theory from SR, and sheds light on the origin and meaning of the QM Principles.

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

or: Why General Relativity (GR) is *NOT* wrong
or: Don’t bet against Einstein ;)
or: QM, the easy way...

Recommended viewing:
via a .PDF Viewer/WebBrowser
with Fit-To-Page & Page-Up/Down
ex. Firefox Web Browser

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)  ver 2019-Aug-01 .01
4-Vectors are a fantastic language/tool for describing the physics of both relativistic and quantum phenomena. They easily show many interesting properties and relations of our Universe, and do so in a simple and concise mathematical way. Due to their tensorial nature, these SR 4-Vectors are automatically coordinate-frame invariant, and can be used to generate *ALL* of the physical SR Lorentz Scalar tensors and higher-index-count SR tensors. **Let me repeat:** You can mathematically build *ALL* the Lorentz Scalars and larger SR tensors from SR 4-Vectors.

4-Vectors are likewise easily shown to be related to the standard 3-vectors that are used in Newtonian classical mechanics, Maxwellian classical electromagnetism, and standard quantum theory.

Why 4-Vectors as opposed to some of the more abstract mathematical approaches to QM? Because the components of 4-Vectors are things that can actually be empirically measured. Experiment is the ultimate arbiter of which theories actually correspond to reality. If your quantum logics and string theories give no testable/measurable predictions, then they are basically useless for real physics.

In this treatise, I will demonstrate how 4-Vectors are used in the context of Special Relativity, and then show that their use in Relativistic Quantum Mechanics is really not fundamentally different. Quantum Principles then emerge in a natural and elegant way.

I also introduce the **SRQM Diagramming Method**, an instructive graphical charting-method, which visually shows how the SRQM 4-Vectors, Lorentz 4-Scalars, and 4-Tensors are all related to each other. This symbolic representation clarifies a lot of physics and is a great tool for understanding and teaching.
Some Physics Abbreviations & Notation

\[ \beta = \text{Relativistic Beta} = \frac{v}{c} = \{0..1\} \hat{n} \]
\[ \gamma = \text{Relativistic Gamma} = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\bm{\beta} \cdot \bm{\beta}}} = \{1..\infty\} \]
\[ D = \text{Relativistic Doppler} = \frac{1}{\gamma(1-|\beta| \cos\theta)} \]
\[ \delta_{ij} = \delta_{i}^{j} = \delta_{ij} = \{1 \text{ if } i=j, \text{ else } 0\} \text{ 3D Kronecker delta} \]
\[ \delta_{\mu\nu} = \delta_{\mu}^{\nu} = \delta_{\mu\nu} = \{1 \text{ if } \mu=\nu, \text{ else } 0\} \text{ 4D Kronecker Delta} \]
\[ \eta_{\mu\nu} \rightarrow \eta_{\mu\nu} \rightarrow \text{Diag}[1,-I_{(3)}]_{\text{rect}} = \text{Minkowski "Flat SpaceTime" Metric} \]
\[ \varepsilon_{ijk} = \text{3D Levi-Civita anti-symmetric permutation symbol} \]
\[ \varepsilon^{\mu\nu\rho\sigma} = \text{4D Levi-Civita Anti-symmetric Permutation Symbol} \]
\[ \varepsilon_{\mu\nu} = \text{Interesting Index Raise:Lower Matchup} \]

Tensor-Index & 4-Vector Notation:
\[ A^i = a = (a^1,a^2,a^3): \text{3-vector} \quad \text{[Latin index} \quad \{1..3\}] \]
\[ A^\mu = A = (a^\mu,a^1,a^2,a^3): \text{4-Vector} \quad \text{[Greek index} \quad \{0..3\}] \]
\[ A^\mu B^\nu = A_\nu B^\mu = A \cdot B: \text{Einstein Sum} : \text{Dot Product} : \text{Inner Product} \]
\[ A^\mu B^\nu = A \otimes B: \text{Tensor Product} : \text{Outer Product} \]
\[ A^\mu B^\nu - A^\nu B^\mu = A \wedge B: \text{Wedge Product} : \text{Exterior Product} \]
\[ A^\mu B^\nu = 0_{\mu\nu}: \text{(2,0)-Zero Tensor} \]
\[ A^\mu B^\nu - B^\nu A^\mu = [A^\mu,B^\nu] = [A,B]: \text{Commutation} \]
\[ A^\mu B^\nu - B^\mu A^\nu = ??? \]

SRQM = The \([\text{SR} \rightarrow \text{QM}]\) Interpretation of Quantum Mechanics, by John B. Wilson
Special Relativity → Quantum Mechanics

The SRQM Interpretation: Links

See also:
http://scirealm.org/SRQM.html (alt discussion)
http://scirealm.org/SRQM-RoadMap.html (main SRQM website)
http://scirealm.org/4Vectors.html (4-Vector study)
http://scirealm.org/SRQM-Tensors.html (Tensor & 4-Vector Calculator)
http://scirealm.org/SciCalculator.html (Complex-capable RPN Calculator)

or Google “SRQM”

http://scirealm.org/SRQM.pdf (this document)

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
The SRQM Diagramming Method shows the properties and relationships of various physical objects in a graphical way. This “flowchart” method aids understanding.

**Representation:** 4-Scalars by ellipses, 4-Vectors by rectangles, 4-Tensors by octagons. Physical/mathematical equations and descriptions inside each shape/object. Sometimes there will be additional clarifying descriptions around a shape/object.

**Relationships:** Lorentz Scalar Products or tensor compositions of different 4-Vectors are on lines with arrows indicating the direction of flow. (ex. multiplication)

**Flow:** Objects that are some function of a Lorentz Scalar with another 4-Vector or 4-Tensor are on lines with arrows indicating the direction of flow. (ex. multiplication)

**Properties:** Some objects will also have a symbol representing its properties nearby, and sometimes there will be color highlighting within the object to emphasize temporal-spatial properties. I typically use blue=Temporal, red=Spatial, purple=mixed TimeSpace.

Alternate ways of writing 4-Vector expressions in physics: (\( A \cdot B \)) is a 4-Vector style, which uses vector-notation (ex. outer product "dot=" or exterior product "wedge="), and is typically more compact, always using **bold** UPPERCASE to represent the 4-Vector, ex. \(( A \cdot B ) = ( A^\nu \eta_{\nu \rho} B^\rho )\), and **bold** lowercase to represent 3-vectors, ex. \(( a \cdot b ) = ( a^\nu \eta_{\nu \rho} b^\rho )\). Most 3-vector rules have analogues in 4-Vector mathematics.

\(( A^\nu \eta_{\nu \rho} B^\rho )\) is a Ricci Calculus style, which uses tensor-index-notation and is useful for more complicated expressions, especially to clarify those expressions involving tensors with more than one index, such as the Faraday EM Tensor \( F^{\mu \nu} = ( \partial^\mu A^\nu - \partial^\nu A^\mu ) = ( \partial ^{\nu} A ) \)

"Alternate ways of writing 4-Vector expressions in physics:"

\(( A \cdot B ) = ( A^\nu \eta_{\nu \rho} B^\rho )\) and **bold** UPPERCASE to represent the 4-Vector, ex. \(( A \cdot B ) = ( A^\nu \eta_{\nu \rho} B^\rho )\), and **bold** lowercase to represent 3-vectors, ex. \(( a \cdot b ) = ( a^\nu \eta_{\nu \rho} b^\rho )\). Most 3-vector rules have analogues in 4-Vector mathematics.
SRQM Study: Physical/Mathematical Tensors

Tensor Types: 4-Scalar, 4-Vector, 4-Tensor

Component Types: Temporal, Spatial, Mixed

Matrix Format

SR 4-Scalar S

SR 4-Vector V^μ

SR 4-Tensor T^{μν}

SRQM Diagram Format

SR 4-Scalar (0,0)-Tensor

4-Scalars, 0 index
4^0 = 1 component

1 Temporal + 3 Spatial
= 4 SpaceTime Dimensions

16 General
10 Symmetric
6 AntiSymmetric

4-Tensors, 2 index
4^2 = 16 components
6 AntiSymmetric
10 Symmetric
16 General

Mixed 4-Vector “Dual” 4-Vector

(0,1)-Tensor aka. One-Form

Mixed TimeSpace region: purple
The mnemonic being red and blue mixed make purple

SR:Minkowski Metric

\[ \delta[R] = \partial^\mu R^\nu = V^{\mu
u} \]

Diag[1,-1,-1,-1] = Diag[1,-\delta]\n
\{in Cartesion form\} "Particle Physics" Convention

SR: Lorentz Transform

\[ \partial_c[R^\mu] = \partial^\nu R^\nu = \Lambda^\nu_\mu \]

\[ \Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\mu_\alpha \Lambda^\alpha_\nu = \eta^\mu_\nu = \delta^\mu_\nu \]

\[ \eta^\mu_\nu \Lambda^\mu_\nu \Lambda^\nu_\nu = \eta_{diag} \]

\[ \det[\Lambda^\mu_\nu] = \pm 1 \]

\[ \Lambda^\mu_\nu \Lambda^\nu_\mu = \eta_{diag} \]

SpaceTime

\[ \partial R = \partial_\mu R^\mu = 4 \]

Dimension

Technically, all these objects are "SR 4-Tensors", but we usually reserve the name "4-Scalar" for objects with 2 or more indices, and use the (m,n)-Tensor notation to specify all the objects more precisely.
One of the extremely important properties of Tensor Mathematics is the fact that there are numerous ways to generate Tensor Invariants. These Invariants lead to Physical Properties that are fundamental in our Universe. They are totally independent of the coordinate systems used to measure them. Thus, they represent symmetries that are inherent in the fabric of SpaceTime. See the Cayley-Hamilton Theorem.

Trace Tensor Invariant: \( \text{Tr}[T^\mu{}_{\nu}] = T^\mu{}_{\mu} \)

Inner Product Tensor Invariant: \( \text{IP}[T^\mu{}_{\nu}] = T^{\mu\nu} T_{\mu\nu} \)

Determinant Tensor Invariant: \( \text{Det}[T^\mu{}_{\nu}] = \text{II}[ \text{EigenValues } ] \) for \( T^\nu{}_{\nu} \)

4-Divergence Tensor Invariant: \( \partial^\mu T^\nu_{\mu} = \partial^\nu \)\( \partial^\nu \)\( = \partial^\nu \)

Lorentz Scalar Product Tensor Invariant: \( \text{LSP}[T^\mu{}_{\nu}, S^\mu{}_{\nu}] = T^\mu{}_{\nu} S^\mu{}_{\nu} = T^\mu{}_{\nu} S^\nu{}_{\nu} = T^\mu{}_{\nu} \cdot S \)

Phase Space Tensor Invariant: \( \text{PS}[T^\mu{}_{\nu}] = \partial^\mu T^\nu_{\mu} \) if \( T^\nu_{\mu} \) is constant

The Ratio of 4-Vector Magnitudes (Ratio of Rest Value 4-Scalars): \( T^\nu_{\nu} S^\nu_{\nu} = (t^2/s^2) \)

other Invariants are possible, including Tensor Eigenvalues, the various Asymmetric products, etc.

a) \( T^\nu_{\nu} = \text{Trace = Sum of EigenValues for } (1,1)-\text{Tensors} \)

b) \( T^\nu_{\nu} T^\mu_{\mu} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product} \)

c) \( T^\nu_{\nu} T^\mu{}_{\mu} T^\alpha{}_{\alpha} = \text{Asymm Tri-Product} \rightarrow \text{?Name?} \)

d) \( T^\nu_{\nu} T^\mu{}_{\mu} T^\alpha{}_{\mu} T^\beta{}_{\nu} = \text{Asymm Quad-Product} \rightarrow 4D \text{ Determinant = Product of EigenValues for } (1,1)-\text{Tensors} \)
**SRQM Study: Physical/Mathematical Tensors**

**Tensor Types: 4-Scalar, 4-Vector, 4-Tensor**

Examples – Venn Diagram

0 index-count Tensors:
- **SR 4-Scalar** (0,0)-Tensors aka. One-Forms
- EM Charge (Q=\[\mathbf{p}\cdot\mathbf{x}\])
- Lorentz Scalar

1 index-count Tensors:
- 4-Position \(\mathbf{R} = R^\mu = (ct, r)\)
- 4-Velocity \(\mathbf{U} = U^\mu = (c, \mathbf{v})\)
- 4-Momentum \(\mathbf{P} = P^\mu = (mc, \mathbf{p})\) aka. 4-Momentum

2 index-count Tensors:
- SR Mixed 4-Tensor (1,1)-Tensors
- Minkowski \(n^\mu n_\mu = -1\), Lorentz \(\mathbf{A} = \mathbf{A}^\mu n_\mu\)
- Faraday EM 4-Vector \(\mathbf{F}^\mu = \mathbf{E}^\mu - \mathbf{B}^\mu\)
- Perfect Fluid 4-Vector \(\mathbf{T}^\mu = \rho(x) \mathbf{V}^\mu + (\rho\mathbf{u})\mathbf{H}^\mu\)

Higher index-count Tensors:
- Lowered Minkowski \(\mathbf{R}_{\mu\nu} = \mathbf{R}_{\mu\nu} = \mathbf{R}_{\mu\nu} = \mathbf{R}_{\mu\nu}\)
- Weyl (Conformal) Curvature Tensor \(C_{\mu\nu} = \text{Traceless part of Riemann} [R_{\mu\nu}]\)

Physical 4-Tensors: Objects which have Invariant 4D SpaceTime properties

- Speed-of-Light \(c = \sqrt{\mathbf{U} \cdot \mathbf{U}}\)
- ProperTime \(\delta \mathbf{R} = \mathbf{d}t = \mathbf{d}t = \mathbf{d}t = \mathbf{d}t\)
- SpaceTime \(\delta \mathbf{R} = \delta \mathbf{R} = \delta \mathbf{R} = \delta \mathbf{R} = \delta \mathbf{R}\)
- Dimension \(\delta [\mathbf{R}^\mu] = \pm 1\)

4-Vector SRQM Interpretation of QM

SR → QM

John B. Wilson
SciRealm.org
SRQM 4-Vectors = (1,0)-Tensors

4-Tensors = (2+ index)-Tensors

4-Tensors can be constructed from the Tensor Products of 4-Vectors. Technically, 4-Tensors refer to all SR objects (4-Scalars, 4-Vectors, etc.), but we typically reserve the name 4-Tensor for SR Tensors of 2 or more indices.
### SRQM 4-Scalars = (0,0)-Tensors = Lorentz Scalars → Physical Constants

#### 4-Scalar = Type (0,0)-Tensor
- RestTime: ProperTime \( (t_o) = (\tau) \)
- Speed of Light \( (c) \)
- RestMass \( (m_o) \)
- RestEnergy \( (E_o) \)
- RestAngularFrequency \( (\omega_o) \)
- RestChargeDensity \( (\rho_o) \)
- RestScalarPotential \( (\phi_o) \)
- ProperTimeDerivative \( (d/d\tau) \)
- RestNumberDensity \( (n_o) \)
- EM Charge \( (q) \)
- Particle # \( (N) \)
- RestVolume \( (V_o) \)
- RestEnergyDensity \( (\rho_{en}) \)
- RestPressure \( (p_o) \)

#### Lorentz Scalars can be constructed from the Lorentz Scalar Product of 4-Vectors

- **Time as measured in the at-rest frame**
- **Speed of Light**
- **RestMass** as Electron RestMass
- **RestEnergy**
- **RestAngularFrequency**
- **RestChargeDensity**
- **RestScalarPotential**
- **ProperTimeDerivative**
- **RestNumberDensity**
- **EM Charge**
- **Particle #**
- **RestVolume**
- **RestEnergyDensity**
- **RestPressure**

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**Formulas and Notes:**

- **4-Scalar** = \( \mathbf{S} \) of Physical 4-Vectors
- **SR 4-Vector** \( \mathbf{V} \) is Lorentz Scalar
- **SR 4-Vector** \( \mathbf{V'} = \mathbf{V} = (v^0, \mathbf{v}) \)
- **SR 4-Scalar** \( \mathbf{S} \) or \( \mathbf{S}_{\text{EM}} \)

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SRQM Study: Physical 4-Vectors

Some SR 4-Vectors and Symbols
SRQM Study: Physical 4-Tensors

Some SR 4-Tensors and Symbols

**Lorentz Identity Transform**
\[ \Lambda^{\mu,\nu} \rightarrow \eta^{\mu,\nu} = I_{(4)} \]

**Lorentz Time-Reverse Transform**
\[ \Lambda^{\mu,\nu} \rightarrow \Lambda_{\nu,\mu} = \Lambda^{\mu,\nu} \]

**Lorentz Space-Parity Transform**
\[ \Lambda^{\mu,\nu} \rightarrow \Lambda_{\nu,\mu} = \Lambda^{\mu,\nu} \]

**Lorentz ComboPT Transform**
\[ \Lambda^{\mu,\nu} \rightarrow \Lambda_{\nu,\mu} = \Lambda^{\mu,\nu} \]

**Perfect Fluid**
\[ T^{ij} = (\rho_0 + p_0)\eta^{ij} \]

**Faraday EM**
\[ F^{0i} = \delta^j A^i - \delta^i A^j = \partial ^A \]

**4-AngularMomentum**
\[ M^{4j} = X^j P^p - X^j P^p = X^j P^p \]

**4-Tensor Symmetric**
\[ T_{\mu,\nu} = T_{\nu,\mu} \]

**4-Tensor Anti-symmetric**
\[ T_{\mu,\nu} = -T_{\nu,\mu} \]

**Faraday EM**
\[ F^{0i} = \delta^j A^i - \delta^i A^j = \partial ^A \]

**4-AngularMomentum**
\[ M^{4j} = X^j P^p - X^j P^p = X^j P^p \]

**Trace**
\[ T^{\mu,\nu} = \eta^{\mu,\nu}T^{\mu,\nu} = T^\mu _\nu = T \]

**4-Vector SRQM Interpretation of QM**

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SRQM Study: Physical 4-Tensors

Some SR 4-Tensors and Symbols

A Tensor Study of Physical 4-Vectors

SR-Vector

4-Vector SRQM Interpretation of QM

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SR → QM

Temporal “(V)ertical” Projection (2,0)-Tensor

$P^{\mu\nu} \rightarrow P^{\mu\nu} = T^{\mu\nu} 
\rightarrow \text{Diag}[1,0,0,0]$

Spatial “(H)orizontal” Projection (2,0)-Tensor

$P^{\mu\nu} \rightarrow P^{\mu\nu} = T^{\mu\nu} 
\rightarrow \text{Diag}[0,1,1,1]$

SR:Minkowski Metric

$\delta[R] = \delta[R'] = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} 
\rightarrow \text{Diag}[1,1,-1,-1]$ = Diag[1, -\delta]

Faraday EM Tensor

$F_{\mu\nu} = \delta^2 A^\mu - \delta^2 A^\nu = \partial A^\mu$

4-Tensor (0,2)-Tensor $T_{\mu\nu} = \text{Tr}[F_{\mu\nu}]$

4-Tensor Perfect Fluid

$T^{\mu\nu} = (\rho_{\text{eo}}) V^{\mu\nu} + (-\rho) H^{\mu\nu}$

$\text{Tr}[T^{\mu\nu}] = \rho_{\text{eo}} - 3 \rho$

$\text{Tr}[T^{\mu\nu}] = \rho_{\text{eo}}$

4-Tensor Null-Dust=Photon Gas

$T^{\mu\nu} = (\rho_{\text{eo}}) V^{\mu\nu} + (-\rho/3) H^{\mu\nu}$

$\text{Tr}[T^{\mu\nu}] = 0$

4-Tensor Lambda Vacuum

$T^{\mu\nu} = (\rho_{\text{eo}}) \eta^{\mu\nu} = (\Lambda) \eta^{\mu\nu}$

$\text{Tr}[T^{\mu\nu}] = \rho_{\text{eo}}$

4-Tensor Zero:Nothing Vacuum

$T^{\mu\nu} = 0$

$\text{Tr}[T^{\mu\nu}] = 0$

Equation of State (EoS) $w = (\rho_{\text{eo}}/\rho_{\text{eo}})$

$\text{Tr}[\Lambda^{\mu\nu}] = \rho_{\text{eo}}$

Note that the Projection Tensors and the Minkowski Metric are dimensionless.

SR 4-Tensor $(2,0)$-Tensor $T_{\mu\nu}$

$V^{\mu\nu} = V = (v^i, v^j)$

SR 4-Vector

$V^{\mu\nu} = V = (V_i, V_j)$

SR 4-Scalar

$(0,0)$-Tensor $S$

$S_{\text{Lorentz Scalar}}$

Interpretation

SciRealm.org
SRQM Study: Physical 4-Tensors

Projection 4-Tensors

Temporal "(V)ertical" Projection (2,0)-Tensor
$P_{\mu\nu} \to V_{\mu\nu} = T\mu T\nu \to \text{Diag}[1,0]$,

Temporal "(V)ertical" Projection (1,1)-Tensor
$P_{\mu\nu} \to V_{\mu\nu} = T\mu T\nu \to \text{Diag}[1,0]$,

Temporal "(V)ertical" Projection (0,2)-Tensor
$P_{\mu\nu} \to V_{\mu\nu} = T\mu T\nu \to \text{Diag}[1,0]$,

Temporal "(V)ertical" Projection (0,1)-Tensor
$P_{\mu\nu} \to V_{\mu\nu} = T\mu T\nu \to \text{Diag}[1,0]$,

Spatial "(H)orizontal" Projection (2,0)-Tensor
$P_{\mu\nu} \to H_{\mu\nu} = \eta_{\mu\nu} - T\mu T\nu \to \text{Diag}[0,-1,0]$,

Spatial "(H)orizontal" Projection (1,1)-Tensor
$P_{\mu\nu} \to H_{\mu\nu} = \eta_{\mu\nu} - T\mu T\nu \to \text{Diag}[0,-1,0]$,

Spatial "(H)orizontal" Projection (0,2)-Tensor
$P_{\mu\nu} \to H_{\mu\nu} = \eta_{\mu\nu} - T\mu T\nu \to \text{Diag}[0,-1,0]$,

Spatial "(H)orizontal" Projection (0,1)-Tensor
$P_{\mu\nu} \to H_{\mu\nu} = \eta_{\mu\nu} - T\mu T\nu \to \text{Diag}[0,-1,0]$,
SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR, although technically SR is itself the low-curvature limiting-case of GR

\{c,\tau,m_o,\hbar,i\}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors:

- **4-Position**: \( R = (ct, r) \) = <Event> \( (R \cdot R) = (ct)^2 \)
- **4-Velocity**: \( U = \gamma(c, u) \) = \((U \cdot \partial)R=(d/d\tau)R=dR/d\tau \) \( (U \cdot U) = (c)^2 \)
- **4-Momentum**: \( P = (E/c, p) \) = \( m_o U \) \( (P \cdot P) = (m_o c)^2 \)
- **4-WaveVector**: \( K = (\omega/c, k) \) = \( P/\hbar \) \( (K \cdot K) = (m_o c/\hbar)^2 \)
- **4-Gradient**: \( \partial = (\partial/c, -\nabla) \) = -iK \( (\partial \cdot \partial) = -(m_o c/\hbar)^2 = KG Eqn\rightarrow RQM\rightarrow QM \)

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn, and thence to QM via the low-velocity limit \( \{|v| << c\} \), giving the Schrödinger Eqn. The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM 4-Vector Topics Covered

SR & QM via 4-Vector Diagrams

Mostly SR Stuff

- 4-Vector Basics
- Paradigm Assumptions, Where is Quantum Gravity?
- Minkowski SpaceTime, <Events>, WorldLines, Minkowski Metric
- 4-Scalars, 4-Vectors, 4-Tensors & Tensor Invariants
- SR 4-Vector Connections
- SR Lorentz Transforms, CPT Symmetry, Trace Identification, Antimatter
- Fundamental Physical Constants = Lorentz Scalar Invariants
- Projection Tensors: Temporal (V) & Spatial (H)
- Stress-Energy Tensors, Perfect Fluids, Special Cases (Dust, Radiation, etc)
- Invariant Intervals, Measurement, Causality, Relativity
- SpaceTime Kinematics & Dynamics, ProperTime Derivative
- Einstein’s $E = mc^2 = \gamma m_0 c^2$, Rest Mass, Rest Energy, Invariants
- SpaceTime Orthogonality: Time-like velocity, Space-like acceleration
- Relativity of Simultaneity, Time Dilation, Length Contraction
- SR Motion * Lorentz Scalar = Interesting Physical 4-Vector
- SR Conservation Laws & Local Continuity Equations, Symmetries
- Relativistic Doppler Effect, Relativistic Aberration Effect
- SR Wave-Particle Relation, Invariant d’Alembertian Wave Eqn, SR Waves
- SpaceTime is 4D: $\partial \cdot R = \partial_\mu R^\mu = 4$, $\Lambda_{\mu\nu}\Lambda^{\mu\nu} = 4$, $\text{Tr}[\eta^{\mu\nu}] = 4$, $A^\mu = (a_0, a_1, a_2, a_3)$
- Minimal Coupling = Interaction with a (Vector)Potential
- Conservation of 4-TotalMomentum
- SR Hamiltonian: Lagrangian Connection
- Lagrangian, Lagrangian Density
- Hamilton-Jacobi Equation (differential), Relativistic Action (integral)
- Euler-Lagrangian Equations
- Relativistic Equations of Motion, Lorentz Force Equation
- $c^2$ Invariant Relations, The Speed-of-Light (c)
- Thermodynamic 4-Vectors, Unruh-Hawking Radiation, Particle Distributions

Mostly QM & SRQM Stuff

- Relativistic Quantum Wave Equations
- Klein-Gordon Equation/Relation
- RoadMap from SR to QM: SR→QM, SRQM 4-Vector Connections
- QM Schrödinger Relation
- QM Axioms? - No, (QM Principles derived from SR) = SRQM
- Relativistic Wave Equations: based on mass & spin & velocity
- Klein-Gordon, Dirac, Proca, Maxwell, Weyl, Pauli, Schrödinger, etc.
- Classical Limits $|v| << c$
- Photon Polarization
- Linear PDE’s→
  - (Principle of Superposition, Hilbert Space, <Bra|\text{Ket}> Notation)
- Canonical QM Commutation Relations – derived from SR
- Heisenberg Uncertainty Principle (due to non-zero commutation)
- Pauli Exclusion Principle (Fermion), Bose Aggregation Principle (Boson)
- Complex 4-Vectors
- CPT Theorem, Lorentz Invariance, Poincaré Invariance, Isometry
- Hermetian Generators, Unitarity, Anti-Unitarity
- QM → Classical Correspondence Principle, similar to SR → Classical
- Quantum Probability
- The Compton Effect = Photon:Electron Interaction (neglecting Spin Effects)
- Photon Diffraction, Crystal-Electron Diffraction, The Kapitza-Dirac Effect
- The ħ Relation, Einstein-de Broglie, Planck:Dirac
- The Aharonov-Bohm Effect, The Josephson Junction Effect
- Noether’s Theorem, Continuous Symmetries, Conservation Laws
- Dimensionless Quantities
- Quantum Relativity: GR is *NOT* wrong, *Never bet against Einstein* :)
- Quantum Mechanics is Derivable from Special Relativity, SR→QM, SRQM

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
There are some paradigm assumptions that need to be cleared up:

Relativistic Physics **IS NOT** the generalization of Classical Physics.
Classical Physics **IS** the low-velocity limiting-case approximation of Relativistic Physics \( \{ |v| \ll c \} \).

This includes (Newtonian) Classical Mechanics and Classical QM, (meaning the non-relativistic Schrödinger QM Equation).

Classical EM is for the most part already compatible with Special Relativity.
However, Classical EM doesn't include intrinsic spin, even though spin is a result of SR Poincaré Invariance, not QM.

So far, in all of my research, if there was a way to get a result classically, then there was usually a much simpler way to get the result using 4-Vectors and SRQM relativistic thinking.
Likewise, a lot of QM results make much more sense when approached from SRQM.

4-Vector formulations are all extremely easy to derive in SRQM and are all relativistically covariant.
There are some paradigm assumptions that need to be cleared up:

SR 4D Physical 4-Vectors *ARE NOT* generalizations of Classical/Quantum 3D Physical 3-vectors. While a “mathematical” Euclidean (n+1)D-vector is the generalization of a Euclidean (n)D-vector, the “physical” analogy ends there.

Minkowskian SR 4-Vectors *ARE* the primitive elements of 4D Minkowski SR SpaceTime. Classical/Quantum Physical 3-vectors are just the spatial components of SR Physical 4-Vectors. There is also a fundamentally-related Classical/Quantum Physical scalar related to each 3-vector, which is just the temporal component scalar of a given SR Physical 4-Vector.

ex. 4-Position $\mathbf{R} = (r^\mu) = (r^0, r) = (ct, r) \rightarrow (ct, x, y, z)$:

4-Momentum $\mathbf{P} = (p^\mu) = (p^0, p) = (E/c, p) \rightarrow (E/c, p_x, p_y, p_z)$

These Classical/Quantum {scalar}+{3-vector} are the dual {temporal}+{spatial} components of a single SR 4-Vector = (temporal scalar $\times c^{\pm 1}$, spatial 3-vector) with SR lightspeed factor ($c^{\pm 1}$) to give correct overall dimensional units.

While different observers may see different "values" of the Classical/Quantum components ($v^0, v^1, v^2, v^3$) from their point-of-view in SpaceTime, each will see the same actual SR 4-Vector $\mathbf{V}$ and its magnitude $|\mathbf{V}|$ at a given <Event> in SpaceTime.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
There are some paradigm assumptions that need to be cleared up:

We will **NOT** be employing the commonly-(mis)used Newtonian classical limits \( \{ c \rightarrow \infty \} \) and \( \{ \hbar \rightarrow 0 \} \).

Neither of these is a valid physical assumption, for the following reasons:

1. Both \( c \) and \( \hbar \) are unchanging Physical Constants and Lorentz Invariants.
2. Taking a limit where these change is non-physical. They are CONSTANT.

Many, many experiments verify that these constants have not changed over the lifetime of the universe. This is one reason for the 2019 Redefinition of SI Base Units on Fundamental Constants \( (c,\hbar,e,k_B,N_A,K_{CD},\Delta \nu_{Cs}) \).

Let \( E = pc \). If \( c \rightarrow \infty \), then \( E \rightarrow \infty \). Then Classical EM light rays/waves have infinite energy.

Let \( E = \hbar \omega \). If \( \hbar \rightarrow 0 \), then \( E \rightarrow 0 \). Then Classical EM light rays/waves have zero energy.

Obviously neither of these is true in the Newtonian limit.

In Classical EM and Classical Mechanics, \( (c) \) remains a large but finite constant.

Likewise, \( (\hbar) \) remains very small but never becomes zero.

The correct way to take the limits is via:

- The low-velocity non-relativistic limit \( \{ |v| \ll c \} \), which is a physically-occurring situation.
- The Hamilton-Jacobi non-quantum limit \( \{ \hbar|\nabla \cdot p| \ll (p \cdot p) \} \), which is a physically-occurring situation.
There are some paradigm assumptions that need to be cleared up:

We will *NOT* be implementing the common \{→ lazy and extremely misguided\} convention of setting physical constants to the value of (dimensionless) unity, often called “Natural Units”, to hide them from equations; nor using mass \( m \) instead of \( (m_0) \) as the RestMass. Likewise for other components vs Lorentz Scalars with naughts, like energy \( E \) vs \( (E_0) \) as the RestEnergy.

One sees this very often in the literature. The usual excuse cited is “For the sake of brevity”.

Well, the “sake of brevity” forsakes “clarity”.

The *ONLY* situation in which setting constants to unity is practical or advisable is in numerical simulation. When teaching physics, or trying to understand physics, it helps when equations are dimensionally correct.

In other words, the technique of dimensional analysis is a powerful tool that should not be disdained.

i.e. Brevity aids speed of computation, Clarity aids understanding.

The situation of using “naught = 0” for rest-values, such as \( (m_0) \) for RestMass and \( (E_0) \) for RestEnergy:

Is intrinsic to SR, is a very good idea, absolutely adds clarity, identifies Lorentz Scalar Invariants, and will be explained in more detail later.

Essentially, the relativistic gamma \((\gamma)\) pairs with a (Lorentz scalar:rest value \( \_0 \)) to make a relativistic component: \( m = \gamma m_0, E = \gamma E_0 \)

Note the multiple equivalent ways that one can write 4-Vectors using these rules:

4-Momentum \( P = P^\mu = (p^\mu) = (mc,p) = (mc,p) = m_0U = m_0\gamma(c,u) = \gamma m_0(c,u) = m(c,u) = (mc,mu) = (mc,p) \)

\[ = (E/c,p) = (E_0/c^2)U = (E_0/c^2)\gamma(c,u) = \gamma(E_0/c^2)(c,u) = (E/c^2)(c,u) = (E/c,Eu/c^2) = (E/c,p) \]

It is damn hard enough just to get the minus-signs right in GR/SR, as there are different metric-conventions available.

BTW, I prefer the “Particle Physics” Metric-Convention (+,-,-,-). \{Makes rest values positive, fewer minus signs to deal with\}

Show the physical constants and naughts in the work. They deserve the respect and you will benefit.

You can always set constants to unity later, when you are doing your numerical simulations.
There are some paradigm assumptions that need to be cleared up:

Many physics books say that the Electric field \( E \) and the Magnetic field \( B \) are the “real” physical objects, and that the EM scalar-potential \( \phi \) and the EM 3-vector-potential \( A \) are just “calculational/mathematical” artifacts.

Neither of these statements is relativistically correct.

All of these physical EM properties: \( \{ E, B, \phi, A \} \) are actually just the components of SR tensors, and as such, their magnitudes will vary in different observers’ reference-frames.

The truly SR invariant physical objects are:

- The 4-Gradient \( \partial \),
- The 4-VectorPotential \( A \),

and their combination via exterior (wedge=\(^{\wedge}\)) product into the Faraday EM Tensor

\[
F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha} = \partial^{\wedge}A
\]

Given this SR knowledge, we demote the physical property symbols, (the tensor components) to their lower-case equivalents \( \{ e, b, \phi, a \} \).

Temporal-spatial components of 4-Tensor \( F_{\alpha\beta} \): electric 3-vector field \( e \).
Spatial-spatial components of 4-Tensor \( F_{\alpha\beta} \): magnetic 3-vector field \( b \).
Temporal component of 4-Vector \( A \): EM scalar-potential \( \phi \).
Spatial components of 4-Vector \( A \): EM 3-vector-potential \( a \).

Note that the speed-of-light (c) plays a prominent role in the component definitions.
Also, QM requires the 4-VectorPotential \( A \) as explanation of the Aharonov-Bohm Effect. Again, all the higher-index-count SR tensors can be built from fundamental 4-Vectors.
There are some paradigm assumptions that need to be cleared up:

A number of QM philosophies make the assertion that particle “properties” do not “exist” until measured. The assertion is based on the Heisenberg Uncertainty Principle, and more specifically on quantum non-zero commutation, in which a measurement on one property of a particle alters a non-commuting property of the same particle.

That is an incorrect analysis. Properties define particles: what they do, how they interact with other particles. Particles and their properties “exist” independently of human intervention or observation. The correct way to analyze this is to understand what a measurement is: the arrangement of some number of fundamental particles in a particular manner as to allow an observer to get information about one or more of the subject particle’s properties. Typically this involves “counting” spacetime events and using SR invariant intervals as a basis of measurement.

Some properties are indeed non-commuting. This simply means that it is not possible to arrange a set of particles in such a way as to measure (ie. obtain “complete” information about) both of the subject particle’s non-commuting properties at the same spacetime event. The measurement arrangement events can be done at best sequentially, and the temporal order of these events makes a difference in observed results. EPR-Bell, however, allows one to “infer” properties on a subject particle by making a measurement on a different {space-like separated but entangled} particle.

So, a better way to think about it is this: The “measurement” of a property does not “exist” until a physical setup event is arranged. Non-commuting properties require different physical arrangements in order for all to be measured, and the temporally-first measurement alters the particle’s properties in a minimum sort of way, which affects the latter measurement. All observers agree on the order of temporally-separated spacetime events. However, individual observers may have different sets of partial information about the same particle(s).

This makes way more sense than the subjective belief that a particle’s property doesn’t exist until it is observed, which is about as unscientific and laughable a statement as I can imagine.

*Relativity is the system of measurement that QM has been looking for*
There are some paradigm assumptions that need to be cleared up:

**Correct Notation is critical for understanding physics**

Unfortunately, there are a number of “sloppy” notations in relativistic and quantum physics.

Incorrect: Using $T^{ii}$ as a Trace of tensor $T^{ij}$, or $T^{μμ}$ as a Trace of tensor $T^{μν}$

$T^{ii}$ is just the diagonal part of 3-tensor $T^{ij}$, the components: $T^{ii} = \text{Diag}[T_{11}, T_{22}, T_{33}]$

$T_{i}^{i}$ is the Trace of 3-tensor $T^{ij}$: $T_{i}^{i} = T_{1}^{1} + T_{2}^{2} + T_{3}^{3} = 3\text{-trace}[T^{ij}] = T_{11} + T_{22} + T_{33}$ in the Euclidean Metric $E^{ij} = \delta^{ij}$

$T^{μμ}$ is just the diagonal part of 4-Tensor $T^{μν}$, the components: $T^{μμ} = \text{Diag}[T_{00}, T_{11}, T_{22}, T_{33}]$

$T_{μ}^{μ}$ is the Trace of 4-Tensor $T^{μν}$: $T_{μ}^{μ} = T_{0}^{0} + T_{1}^{1} + T_{2}^{2} + T_{3}^{3} = 4\text{-Trace}[T^{μν}] = T_{00} - T_{11} - T_{22} - T_{33}$ in the Minkowskian Metric $η^{μν}$

Incorrect: Hiding factors of ($c$) in relativistic equations, ex. $E = m$

The use of “natural units” leads to a lot of ambiguity, and one loses the ability to do dimensional analysis.

Wrong: $E = m$: Energy is *not* identical to mass.

Correct: $E = mc^2$: Energy is related to mass via the speed-of-light, ie. mass is a type of concentrated energy.

Incorrect: Using $m$ instead of $m_0$ for rest mass, Using $E$ instead of $E_0$ for rest energy

Correct: $E = mc^2 = \gamma m_0 c^2 = \gamma E_0$

$E$ & $m$ are relativistic internal components of 4-Momentum $P = (mc, p) = (E/c, p)$ which vary in different reference-frames.

$E_0$ & $m_0$ are Lorentz Scalar Invariants, the rest values, which are the same, even in different reference-frames: $P = m_0 U = (E_0/c^2) U$
There are some paradigm assumptions that need to be cleared up:

Incorrect: Using the same symbol for a tensor-index and a component
The biggest offender in many books for this one is quantum commutation. Unclear because (i) means two different things in one equation. Better: (i = \(\sqrt{-1}\)) is the imaginary unit; \{j,k\} are tensor-indicies
In general, any equation which uses complex-number math should reserve (i) for the imaginary, not as a tensor-index.

Incorrect: Using the 4-Gradient notation incorrectly
The 4-Gradient is a 4-Vector, a (1,0)-Tensor, which uses an upper index, and has a negative spatial component in SR. The Gradient One-Form, its natural tensor form, a (0,1)-Tensor, uses a lower index in SR.
4-Gradient: \(\partial = \partial\mu = (\partial_t/c, -\nabla)\) Gradient One-Form: \(\partial_\mu = (\partial_t/c, \nabla)\)

Incorrect: Mixing styles in 4-Vector naming conventions
There is pretty much universal agreement on the 4-Momentum \(P^\mu = (E/c, p) = (mc, p)\)
Do not in the same document use 4-Potential \(A = (\phi, A)\): This is wrong on many levels.
The correct form is 4-VectorPotential \(A^\mu = (\phi/c, a)\), with \(\phi\) as the scalar-potential & \(a\) as the 3-vector-potential
For all 4-Vectors, one should use a consistent notation:
The Upper-Case SpaceTime 4-Vector Names match the lower-case spatial 3-vector names
There is a \((c)\) factor in the temporal component to give overall matching dimensional units for the entire 4-Vector
4-Vector components are typically lower-case with a few historical exceptions, mainly energy \(E\), energy-density \(e\) or \(\rho\)
Old Paradigm: QM (as I was taught)

SR and QM as separate theories

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g^{\mu\nu}$
- $c, G$ = physical constants

GR limiting-case: $g^{\mu\nu} \to \eta^{\mu\nu}$
Minkowski “Flat” SpaceTime Metric = (Curvature ~ 0)

Obscure QM Axioms:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle
- Hermitian Generators
- Correspondence Principle to CM
- Born Probability Interpretation
- $\hbar, \hbar =$ physical constants

SR and QM as separate theories

This was the QM paradigm that I was taught while in Grad School; everyone trying for Quantum Gravity
Old Paradigm: QM (years later)

SR and QM still as separate theories
QM limiting-case better defined, still no QG

Obscure QM Axioms:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle
- Correspondence Principle to CM
- Born Probability Interpretation
- $h, \hbar = \text{physical constants}$

Simple GR Axioms:
- Principle of Equivalence
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SR and QM still as separate theories
QM limiting-case better defined, still no QG

It is known that QM + SR “join nicely” together to form RQM, but problems with RQM + GR...

Yet another “would be” fortuitous merging??
50+ years searching for QG with no success...

A Tensor Study of Physical 4-Vectors
SciRealm.org
John B. Wilson
Physical Theories as Venn Diagram
Which regions are real?

GR: General Relativity
SR: Special Relativity
QM: Quantum Mechanics
CM: Classical Mechanics
RQM: Relativistic QM

Reality

SR limiting-case: $|v| \ll c$
QM limiting-case: $\hbar |\nabla \cdot p| \ll (p \cdot p)$

GR limiting-case: $g^{\mu \nu} \rightarrow \eta^{\mu \nu}$ Minkowski "Flat" SpaceTime = (Curvature $\sim 0$)

QM physicists think these areas, anything outside of QM, doesn’t exist...
Hence the attempt to Quantize Gravity...
Unsuccessful for 50+ years...

Many-QM physicists believe that the regions outside of QM don’t exist...
SRQM Interpretation would say that the regions outside of GR probably don’t exist...

Many-Worlds Interpretations
Non-local interactions
Instantaneous QM entangled connections
Instantaneous Physical Wavefunction Collapse
Spacetime Dimensions $>4$
Hidden:Alternate Dimensions
Super-Symmetry
String Theory
Alternate Gravity Theories
etc.

Quantum Mysticism...

Basically lots of stuff for which there is little to no empirical evidence...
Physical Limit-Cases as Venn Diagram

Which limit-regions use which physics?

Quantum Gravity? Actual GR?

SR limit-case: \(|v| << c|

QM limit-case: \(|\nabla \cdot p| << (p \cdot p)| or \psi \rightarrow \text{Re}[\psi]|

Change by a few quanta has negligible effect on overall state

GR limit-case: \(g^{\mu \nu} \rightarrow \eta^{\mu \nu}|

Minkowski “Flat” SpaceTime = (Curvature \sim 0)

SRQM: Special Relativity \rightarrow Relativistic QM

Classical SR
Classical (non-QM) Special Relativity

Classical GR
Classical (non-QM) General Relativity

RQM
Relativistic QM

CM
Classical Mechanics

QM
Non-relativistic Quantum Mechanics

SRQM: A treatise of SR \rightarrow QM by John B. Wilson (SciRealm@aol.com)
Special Relativity → Quantum Mechanics

Background: Proven Physics

Both General Relativity (GR) and Special Relativity (SR) have passed very stringent tests of multiple varieties. Likewise, Relativistic Quantum Mechanics (RQM) and Quantum Mechanics (QM) have passed all tests within their realms of validity: generally micro-scale systems, but a few special macro-scale systems ex. Bose-Einstein condensates, superfluids, etc.

To date, however, there is no experimental indication that quantum effects "alter" the fundamentals of either SR or GR. Likewise, there are no known violations, QM or otherwise, of Local Lorentz Invariance (LLI) nor of Local Position/Poincaré Invariance (LPI). In fact, in all known experiments where both SR/GR and QM are present, QM respects the principles of SR/GR, whereas SR/GR modify the results of QM. All tested quantum-level particles, atoms, isotopes, super-positions, spin-states, etc. obey GR's Universality of Free Fall & Equivalence Principle and SR's \( E = mc^2 \) and speed-of-light (c) communication limit. Quantum-level atomic clocks are used to measure gravitational red:blue-shift effects. i.e. GR gravitational frequency-shift (time-dilation) alters atomic=quantum-level timing.

Some might argue that QM modifies the results of SR, such as via non-commuting measurements. However, that is an alteration of CM expectations, not SR expectations. In fact, there is a basic non-zero commutation relation fully within SR: \([\partial^\mu, X^\nu] = \eta^{\mu\nu}\) which will be derived from purely SR Principles in this treatise.

On the other hand, GR Gravity *does* induce changes in quantum interference patterns and hence modifies QM: See the COW gravity-induced neutron QM interference experiments and the LIGO gravitational-wave detections via QM interferometry. Likewise, SR induces fine-structure splitting of spectral lines of atoms, “quantum” spin, spin magnetic moments, spin-statistics (fermions & bosons), antimatter, QED, Lamb shift, etc. - essentially requiring QM to be RQM to be valid.

Some QM scientists say that quantum entanglement is "non-local", but you still can't send any real messages/signals/information/particles faster than SR's speed-of-light (c). The only "non-local" aspect is the alteration of probabilities based on knowledge gained via measurement. A local measurement can alter the "partial information" known about a distant (entangled) system. There is no FTL communication to or alteration of the distant particle. Getting a Stern-Gerlach “up” here doesn’t cause the distant entangled particle to suddenly start moving “down”. One only knows “now” that it “will” go down if the distant experimenter actually performs the measurement.

QM respects the principles of SR/GR, whereas SR/GR modify the results of QM
Special Relativity → Quantum Mechanics

Background: GR Principles

Principles/Axioms and Mathematical Consequences of GR:

**Equivalence Principle:** Inertial Motion = Geodesic Motion, Universality/Equivalency of Free-Fall, \( \text{Mass}_{\text{inertial}} = \text{Mass}_{\text{gravitational}} \)

**Relativity Principle:** SpaceTime M has a Lorentzian Metric \( g^{\mu \nu} \), SR:Minkowski Space rules apply locally

**General Covariance Principle:** Tensors describe Physics, Laws of Physics are independent of Coordinate System

**Invariance Principle:** Invariant Interval Measure comes from Tensor Invariance Properties

**Causality Principle:** Minkowski Diagram/Light-Cone give {Time-Like, Light-Light(null), Space-Like} Measures and Causality Conditions

Einstein: Riemann’s Ideas about Matter & Curvature:
Riemann\((g)\) has 20 independent components = too many
Ricci\((g)\) has 10 independent components = enough to describe gravitational field

\{c,G\} are Fundamental Physical Constants

To date, there are no violations of any of these GR Principles.

GR limiting-case: \( g^{\mu \nu} \rightarrow \eta^{\mu \nu} \)
Minkowski “Flat” SpaceTime Metric = (Curvature ~ 0)

All known experiments to date comply with all of these Principles, including QM
Old Paradigm: QM (for comparison)
SR and QM still as separate theories
QM limiting-case better defined, still no QG

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g^\mu{}_{\nu}$
- $c, G =$ physical constants

SR limiting-case: $|v| \ll c$

Obscure QM Axioms:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle
- Correspondence Principle to CM
- Born Probability Interpretation
- $\hbar, \hbar =$ physical constants

| QM limiting-case: $\hbar |\nabla \cdot p| \ll (p \cdot p)$
or $\psi \to \text{Re}[\psi]$ |
|-----------------------------------------------|

GR limiting-case: $g^\mu{}_{\nu} \rightarrow \eta^\mu{}_{\nu}$
Minkowski “Flat” SpaceTime Metric = (Curvature $\sim 0$)

Yet another “would be” fortuitous merging???
50+ years searching for QG with no success...

SR limiting-case: $|v| \ll c$

A fortuitous merging?

QM → CM

It is known that QM + SR “join nicely” together to form RQM, but problems with RQM + GR...
New Paradigm: SRQM or [SR→QM]
QM derived from SR + a few empirical facts

This new paradigm explains why RQM “miraculously fits” SR, but not necessarily GR
New Paradigm: SRQM w/ EM

QM, EM, CM derived from SR + a few empirical facts

This new paradigm explains why RQM “miraculously fits” SR, but not necessarily GR
The entire classical SR → EM, CM structure is based on the limiting-case of quantum effects being negligible.

Notice that only the SR 4-Vector relation: \( K = (1/\hbar)P \) is missing from the Classical Interpretation...

All of the SR 4-Vectors, including \( (K, \partial) \), are still present in the Classical setting.

\( K \) is used in the Relativistic Doppler Effect and EM waves.
\( \partial \) is used in the SR Conservation/Continuity Equations, Maxwell Equations, Hamilton-Jacobi, Lorenz Gauge, etc.

\( \partial = (\partial t/c, -\nabla) \) may be somewhat controversial, but it is the equation for complex plane-waves, which are in classical EM (in real form).

This \{Classical=non-QM\} SR → EM, CM paradigm has been working successfully for decades
New Paradigm: SRQM View as Venn Diagram

The SRQM view: Each level (range of validity) is a subset of the larger level.

- **GR**
  General Relativity

- **SRQM**
  Special Relativity → Relativistic QM
  GR limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$ Minkowski “Flat” SpaceTime = (Curvature ~ 0)

- **QM**
  Non-relativistic Quantum Mechanics
  SR limiting-case: $|v| << c$

- **CM**
  Classical Mechanics
  QM limiting-case: $\hbar|\nabla \cdot p| << (p \cdot p)$ or $\psi \rightarrow \text{Re}[\psi]$
  Change by a few quanta has negligible effect on overall state
New Paradigm:
SRQM View w/ EM as Venn Diagram

The SRQM view: Each level (range of validity) is a subset of the larger level
Newton's laws of classical physics are greatly simplified by the use of physical 3-vector notation, which converts 3 separate space components, which may be different in various coordinate systems, into a single invariant object, a vector:

The basis values of these components can differ, yet still refer to the same overall 3-vector object.

→ (a x, a y, a z) Cartesian/Rectangular 3D basis

→ (a r, a θ, a z) Polar/Cylindrical 3D basis

→ (a r, a θ, a Φ) Spherical 3D basis

The scalar products of either type are basis-independent

→ (a t, a x, a y, a z) Cartesian/Rectangular 4D basis

→ (a t, a r, a θ, a z) Polar/Cylindrical 4D basis

→ (a t, a r, a θ, a φ) Spherical 4D basis

SR is able to expand the concept of mathematical vectors into the Physical 4-Vector, which combines both (time) and (space) components into a single TimeSpace object:

These 4-Vectors are elements of Minkowski 4D SR SpaceTime.

Typically there is a speed-of-light factor (c) in the temporal component to make the dimensional units match.

eg. R = (ct, r): overall dimensional units of [length] = SI Unit [m]

This also allows the 4-Vector name to match up with the 3-vector name.

In this presentation:
I use the (+,−,−,−) metric signature, giving $A\cdot A = A^\mu \eta_{\mu\nu} A^\nu = (a_0^2 - a\cdot a) = (a_0^2)$. 4-Vectors will use Upper-Case Letters, ex. A; 3-vectors will use lower-case letters, ex. a

Vectors of both types will be in bold font; components and scalars in normal font and usually lower-case. 4-Vector name will match 3-vector name.

Tensor form will usually be normal font with a tensor index, ex. Aμ or a i, with Greek TimeSpace index (0,1..3); Latin SpaceOnly index (1..3)

SR is able to expand the concept of mathematical vectors into the Physical 4-Vector, which combines both (time) and (space) components into a single TimeSpace object:

These 4-Vectors are elements of Minkowski 4D SR SpaceTime.

Typically there is a speed-of-light factor (c) in the temporal component to make the dimensional units match.

eg. $R = (ct, r) = (ct, r) = (ct, r_1, r_2, r_3)$

$R = \langle x, y, z, t \rangle$

SR 4-Vector

Classical 3-vector

Classical scalar

Lorentz 4-Scalar

4-Position

[m/s]

[s]

[m]

SR 4-Vector

4-Vector SRQM Interpretation

of QM

SciRealm.org

John B. Wilson

A Tensor Study of Physical 4-Vectors

SR → QM

4-Vector SRQM

Interpretation of QM

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John B. Wilson

A Tensor Study of Physical 4-Vectors

SR language beautifully expressed with Physical 4-Vectors

Classical 3D objects styled this way to emphasize that they are actually just the separated components of SR 4-Vectors. The triangle/wedge (3 sides) represents splitting the components into a scalar and 3-vector.
4-Vectors are type (1,0)-Tensors, Lorentz 4-Scalars are type (0,0)-Tensors, 4-CoVectors are type (0,1)-Tensors, (m,n)-Tensors have (m) upper-indices and (n) lower-indices. $V^\mu$, $S$, $C_\mu$, $T^{\mu\nu\alpha\beta\gamma\ldots\mu}$ have (m) upper-indices and (n) lower-indices.

Any equation which employs only Tensors, such as those with only 4-Vectors and Lorentz 4-Scalars, (ex. $P = m_o U$) is automatically Frame-Invariant, or coordinate-frame-independent. One’s frame-of-reference plays no role in the form of the overall equations. This is also known as being “Manifestly-Invariant”. This is exactly what Einstein meant by his postulate: “The laws of physics should have the same form for all inertial observers”. Use of the RestFrame-naught ($o$) helps show this.

The components ($a^0, a^1, a^2, a^3$) of the 4-Vector $A$ can vary depending on the observer and their choice of coordinate system, but the 4-Vector $A$ itself is invariant. Equations using only 4-Tensors, 4-Vectors, and Lorentz 4-Scalars are true for all inertial observers. The SRQM Diagramming Method makes this easy to see in a visual format, and will be used throughout this treatise.

The following examples are frame-invariant equations:

- $U \cdot U = (c)^2$
- $U = \gamma(c,u)$
- $P = (mc,p) = (E/c,p) = m_o U = (E_o/c^2) U$
- $K = (\omega/c,k) = (\omega_o/c^2) U$
- $P \cdot U = E_o$

The SRQM Diagram Form has all of the info of the Equation Form, but shows overall relationships and symmetries among the 4-Vectors much more clearly. Blue: Temporal components Red: Spatial components Purple: Mixed TimeSpace components

4-Vector $= 4D (1,0)$-Tensor $A = A^\mu = (a^\mu) = (a^0, a^i) = (a^0, a^1, a^2, a^3) \to (a^0, a^1, a^2, a^3)$

$A \cdot A = A^\mu \eta_{\mu\nu} A^\nu = (a^0)^2 - a\cdot a = (a^0_o)^2$

The components ($a^0, a^1, a^2, a^3$) of the 4-Vector $A$ can vary depending on the observer and their choice of coordinate system, but the 4-Vector $A$ itself is invariant. Equations using only 4-Tensors, 4-Vectors, and Lorentz 4-Scalars are true for all inertial observers. The SRQM Diagramming Method makes this easy to see in a visual format, and will be used throughout this treatise.

The following examples are frame-invariant equations:

- $U \cdot U = (c)^2$
- $U = \gamma(c,u)$
- $P = (mc,p) = (E/c,p) = m_o U = (E_o/c^2) U$
- $K = (\omega/c,k) = (\omega_o/c^2) U$
- $P \cdot U = E_o$

The SRQM Diagram Form has all of the info of the Equation Form, but shows overall relationships and symmetries among the 4-Vectors much more clearly. Blue: Temporal components Red: Spatial components Purple: Mixed TimeSpace components

4-Vector $= 4D (1,0)$-Tensor $A = A^\mu = (a^\mu) = (a^0, a^i) = (a^0, a^1, a^2, a^3) \to (a^0, a^1, a^2, a^3)$

$A \cdot A = A^\mu \eta_{\mu\nu} A^\nu = (a^0)^2 - a\cdot a = (a^0_o)^2$
We want to be clear, however, that SR 4-Vectors are NOT generalizations of Classical or Quantum 3-vectors. SR 4-Vectors are the primitive elements of Minkowski SpaceTime (4D) which incorporate both:

- a {temporal scalar element} and a {spatial 3-vector element} as components.

4-Vector \( \mathbf{A} = (a_0, a_1, a_2, a_3) \rightarrow (a^t, a^x, a^y, a^z) \) with scalar \( a^t \) & 3-vector \( a^x \rightarrow (a^x, a^y, a^z) \)

It is the Classical or Quantum 3-vector \( a^x \) which is a limiting-case approximation of the spatial part of SR 4-Vector \( \mathbf{A} \) for \( |v| << c \).

i.e. The Energy (E) and 3-momentum (p) as “separate” entities occurs only in the low-velocity limit \( |v| << c \) of the Lorentz Boost Transform. They are actually part of a single 4D entity: the 4-Momentum \( \mathbf{P} = (E/c, \mathbf{p}) \); with the components: temporal (E), spatial (p), dependent on a frame-of-reference, while the overall 4-Vector \( \mathbf{P} \) is invariant. Likewise with (t) and (r) in the 4-Position \( \mathbf{R} \).

SR is Minkowskian; obeys Lorentz Invariance.

CM is Euclidean; obeys Galilean Invariance.
Relations among 4-Vectors and Lorentz 4-Scalars are Manifestly Invariant, meaning that they are true in all inertial reference frames.

Consider a particle at a SpaceTime <Event> that has properties described by 4-Vectors \( \mathbf{A} \) and \( \mathbf{B} \):

One possible relationship is that the two 4-Vectors are related by a Lorentz 4-Scalar (S): \( \mathbf{B} = (S) \mathbf{A} \).

How can one determine this? Answer: Make an experiment that empirically measures the tensor invariant \( \mathbf{B} \cdot \mathbf{A} / \mathbf{A} \cdot \mathbf{A} \).

If \( \mathbf{B} = (S) \mathbf{A} \)
\( \mathbf{B} \cdot \mathbf{A} = (S) \mathbf{A} \cdot \mathbf{A} \)
\( (S) = \mathbf{B} \cdot \mathbf{A} / \mathbf{A} \cdot \mathbf{A} \)
Note that this basically a vector projection.

Run the experiment many times. If you always get the same result for \( (S) \), then it is likely that the relationship is true, and invariant.

Example: Measure \( (S_P) = \mathbf{P} \cdot \mathbf{U} / \mathbf{U} \cdot \mathbf{U} \) for a given particle type.
Repeated measurement always give \( (S_P) = m_o \)
This makes sense because we know \( \mathbf{P} \cdot \mathbf{U} = \gamma(E - \mathbf{p} \cdot \mathbf{u}) = E_o \) and \( \mathbf{U} \cdot \mathbf{U} = c^2 \)
Thus, 4-Momentum \( \mathbf{P} = (E_o/c^2)\mathbf{U} = (m_o)\mathbf{U} = (m_o)^*4\)-Velocity \( \mathbf{U} \)

Example: Measure \( (S_K) = \mathbf{K} \cdot \mathbf{U} / \mathbf{U} \cdot \mathbf{U} \) for a given particle type.
Repeated measurement always give \( (S_K) = (\omega_o/c^2) \)
This makes sense because we know \( \mathbf{K} \cdot \mathbf{U} = \gamma(\omega - \mathbf{k} \cdot \mathbf{u}) = \omega_o \) and \( \mathbf{U} \cdot \mathbf{U} = c^2 \)
Thus, 4-WaveVector \( \mathbf{K} = (\omega_o/c^2)\mathbf{U} = (\omega_o/c^2)^*4\)-Velocity \( \mathbf{U} \)

Since \( \mathbf{P} \) and \( \mathbf{K} \) are both related to \( \mathbf{U} \), this would also mean that the 4-Momentum \( \mathbf{P} \) is related to the 4-WaveVector \( \mathbf{K} \) in a particular manner for each given particle type...
Some SR Mathematical Tools

Definitions and Approximations

β = v/c: dimensionless Velocity Beta Factor
γ = 1/√(1-β²): dimensionless Lorentz Relativistic Gamma Factor

(1+x)ⁿ ≈ (1 + nx + O[x²]) for { |x| << 1 } Approximation used for SR→Classical limiting-cases

Lorentz Transformation \( \Lambda^\mu_v = \partial X^\mu / \partial X^v = \partial \{X^\nu\} \): a relativistic frame-shift, such as a rotation or velocity boost

It transforms a 4-Vector in the following way: \( X^\nu = \Lambda^\nu_v X^v \) : with Einstein summation over the paired indicies

A typical Lorentz Boost Transformation \( \Lambda^{\nu}_v \rightarrow B^{\nu}_v \) for a linear-velocity frame-shift \((x,t)\)-Boost in the \( x \)-direction:

\[
\Lambda^{\nu}_v = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \beta & 0 \\
0 & -\beta & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Original \( A^\nu = (a^1, a^2, a^3, a^4) \)

Boosted \( A^{\nu}_v = (a^{\nu}_1, a^{\nu}_2, a^{\nu}_3, a^{\nu}_4) = \Lambda^{\nu}_v A^\nu \rightarrow B^{\nu}_v A^\nu = (\gamma a^1 - \gamma \beta a^3, -\gamma \beta a^1 + \gamma a^3, a^2, a^4) \) (for \( x \)-boost Lorentz Transform)

\[
A \cdot B' = (\Lambda^{\nu}_v A^\nu) \cdot (\Lambda^{\sigma}_w B^\sigma) = A^\nu \Lambda^{\nu}_v \Lambda^{\sigma}_w B^\sigma = A^\mu B^\nu = \sum_{\nu=0..3} [a^\nu b^\nu] = \sum_{\nu=0..3} [a^\nu b^\sigma] = (a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3)
\]

using the Einstein summation convention where upper:lower paired-indices are summed over

\[
\delta[X] = \partial[X]\]

\( \partial[R] = \partial R^\mu = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1] = \text{Diag}[1,-\delta^3] \)

\( \eta^{\mu\nu} = 1/\{\delta_\nu^\mu\} : \delta_\nu^\mu = \delta^\nu_v \text{ Tr}[\eta^{\mu\nu}] = 4 \)

\( \delta[R]^{\mu\nu} = \partial R^{\mu\nu} = \Lambda^{\mu\nu} \)

\( \eta^{\mu\nu} = \delta^\mu_v \Lambda^{\nu}_v \)

Det[\Lambda^{\mu\nu}] = \pm 1 \text{ Dimension}

\[
\text{Minkowski Metric}
\]

\[
\text{Lorentz Scalar}
\]

\[
\text{Trace}[T^{\mu\nu}] = \eta^{\mu\nu} T^{\mu\nu} = T_\nu^\nu = T

V \cdot V = V^{\mu\nu} \Lambda^{\nu}_v \Lambda^{\sigma}_w = [V^{\mu\nu} - V^{\mu\nu}] = (V^{\mu\nu})^2 = \text{Lorentz Scalar}
\]
SRQM Diagram:
The Basis of Classical SR Physics Special Relativity via 4-Vectors

Focus on three of the main SR Physical 4-Vectors.

4-Position
\[ R = R^\alpha = (ct, r) = \text{<Event>} \]

4-Velocity
\[ U = U^\mu = \gamma(c, u) \]

4-Gradient
\[ \partial = \partial' = \left( \frac{\partial}{c}, \mathbf{\nabla} \right) \]

These three give some of the main classical results of Special Relativity, including SR concepts like:

- The Minkowski Metric, SpaceTime Dimension = 4, Lorentz Transformations
- <Events>, Invariant Interval Measure, Causality (Temporal Ordering)
- The Invariant Speed-of-Light (c)
- Invariant ProperTime (clock at rest), Invariant ProperLength (ruler at rest)
- Time Dilation (clock moving), Length Contraction (ruler moving)
- Relativity of Simultaneity, Minkowski Diagrams, Light Cone
- Use of the Lorentz Scalar Product to make Lorentz Invariants
- Invariant SR Wave Equations, via the d'Alembertian Continuity Equations

etc.
The Basis of Classical SR Physics
Special Relativity via 4-Vectors

SRQM Diagram:

The Basis of all Classical SR Physics is in the SR Minkowski Metric of “Flat” SpaceTime, which can be generated from the 4-Position and 4-Gradient, and determines the measurement between <Event>’s.

This Metric provides the relations between the main 4-Vectors of SR: 4-Position $R$, 4-Gradient $\partial$, 4-Velocity $\dot{U}$.

The Tensor Invariants of these 4-Vectors give the: Invariant Interval Measures & Causality, from $R\cdot R$ Invariant d’Alembertian Wave Equation, from $\partial\partial$ Invariant Magnitude LightSpeed $(c)$, from $U\cdot U$

The relation between 4-Gradient $\partial$ and 4-Position $R$ gives the Dimension of SpaceTime $(4)$, the Minkowski Metric $\eta_{\mu\nu}$, and the Lorentz Transformations $\Lambda^{\mu\nu}$.

The relation between 4-Gradient $\partial$ and 4-Velocity $\dot{U}$ gives the ProperTime Derivative $d\tau/dt$; Rearranging gives the ProperTime Differential $d\tau$, which leads to Time Dilation & Length Contraction.

The ProperTime Derivative $d\tau/dt$ acting on 4-Position $R$ gives 4-Velocity $\dot{U}$ acting on the SpaceTime Dimension Lorentz Scalar gives the Continuity of 4-Velocity Flow.

The relation between 4-Displacement $\Delta R$ and 4-Velocity $\dot{U}$ gives Relativity of Simultaneity.

One of the most important properties is the Tensor Invariant Lorentz Scalar Product $(\dot{v}\cdot\dot{v})$, provided by the lowered- index form of the Minkowski Metric $\eta_{\mu\nu}$.

From here, each object will be examined in turn...
The 4-Position is essentially one of the fundamental 4-Vectors of SR. It is the SpaceTime location of an \(<\text{Event}\), the basic element of Minkowski SpaceTime: a time \((t)\) & a place \((r)\) \(\rightarrow\) \(\text{when, where} = (ct,r)\).

The 4-Position relates time to space via the fundamental physical constant \((c)\): the speed-of-light = "(c)elerity ; (c)eleritas", which is used to give consistent dimensional units across all SR 4-Vectors.

The 4-Position is a specific type of 4-Displacement, for which one of the endpoints is the origin, or 4-Zero.

\[ \Delta R = R_2 - R_1 → R - Z = R \]

As such, the 4-Position and 4-Zero are Lorentz Invariant (point rotations and boosts), but not Poincaré Invariant (Lorentz + time & space translations), which can move the \(<\text{Origin}\>.

The general 4-Displacement and 4-Differential(Displacement) are invariant under both Lorentz and Poincaré transformations, since neither of their endpoints are pinned this way.

The 4-Differential(Displacement) is just the infinitesimal version of the finite 4-Displacement, and is used in the calculus of SR. \(U = dR/d\tau : dR = Ud\tau\)

\[ 4-\text{Position} \quad \mathbf{R} = \langle ct, r \rangle = \langle \Delta x, \Delta y, \Delta z, \Delta t \rangle \]

4-Position

\[ 4-\text{Displacement} \quad \Delta \mathbf{R} = (\Delta x, \Delta y, \Delta z, \Delta t) \]

Is used in the calculus of SR.

The 4-Differential is just the infinitesimal version of the finite 4-Displacement, and is used in the calculus of SR. \(U = dR/d\tau : dR = Ud\tau\)

\[ \mathbf{R} = \int d\mathbf{R} = \int U d\tau = \int \mathbf{U} d\tau = \int (c, u) dy dt + \int (c,\mathbf{u}) dy dt = (ct, r) \]

\[ \mathbf{R} = \Sigma \Delta \mathbf{R} = \Sigma U \Delta t = \Sigma (c, u) \Delta t = (c, u) \Delta t = (c, r) \]

\[ \mathbf{R} = \langle ct, r \rangle = \langle \Delta x, \Delta y, \Delta z, \Delta t \rangle \]

SRQM Diagram: The Basis of Classical SR Physics

4-Position, 4-Displacement, 4-Differential

SRQM Diagram: 4-Vector SRQM Interpretation of QM

\[ \mathbf{V} \cdot \mathbf{V} = \mathbf{V}^2 = \mathbf{V}^* \mathbf{V} = (\mathbf{v}^\times)^2 - \mathbf{v}^2 \]

\[ \text{Lorentz Scalar} \]

\[ \text{SRQM Diagram:}\]

\[ \mathbf{V} = \mathbf{V}^* \mathbf{V} = (\mathbf{v}^\times)^2 - \mathbf{v}^2 \]

\[ \text{Lorentz Scalar} \]

\[ \mathbf{V} \cdot \mathbf{V} = \mathbf{V}^2 = \mathbf{V}^* \mathbf{V} = (\mathbf{v}^\times)^2 - \mathbf{v}^2 \]

\[ \text{Lorentz Scalar} \]
The Invariant Interval is the Lorentz Scalar Product of the (4-Position, 4-Displacement, 4-Differential) with itself, giving a magnitude-squared.

\[ \Delta R \cdot \Delta R = (c\Delta t)^2 - c^2 \Delta r^2 \]

The Invariant Interval is the Lorentz Scalar Product of Physical 4-Vectors, giving a magnitude-squared.

\[ \Delta R \cdot \Delta R = (c\Delta t)^2 - c^2 \Delta r^2 \]

In the context of Classical SR Physics, the 4D intervals are invariant, meaning that all observers must agree on their magnitudes, regardless of differing reference frames. This leads to the idea of ProperTime (\( \Delta t \)), which is the time-displacement measured by a clock at-rest. This also leads to the various Causality Conditions of SR, and the concept of the (Minkowski Diagram) Light Cone. The differential form \( d\mathbf{R} \cdot d\mathbf{R} \) is apparently also still true in GR.

\[ \Delta \mathbf{R} \cdot \Delta \mathbf{R} = (c\Delta t)^2 - c^2 \Delta \mathbf{r}^2 \]

The 4-Displacement \( \Delta \mathbf{R} = (c\Delta t, \Delta \mathbf{r}) = \mathbf{U} \Delta t - \mathbf{R} = (c\Delta t, \Delta \mathbf{r}) \):

- Time-like: Temporal
- Light-like: Null: Photonic
- Space-like: Spatial

\[ \Delta \mathbf{R} \cdot \Delta \mathbf{R} = (c\Delta t)^2 - c^2 \Delta \mathbf{r}^2 \]

(+) {causal = temporally-ordered}
(0) {causal, maximum signal speed (|\( \Delta \mathbf{r} / \Delta t \)|=c)}
(-) {non-causal, spatially-extended}

\[ \Delta \mathbf{R} \cdot \Delta \mathbf{R} = (c\Delta t)^2 - c^2 \Delta \mathbf{r}^2 \]

The 4-Displacement \( \Delta \mathbf{R} = (c\Delta t, \Delta \mathbf{r}) = \mathbf{U} dt - \mathbf{R} = (c\Delta t, \Delta \mathbf{r}) \):

- \( 4\text{-Position} \)
- \( \mathbf{R} = (ct, \mathbf{r}) \)

The Basis of Classical SR Physics

\[ \Delta \mathbf{R} \cdot \Delta \mathbf{R} = (c\Delta t)^2 - c^2 \Delta \mathbf{r}^2 \]

Note: The diagram illustrates the Lorentz transformation and the concept of ProperTime, which is the time-displacement measured by a clock at-rest. This leads to the idea of Causality Conditions of SR, and the concept of the Light Cone. The differential form \( d\mathbf{R} \cdot d\mathbf{R} \) is apparently also still true in GR.
SRQM Diagram:
The Basis of Classical SR Physics
SpaceTime Dimension = 4D

4-Gradient
\[ \partial = (\partial/c, \nabla) \]
4-Position
\[ R = (ct, r) = \text{<Event>} \]

\( \partial \cdot R = 4 \) : The 4-Divergence SpaceTime Dimension Relation

\[ \partial = \partial/c, \nabla (ct, r) = \begin{pmatrix} \partial, & [\partial/c] \cdot (ct, \nabla (ct, r)) \end{pmatrix} \]

\[ \nabla \cdot R = (\partial/c, \nabla) \cdot (ct, r) = (\partial/c, \nabla) \cdot (ct, \nabla (ct, r)) \]

All empirical evidence to-date indicates that there are only the 4 known dimensions:
SpaceTime is 4D

(1,0)-Tensor V
(0,1)-Tensor V
(1,1)-Tensor T
(0,0)-Tensor S

4-Vector SRQM Interpretation of QM
SciRealm.org
John B. Wilson

SR : Minkowski Space Time is 4D
Trace[\( T^\mu_\nu \)] = \( \eta_{\mu\nu} T^\mu_\nu = T^\mu_\mu = T \)
\( V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = (|v|^2 - V^0) = (V^0)^2 = \text{Lorentz Scalar} \)
The SR Minkowski Metric = "Flat" SR SpaceTime

\[ \eta_{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu \\ -1 & \text{for } \mu \neq \nu \end{cases} \]

The Minkowski Metric is a fundamental 4-dimensional metric tensor in Special Relativity, often denoted as $\eta_{\mu\nu}$, which plays a crucial role in defining the geometry of spacetime. It is central to the formulation of the Lorentz transformation and the propagation of light as well as other physical phenomena.

**Derivation:**

The Minkowski Metric will differ with the chosen basis, just like with 4-Vectors.

- $\eta_{\mu\nu}$ will differ with the chosen basis, just like with 4-Vectors.
- $\eta_{\mu\nu}$ will differ with the chosen basis, just like with 4-Vectors.

**SRQM Diagram:**

The SRQM Diagram is a visual representation of the fundamental concepts in Special Relativity and Quantum Mechanics, integrating the Minkowski Metric and other key components to illustrate the relationships and transformations.

**SRQM Diagram Elements:**

- **SR: Minkowski Metric**
  \[ \eta_{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu \\ -1 & \text{for } \mu \neq \nu \end{cases} \]

- **SR: Temporal Projection**
  "Vertical" $V^\mu = T^T = 4$ (T = Temporal)
  $\text{Diag}[1,0,0,0] = \text{Diag}[1,0,0,0]$.

- **SR: Spatial Projection**
  "Horizontal" $H^\mu = \text{c}^4 T^T = \text{Diag}[0,1,0,0] = \text{Diag}[0,1,0,0]$

- **SR: Temporal Projection**
  "Horizontal" $H^\mu = \text{c}^4 T^T = \text{Diag}[0,1,0,0] = \text{Diag}[0,1,0,0]$

- **SR: Spatial Projection**
  "Vertical" $V^\mu = T^T = 4$ (T = Temporal)
  $\text{Diag}[1,0,0,0] = \text{Diag}[1,0,0,0]$.

**Derivation:**

It is important to note that the low-mass case is a curve that gives the curvature of the structure of space-time. The Minkowski Metric can be used to raise/lower indices on other tensors and 4-Vectors. The derivation of the Minkowski Metric involves understanding the invariance of the Metric under Lorentz transformations.

**SRQM Diagram Elements:**

- **SR: Temporal Projection**
  "Vertical" $V^\mu = T^T = 4$ (T = Temporal)
  $\text{Diag}[1,0,0,0] = \text{Diag}[1,0,0,0]$

- **SR: Spatial Projection**
  "Horizontal" $H^\mu = \text{c}^4 T^T = \text{Diag}[0,1,0,0] = \text{Diag}[0,1,0,0]$

- **SR: Temporal Projection**
  "Vertical" $V^\mu = T^T = 4$ (T = Temporal)
  $\text{Diag}[1,0,0,0] = \text{Diag}[1,0,0,0]$

- **SR: Spatial Projection**
  "Horizontal" $H^\mu = \text{c}^4 T^T = \text{Diag}[0,1,0,0] = \text{Diag}[0,1,0,0]$

The SR: Minkowski Metric is the "Flat SpaceTime" low-curvature limiting-case of the more general GR Metric $g_{\mu\nu}$. It can be divided into temporal and spatial parts. The Minkowski Metric is used to define the geometry of spacetime and is fundamental in the formulation of the Lorentz transformation and the propagation of light as well as other physical phenomena.
SRQM Diagram:

The Basis of Classical SR Physics

The Lorentz Transform \( \partial_{\nu}[R^{\mu}_{\nu}] = \Lambda^{\mu}_{\nu} \)

4-Vector SRQM Interpretation of QM

SciRealm.org

John B. Wilson
SRQM Diagram:
The Basis of Classical SR Physics
SpaceTime Dimension = 4D, again!

\[\delta R = Tr[\eta^\mu\nu] = \Lambda^\mu_{\alpha}\Lambda^\alpha_{\mu} = 4: \text{The SpaceTime Dimension Relations}\]

Tensor Invariants include: Trace, InnerProduct, Determinant, etc. The Trace of the Minkowski Metric \(\eta\) and the InnerProduct of any of the Lorentz Transforms give the Dimension of SR SpaceTime = 4D.

Minkowski Metric 4-Divergence
Trace Invariant of 4-Position
\[
\delta R = Tr[\eta^\mu\nu] = \eta_{\mu\nu}
\]
\[
\Sigma = \eta^\mu_{\nu} = \eta_{\mu\nu}
\]
\[
\Lambda^\mu_{\alpha} = \delta^\mu_{\alpha}
\]
\[
\eta^\mu_{\nu} = 4
\]

Lorentz Transform
Begin Tensor Invariant
\[
\delta - \eta^\mu_{\nu} = \eta_{\mu\nu}
\]
\[
\Lambda^\mu_{\alpha} = \delta^\mu_{\alpha}
\]
\[
\eta^\mu_{\nu} = 4
\]

\[\text{4-Gradient} \quad \partial = (\partial_0, \partial_1, \partial_2, \partial_3) = \frac{1}{c} \nabla\]

\[\text{SRQD Diagram} \quad \text{SR} : \text{Minkowski SpaceTime is 4D} \]

\[\text{SR 4-Tensor (2,0)-Tensor } T^\mu_\nu \]
\[\text{SR 4-Vector (1,0)-Tensor } V^\mu = (v^0, v) \]
\[\text{SR 4-Scalar (0,0)-Tensor } S \]

\[\text{SR 4-Covector (0,1)-Tensor } V_\mu = (v_0, -v) \]

\[\text{SR 4-Gradient} \quad \partial = (\partial_0, \partial_1, \partial_2, \partial_3) \]

\[\text{SRQD Diagram} \quad \text{SR} = \text{Minkowski SpaceTime is 4D} \]

\[\text{SRQD Diagram} \quad \text{SR} : \text{Minkowski SpaceTime is 4D} \]

\[\text{SRQD Diagram} \quad \text{SR} = \text{Minkowski SpaceTime is 4D} \]
A Tensor Study

The Basis of Classical SR Physics

Lorentz Scalar (Dot) Product \( \eta_{\mu\nu} = \cdot \)
A Tensor Study of Physical 4-Vectors

SR

(1,1)-Tensor T
4-Momentum

4-Velocity
absorbed into the Lorentz 4-Scalar factor that goes into their components.

\[ U \cdot U \]

This is due to the constraint placed by the Tensor Invariant of the 4-Velocity.

The 4-Velocity is unlike most of the other SR 4-Vectors in that it only but changes direction as the WorldLine bends.

For an un-accelerated observer,

4-Velocity

\[ \text{for} \; \text{d} \tau = \frac{dR}{c} \]

has a constant magnitude, the speed-of-light (c) in SpaceTime.

\[ 3 \text{ independent } + 1 \text{ independent } = 4 \text{ independent} \]

Components:

4-Velocity

\[ 4-Velocity \; U = \gamma(c, u) = (c, \gamma u) \]

4-Momentum

\[ 4-Momentum \; P = (mc, p) = (E/c, p) \]

The 4-Velocity also usually has the Relativistic Gamma factor (\( \gamma \)) exposed in component form, whereas most of the other temporal 4-Vectors have it absorbed into the Lorentz 4-Scalar factor that goes into their components.

\[ U \cdot U = c^2 \]

\[ U \cdot U = E/c^2 \]

\[ P \cdot P = (mc)^2 = (E/c)^2 \]

The 4-Velocity is unlike most of the other SR 4-Vectors in that it only has 3 independent components, whereas the others usually have 4. This is due to the constraint placed by the Tensor Invariant of the 4-Velocity. \( U \cdot U \) has a constant magnitude, the speed-of-light (c) in SpaceTime.

4-Displacement

\[ \Delta R = (c \Delta t, \Delta r) \]

4-Momentum

\[ dR = (cdt, dr) \]

4-Position

\[ R = (ct, r) \]

\[ \delta[R] = \delta^\nu_R \eta^{\nu\mu} \rightarrow \text{Diag}[1,-1,-1,-1] = \text{Diag}[1,-\delta^0, \delta^1, \delta^2, \delta^3] \]

Minkowski Metric

\[ \partial \cdot \partial \equiv \text{d'Alembertian} \]

\[ T^\nu_\mu = \partial_t T^\nu_\mu = 0 \]

\[ (\partial /c) \partial_t \rightarrow (\partial /c) \partial_t = \partial^t \]

\[ (\partial /c) \partial^t \rightarrow (\partial /c) \partial^t = \partial^t \]

\[ \text{Relativity of Simultaneity} \]

\[ U \cdot \Delta R = (c, u) \cdot (c \Delta t, \Delta r) = (c^2 \Delta t - u \cdot \Delta r) = c^2 \Delta t_o - c^2 \Delta r \]

\[ \gamma \frac{d}{dt} \]

\[ \text{ProperTime Differential} \]

\[ dt = \gamma \frac{dR}{c} = \gamma \frac{dt}{c} \]

\[ \text{Continuity of 4-Velocity Flow} \]

\[ \Delta \cdot U = 0 \]

\[ \text{SRQM Diagram} \]

The temporal components give Einstein’s famous

\[ E = mc^2 = \gamma m_o c^2 = \gamma E_o \]

\[ \text{The spatial components give} \]

\[ p = mu \]

\[ \gamma m_o u \]

\[ \text{SR 4-Tensor} \]

\[ (2,0)-\text{Tensor T}^{\mu \nu} \]

\[ (1,1)-\text{Tensor T}_\nu^\mu \]

\[ (0,2)-\text{Tensor T}_{\mu \nu} \]

\[ \text{SR 4-Scalar} \]

\[ (0,0)-\text{Tensor S} \]

\[ \text{Lorentz Scalar} \]

\[ \text{SR 4-Vector} \]

\[ (1,0)-\text{Vector V}^\nu = (v^\nu) \]

\[ (0,1)-\text{Vector V}_\nu = (v_\nu) \]

\[ \text{SR 4-CoVector} \]

\[ (0,1)-\text{Vector V}_\nu = (v_\nu) \]

\[ (1,0)-\text{Vector V}^\nu = (v^\nu) \]

\[ \text{SciRealm.org} \]

John B. Wilson
SRQM Diagram: The Basis of Classical SR Physics

4-Velocity Magnitude = Invariant Speed-of-Light (c)

4-Velocity U = γ(c,u) = (γ(c,u), γ(ω,c)) = (U, dR/dτ)R = -dR/dτ = (dR/dc)(dR/dτ) = γ(c, u)

with Relativistic Gamma γ = 1/√(1 - β·β), β = u/c

The Lorentz Scalar Product of the 4-Velocity gives the Invariant Magnitude Speed-of-Light (c), one of the fundamental SR physical constants of physics. Technically, it is the maximum speed of SR causality, which any massless particle, ex. the photon, travel at.

This allows one to make new 4-Vectors related to 4-Velocity U, making it have only 3 independent components (u).

If (c) was not a constant, but varied somehow, then all 4-Vectors made from the 4-Velocity would have more than 4 independent components, which is not observed. It seems a compelling argument against variable light-speed theories.
SRQM Diagram:
The Basis of Classical SR Physics
Relativity of Simultaneity

Relativity of Simultaneity:
\[ \mathbf{U} \Delta \mathbf{X} = \gamma(c, u)(c \Delta t, \Delta \mathbf{X}) = \gamma(c^2 \Delta t - u \cdot \Delta \mathbf{X}) = c^2 \Delta t - \frac{u \cdot \Delta \mathbf{X}}{c^2} \]

If Lorentz Scalar (\( \mathbf{U} \cdot \Delta \mathbf{X} = 0 = c^2 \Delta t \)),
then the ProperTime displacement (\( \Delta t \)) is zero,
and the \(<\text{Events}>\)'s separation (\( \Delta \mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2 \)) is orthogonal
to the worldline at \( \mathbf{U} \).

Examining the equation we get \( \gamma(c^2 \Delta t - u \cdot \Delta \mathbf{X}) = 0 \).
The coordinate time difference between the events is (\( \Delta t = u \cdot \Delta \mathbf{X}/c^2 \))
The condition for simultaneity in an alternate frame
(moving at \( 3-\text{velocity} \))
The coordinate time difference between the events is (\( \Delta t = 0 \)),
which implies (\( u \cdot \Delta \mathbf{X} = 0 \)).

This condition can be met by:
(\( u = 0 \)), the alternate observer is not moving wrt. the events,
 i.e. on worldline \( \mathbf{U} \) or on a worldline parallel to \( \mathbf{U} \).
(\( \Delta \mathbf{X} = 0 \)), the events are at the same spatial location (co-local).
(\( u \cdot \Delta \mathbf{X} = 0 = |u| \Delta \mathbf{X} \cos(8) \)), the alternate observer's motion is
perpendicular (orthogonal \( \theta = 90^\circ \)) to the spatial separation \( \Delta \mathbf{X} \)
of the events in that frame.

If none of these conditions is met,
then the events will not be simultaneous
in the alternate reference-frame.

\[ \Delta t = 0 \quad \text{Simultaneous in } (t', \mathbf{x}') \]
\[ \Delta t \neq 0 \quad \text{Not Simultaneous in } (t, \mathbf{x}) \]

\( \Delta \mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2 \)

SR 4-Tensor
(2,0)-Tensor \( T^{\mu}_{\nu} \)
(1,1)-Tensor \( T^\mu_\nu \), or \( T_{\mu \nu} \)
SR 4-Vector
(1,0)-Tensor \( V^\nu = V = (\mathbf{v}, \gamma) \)
SR 4-Coordinate Vector
(0,1)-Tensor \( V'_\nu = (\mathbf{v'}, \gamma) \)
SR 4-Scalar
(0,0)-Tensor \( S \)
Lorentz Scalar

Relativity of Simultaneity
\( \mathbf{U} \cdot \Delta \mathbf{X} = \gamma(c, u)(c \Delta t, \Delta \mathbf{X}) = \gamma(c^2 \Delta t - u \cdot \Delta \mathbf{X}) = c^2 \Delta t - \frac{u \cdot \Delta \mathbf{X}}{c^2} \)

ProperTime Derivative
\( \mathbf{U} \cdot \gamma = \gamma(c, u)(\partial/c, \nabla) = \gamma(\partial + u \cdot \nabla) \)

Continuity of 4-Velocity Flow
\( \partial \cdot \mathbf{U} = 0 \)

ProperTime Differential
\( \partial t = (1/\gamma) dt = \text{Time Dilation} \)

\[ \text{Trace}[T^{\mu}_{\nu}] = \eta_{\mu \nu} T^{\mu}_{\nu} = T^{\mu}_{\mu} = T \]
\[ V \cdot V = V' \eta_{\mu \nu} V'^{\mu} = (\mathbf{v} \cdot \mathbf{v}) = (\mathbf{v'})^2 \]

Lorentz Scalar
The Basis of Classical SR Physics

The ProperTime Derivative \( (d/d\tau) \)

4-Velocity SR 4-Vector
\[ \mathbf{U} = \gamma(c, u) \]

ProperTime Derivative
\[ \delta = \frac{\delta}{c \mathbf{\Delta}} \]

4-Position
\[ \mathbf{R} = (ct, r) \]

Invariant Interval
\[ R = R = (ct)^2 - r^2 = (ct)^2 \]

\[ \mathbf{U} \cdot \Delta \mathbf{R} = \gamma(c, u) \cdot (\Delta \mathbf{ct}, \Delta \mathbf{r}) = \gamma(c, u) \cdot (\Delta \mathbf{ct}, \Delta \mathbf{r}) \]

4-Gradient
\[ \mathbf{V} = \nabla = (\partial \mathbf{u}/\partial \mathbf{r}) \]

4-Acceleration
\[ \mathbf{A} = \gamma(c, u) \cdot \frac{\partial \mathbf{U}}{\partial \mathbf{t}} \]

4-Momentum
\[ \mathbf{P} = (E/c, p) = (mc, p) \]

4-Force
\[ \mathbf{F} = \gamma(E/c, f) \]

4-Displacement
\[ \Delta \mathbf{R} = (\Delta \mathbf{ct}, \Delta \mathbf{r}) \]

SR 4-Tensor
\[ (2,0) \text{-Tensor} \]

SR 4-Vector
\[ (1,0) \text{-Tensor} \]

SR 4-CoVector
\[ (0,1) \text{-Tensor} \]

SR 4-Scalar
\[ (0,0) \text{-Tensor} \]

\[ \mathbf{S} = \mathbf{\eta} \]

\[ \mathbf{V} \cdot \mathbf{V} = V^2 = (\mathbf{V} \cdot \mathbf{V}) = (\mathbf{V} \cdot \mathbf{V})^2 \]

\[ \text{Trace}[T^\mu \nu] = \eta_{\mu \nu} T^\mu \nu = T^\mu \nu = T \]

\[ \mathbf{V} \cdot \mathbf{V} = V^2 \]

\[ \mathbf{V} \cdot \mathbf{V} = (\mathbf{V} \cdot \mathbf{V})^2 \]

\[ \text{Lorentz Scalar} \]

\[ \gamma = 1 / \sqrt{1 - \beta^2} \]

Relativistic Gamma
\[ \gamma = 1 / \sqrt{1 - \beta^2} \]

\[ \beta = \frac{u}{c} \]

with Relativistic Gamma \( \gamma = 1 / \sqrt{1 - \beta^2} \), \( \beta = \frac{u}{c} \)

The derivation shows that the ProperTime Derivative is an Invariant Lorentz Scalar. Therefore, all observers must agree on its magnitude, regardless of their frame-of-reference.

It can be used to make new 4-Vectors from existing 4-Vectors, as it is taking the derivative of an existing 4-Vector by a Lorentz Scalar: the ProperTime \( \tau \).
SRQM Diagram:
The Basis of Classical SR Physics

ProperTime Derivative on SR 4-Vectors and Scalars

The ProperTime Derivative acting on SR 4-Vectors:

\[ U \cdot \partial = \gamma(c, u) \cdot (\partial/c, -\nabla) = \gamma(\partial + u \cdot \nabla) = \gamma \frac{d}{dt} = \frac{d}{dt} \]

4-Vectors:
- 4-Position \( R \rightarrow \text{<Event>}, R = (ct, \mathbf{r}) \)
- 4-Velocity \( U = dU/dt \)
- 4-Acceleration \( A = dU/dt \)
- 4-Momentum \( P = m_u U \)
- 4-Force \( F = dP/dt \)

As one can see from the list, the ProperTime Derivative gives the 4-Vectors that are the change in status of the 4-Vector that ProperTime Derivative acts on. It can also act on Scalar Values to give deep SR results.

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SRQM Diagram

4-Vector SRQM Interpretation of QM

The Basis of Classical SR Physics

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SR → QM

A Tensor Study of Physical 4-Vectors

There are several ways to derive Time Dilation.

ProperTime Derivative (Lorentz 4-Scalar):

\[ U \cdot \partial t = c \]

ProperTime Differential (Lorentz 4-Scalar): \( \frac{dR}{d\tau} = \frac{d}{d\tau} R \cdot R = c \)

There are several ways to derive Time Dilation.

ProperTime Derivative (Lorentz 4-Scalar):

\[ U \cdot \partial t = c \]

ProperTime Differential (Lorentz 4-Scalar): \( \frac{dR}{d\tau} = \frac{d}{d\tau} R \cdot R = c \)

The coordinate time \( \Delta t \) measured by an observer is "dilated", compared to the ProperTime as measured by a clock moving with the object. This has the effect that moving objects appear to age more slowly than at-rest objects. The effect is reciprocal as well. Since velocity is relative, each observer will see the other as ageing more slowly, similarly to the effect that each will appear smaller to the other when seen at a distance.

Now multiply both sides by the moving-frame speed \([v]\):

\[ v \Delta t = \gamma v \Delta t \]

\[ v \Delta t = \text{distance } L, \text{ the moving clock travels wrt. frame, which is a proper (fixed-to-frame) displacement length.} \]

\[ L = \frac{\gamma L}{(1/v) L} = \text{Length Contraction!} \]

SR 4-Tensor

\[ 2,0) - \text{Tensor } T^{\mu}_{\nu} \]

(1,1) - Tensor \( T^\nu_{\nu} \), or \( T^\nu_{\nu} \)

(0,2) - Tensor \( T^{\mu}_{\nu \mu} \)

SR 4-Vector

\[ (1,0) - \text{Vector } \mathbf{V} = (\mathbf{V}^\nu, \mathbf{V}) \]

SR 4-Scalar

\[ (0,0) - \text{Tensor } s \]

Lorentz Scalar

SR 4-Vector

\[ \mathbf{V} = (\mathbf{V}^\nu, \mathbf{V}) \]

SR 4-Scalar

\[ s = (s^\nu, s) \]

SR 4-Vector

\[ s = (s^\nu, s) \]

SR 4-Scalar

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SR 4-Vector

\[ \mathbf{V} = (\mathbf{V}^\nu, \mathbf{V}) \]

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\[ s = (s^\nu, s) \]

SR 4-Vector

\[ \mathbf{V} = (\mathbf{V}^\nu, \mathbf{V}) \]

SR 4-Scalar

\[ s = (s^\nu, s) \]
The 4-Gradient ($\partial^\mu$) is the index-raised version of the SR Gradient One-Form ($\partial_\mu$). It is a 4D version of the partial derivative function of calculus.

It is a 4-Vector function that can act on other 4-Vectors and 4-Scalars. The 4-Gradient tells how things change wrt. time and space.

It is instrumental in creating the ProperTime Derivative $U \cdot \partial = \gamma d\nu/d\tau$.

The 4-Gradient plays a major role in advanced physics, showing how SR waves are formed, creating the Hamilton-Jacobi equations, the Euler-Lagrange equations, Conservation equations, Maxwell’s Equations, the Lorentz Transformations, and the Lorentz Transformations. In QM, it provides the Schrödinger relations.

It is fundamental in connecting SR to QM.
The Lorentz Scalar Invariant of the 4-Gradient gives the Invariant d’Alembertian Wave Equation, describing SR wave motion. It is seen in the SR Maxwell Equation for EM light waves.

It provides for the introduction of an SR Wave 4-Vector, it will be seen again in the Klein-Gordon QM equation.

It is seen in the SR Maxwell Equation for EM light waves.

The Lorentz Scalar Invariant of the 4-Gradient gives the
Invariant d’Alembertian Wave Equation, describing SR wave motion.

4-Vector SRQM Interpretation of QM

SRQM Diagram:
The Basis of Classical SR Physics
Invariant d’Alembertian Wave Equation (\(\partial \cdot \partial\))
Conservation of Charge: by the flow of that quantity into or out of a local region. These are local continuity equations which basically say that the temporal change in a quantity is balanced.

All of the Physical Conservation Laws are in the form of equation.

This leads to all the SR Conservation Laws. Continuity of 4-Velocity Flow \( \partial \cdot U = 0 \)

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local region.

Conservation of Charge: \( \partial \cdot U = 0 \)

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local region.
The Basis of Classical SR Physics

<Event> Substantiation

Now focus on six of the main SR 4-Vectors.

- **4-Position** \( R = (ct, r) \)
- **4-Velocity** \( U = \gamma (c, u) \)
- **4-Gradient** \( \partial = (\partial / c \partial t, -\partial / \partial x, -\partial / \partial y, -\partial / \partial z) \)
- **4-Momentum** \( P = (E/c, p) = (mc, p) = (mc, mu) \)
- **4-WaveVector** \( K = (\omega / c, k) = (\omega / c, \omega \hat{n} / \nu \text{phase}) \)
- **4-CurrentDensity:ChargeFlux** \( J = (pc, j) = (pc, pu) \)
- **4-(Dust)NumberFlux** \( N = (nc, n) = (nc, nu) \)

These seven give more of the main classical results of Special Relativity, including SR concepts like:

- SR Particles and Waves, Matter-Wave Dispersion
- Einstein's \( E = mc^2 = \gamma m_o c^2 = \gamma E_o \), Rest Mass, Rest Energy
- Conservation of Charge (Q), Conservation of Particle Number (N), Continuity Equations

SR Particles and Waves:

- **SRQM 4-Tensor** \( T^{\mu\nu} \)
- **SRQM 4-Vector** \( V^{\mu} = V = (\hat{v}, v) \)
- **SR 4-CoVector** \( V_{\nu} = (v_0, -v) \)
- **SR 4-Scalar** \( S \)
- Lorentz Scalar

\[ \text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T^{\mu\mu} = T \]
\[ V \cdot V = V^{\mu} \eta_{\mu\nu} V^{\nu} = [(v^0)^2 - v \cdot v] = (v^0)^2 \]

Lorentz Scalar
SRQM Diagram:
The Basis of Classical SR Physics
4-Momentum, Einstein’s $E = mc^2$

4-Position $R = (ct, r)$
4-Gradient $\partial = (\partial / c, -V)$
4-Velocity $U = \gamma (c, u)$

4-Momentum $P = (E/c, p) = m_c U = \gamma m_o (c, u) = m (c, u)$

**Temporal part:**
- $E = \gamma E_0 = \gamma m_c c^2 = mc^2$
- $E = m_c c^2 + (\gamma - 1)m_o c^2$
  (rest) + (kinetic)

**Spatial part:**
- $p = \gamma m_o u = mu$

4-Momentum $P = (E/c, p) = -\partial [S_{\text{action,free}}] = -\partial / (\partial c, -V) [S_{\text{action,free}}]$

**Temporal part:**
- $E = -\partial [S_{\text{action,free}}]$  

**Spatial part:**
- $p = +\nabla [S_{\text{action,free}}]$

Einstein’s Equivalence Rest Mass Energy/Mass
$E = E_0 = m_o c^2$

$E^2 = (|p|c)^2 + (m_c c^2)^2$

Relativistic Energy $E$: Mass $m_o$ vs Invariant Rest Energy $E_0$:
$E = (|p|c)^2 + (E_0)^2$  

$P \cdot P = (E/c)^2 - (p \cdot p) = (m_c c)^2$

Hamilton-Jacobi Equation
$P_\tau = -\partial [S_{\text{action,free}}]$

$P \cdot P = (m_c c)^2$

$P = (E/c, p)$

$P = (E/c, p) = (mc, p)$

ProperTime Derivative
$U \cdot \partial = -\gamma d/d\tau = d/d\tau$

4-Displacement
$\Delta R = (c \Delta t, \Delta r)$
$
\frac{dR}{d\tau} = (c dt, dr)$

$\Delta (ct, r) = (c \Delta t, \Delta r)$

$\Delta = \int (E_0 c)\,d\tau$

$S_{\text{act}} = -\int (m_o c^2 + V)\,d\tau$

for a free particle

$S_{\text{act}} = -\int (m_o c^2 + V)\,d\tau$

in a potential

which matches:

$S_{\text{act}} = -\int (E_0 + V)\,d\tau$

Trace $T^{\mu \nu} = \eta_{\mu \nu} T^{\mu \nu} = T_{\mu}^{\nu} = T$

$V \cdot V = V^\nu (\eta_{\nu \nu} V^\nu) = (\langle v^2 \rangle - v \cdot v) = (v^0)^2$

$= \text{Lorentz Scalar}$
SRQM Diagram: The Basis of Classical SR Physics

**4-WaveVector, \( u \ast v_{\text{phase}} = c^2 \)**

4-Position \( R = (ct, r) \)

4-Gradient \( \partial = (\partial/c, -\nabla) \)

4-Velocity \( U = \gamma(c, u) \)

Temporal part:

\[ \omega = \gamma \omega_0 \]

Spatial part:

\[ k = \gamma(\omega/c^2)u = (\omega/c^2)u = \omega \hat{n}/v_{\text{phase}} \]

4-WaveVector \( K = (\omega/c, k) = (\omega/c^2)U = \gamma(\omega/c^2)(c, u) \)

SRQM Diagram:

Wave Phase Equation

\[ K \cdot K = (\omega/c)^2 - (k \cdot k) = (\omega/c)^2 \]

\[ \omega^2 = (|k|c)^2 + (\omega_0)^2 : \text{Matter-Wave Dispersion Relation} \]

Relativistic AngFreq(\( \omega_0 \)) vs Invariant Rest AngFreq(\( \omega_0 \))
SRQM Diagram: The Basis of Classical SR Physics
4-CurrentDensity, Charge Conservation

4-Position \( R=(ct,r) \)
4-Gradient \( \partial=\left(\partial/c,-\nabla\right) \)
4-Velocity \( U=\gamma(c,u) \)

4-CurrentDensity \( J=(pc,j)=\rho_o U=\gamma\rho_o(c,u)=\rho(c,u) \)
4-ChargeFlux \( J \)

Temporal part: \( \rho = \gamma\rho_o \) {charge-density}

Spatial part: \( j = \gamma\rho_o u = \rho u \) {3-current-density}

Conservation of Charge \( (Q) \)
\[
\partial\cdot J = (\partial/c,-\nabla)\cdot(pc,j) = (\partial\rho + \nabla\cdot j) = 0
\]

Continuity Equation: Noether's Theorem
The temporal change in charge density is balanced by the spatial change in current density. Charge is neither created nor destroyed. It just moves around as charge currents...

Relativistic ChargeDensity(\( \rho \)) vs Invariant Rest ChargeDensity(\( \rho_o \))

\[
(J\cdot J) = (pc)^2 - (j\cdot j) = (\rho_o c)^2
\]
\[
\rho^2 = (\|j\|/c)^2 + (\rho_o)^2
\]
Conservation of Particle # (N)
\[ \partial \cdot N = (\partial t/c, \nabla) \cdot (nc, n) = (\partial n + \nabla \cdot n) = 0 \]

Continuity Equation: Noether's Theorem
The temporal change in number density is balanced by the spatial change in number-flux.
Particle # is neither created nor destroyed. It just moves around as number currents...

\[ N = \int n \, d^3x = \int \gamma n_o \, d^3x \]
\[ \rightarrow n_o V_o \]
\[ \int dT \cdot N = -cn/V_o \]

Relativistic Number Density (n) vs Invariant Rest Number Density (n_o)
\[ B = (nc)^2 - (n \cdot n) = (n_o c)^2 \]
\[ n^2 = (|n|/c)^2 + (n_o)^2 \]

SRQM Diagram: The Basis of Classical SR Physics
4-(Dust)NumberFlux, Particle # Conservation

Temporal part:
\[ n = \gamma n_o \]
{number-density}

Spatial part:
\[ n = \gamma n_o u = nu \]
{3-number-flux}

\[ 4-\text{Position} \quad R = (ct, r) \]
\[ 4-\text{Gradient} \quad \partial = (\partial t/c, -\nabla) \]
\[ 4-\text{Velocity} \quad U = \gamma(c, u) \]
\[ 4-\text{NumberFlux} \quad N = (nc, n) = n_o U = \gamma n_o (c, u) = n(c, u) \]

\[ 4-\text{Displacement} \quad \Delta R = (c\Delta t, \Delta r) \]
\[ 4-\text{NumberFlux} \quad N = (nc, n) = (nc, nu) \]

\[ 4-\text{Gradient} \quad \partial = (\partial t/c, -\nabla) \]
\[ = (\partial t/c, -\partial x, -\partial y, -\partial z) \]
\[ = (\partial /c \partial t, \partial /\partial x, \partial /\partial y, \partial /\partial z) \]
Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]$ (Continuous) vs (Discrete) (Proper Det=+1) vs (Improper Det=-1)

The main idea that makes a generic 4-Vector into an SR 4-Vector is that it must transform correctly according to an SR Lorentz Transformation $\{ \Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}] \}$, which is basically any linear, unitary or antiunitary, transform (Determinant[$\Lambda^{\mu'}_{\nu}$] = ±1) which leaves the Invariant Interval unchanged.

The SR continuous transforms (variable with some parameter) have $\{ \text{Det} = +1, \text{Proper} \}$ and include:
- "Rotation" (a mixing of space-space coordinates) and "Boost" (a mixing of time-space coordinates).
- The SR discrete transforms can be $\{ \text{Det} = +1, \text{Proper} \}$ or $\{ \text{Det} = -1, \text{Improper} \}$ and include:
  - "Space Parity-Inversion" (reversal of the spatial coordinates), "Time-Reversal" (reversal of the temporal coordinate).
  - The "Identity" (no change), and various single dimension Flips and their combinations.

Typical Lorentz Boost Transformation:

For a linear-velocity frame-shift (x,t)-Boost in the $\hat{x}$-direction:

$A' = (a^1, a^2, a^3, a^0)$

$A' = (a^1, a^0, a^y, a^x)$

$= B^0 A^\gamma$

$= (\gamma a^0 - \gamma \beta a^x, -\gamma \beta a^y + \gamma a^x, a^0, a^x)$

(for $\gamma |\beta| < 1$ transformation)

Lorentz Parity-Inversion Transformation:

$A' = (a^1, a^2, a^3, a^0)$

$A' = (a^1, a^0, a^y, a^x)$

$= P^0 A^\gamma$

$= (a^1, -a^2, -a^3, -a^0)$

(for Parity Inverse Lorentz Transform)

Continuous: ex. Boost depends on variable parameter $\beta$, with $\gamma = 1/\sqrt{1-\beta^2}$

Boosted 4-Vector $A' = A' = (a^0, a^1, a^2, a^3)$

ex. for $\gamma x$-boost

$\rightarrow (\gamma a^0 - \gamma \beta a^x, -\gamma \beta a^y + \gamma a^0, a^x, a^y)$

Det[$B^\mu_{\nu}$] = +1, Proper

$\gamma^x - \beta^2 \gamma = +1$

Proper: preserves orientation of basis

Discrete: ex. Parity has no variable parameters

Parity-Inversed 4-Vector $A' = A' = (a^0, a^1, a^2, a^3)$

$\rightarrow (a^0, -a^1, -a^2, -a^3)$

Det[$P^\mu_{\nu}$] = -1, Improper

$(\gamma - 1)^2 = -1$

Improper: reverses orientation of basis

---

SR 4-Tensor:

(2,0)-Tensor $T^{\mu\nu}$ = Lorentz Scalar (0,0)-Tensor $S$

(1,1)-Tensor $P^\mu_{\nu}$ = Lorentz Parity-Inversion Transformation

SR 4-Vector:

(2,0)-Tensor $V^\mu$ = Lorentz Scalar (0,0)-Tensor $S$

(1,1)-Tensor $P^\mu_{\nu}$ = Lorentz Parity-Inversion Transformation
Lorentz Transforms $\Lambda_{\nu}^{\mu'} = \partial_{\nu}[X_{\mu'}^{\nu}]$

Proper Lorentz Transforms ($\text{Det}=+1$): Continuous: (Boost) vs (Rotation)

Typical Lorentz Boost Transform (symmetric): for a linear-velocity frame-shift $(x,t)$-Boost in the $\hat{x}$-direction:

$$A' = (a', a', a', a^2) = R_{\nu}^{\mu'}A' = (\gamma a' - \beta a', \gamma a', a', a')$$

Typical Lorentz Rotation Transform (non-symmetric): for an angular-displacement frame-shift $(x,y)$-Rotation about the $\hat{z}$-direction:

$$A' = (a', a', a', a^2) = R_{\nu}^{\mu'}A' = (a', \cos[\theta]a' - \sin[\theta]a', \sin[\theta]a' + \cos[\theta]a', a')$$

Lorentz Boost Transform:

$$\Lambda_{\nu}^{\mu'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = e^{\Lambda[\beta \gamma]}$$

Lorentz Rotation Transform:

$$\Lambda_{\nu}^{\mu'} = \begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 & 0 \\ \sin[\theta] & \cos[\theta] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e^{\Lambda[\theta]}$$

Trace $[\Lambda_{\nu}^{\mu'}] = \Lambda_{\nu}^{\mu'} = \sum_{\nu} \Lambda_{\nu}^{\mu'} = 4$
Lorentz Transforms $\Lambda_{\mu'}^\nu = \partial_\nu [X_{\mu'}^\nu]$

Proper Lorentz Transforms (Det=+1): (Boost) vs (Rotation) vs (Identity)

General Lorentz Boost Transform (symmetric, continuous):
for a linear-velocity frame-shift (Boost) in the $\nu=c=\beta(\beta',\beta)\text{-direction}$:

$$\Lambda_{\mu'}^\nu = \frac{1}{\sqrt{1 - \beta^2}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta_0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

General Lorentz Rotation Transform (non-symmetric, continuous):
for an angular-displacement frame-shift (Rotation) angle $\theta$ about the $\hat{n}=(n_1,n_2,n_3)$-direction:

$$\Lambda_{\mu'}^\nu = \frac{1}{\sqrt{1 - \beta^2}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz Identity Transform (symmetric, "discrete, continuous"):
for a non-frame-shift (Identity) in any direction

$$\Lambda_{\mu'}^\nu = \eta_{\mu'\nu} = \delta_{\mu'\nu} = \text{Diag}[1,\delta] = I_{4\times4}$$

**SR:Lorentz Transform**

$$\Lambda_{\mu'}^\nu = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$$\Lambda_{\mu'}^\nu = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$\beta = v/c$: dimensionless Velocity Beta Factor \{ $\beta=(0..1)$, with speed-of-light (c) at $\beta=1$ \}

$\gamma = 1/\sqrt{1-\beta^2}$: dimensionless Lorentz Relativistic Gamma Factor \{ $\gamma=(1..\infty)$ \}

Identity transformation for zero relative motion/rotation: $B[0]=R[0]=I_{4\times4}$

Proper Transformation = positive unit determinant: $\det[B]=\det[R]=\det[\eta]=+1$

Inverses: $B(v)^{-1} = B(-v)$ (relative motion in the opposite direction), and $R(\theta)^{-1} = R(-\theta)$ (rotation in the opposite sense about the same axis)

Matrix symmetry: B is symmetric (equals transpose, $B=B^T$), while R is nonsymmetric but orthogonal (transpose equals inverse, $R^T = R^{-1}$)

**4-Vector SRQM Interpretation of QM**

$TR \rightarrow QM$

**A Tensor Study of Physical 4-Vectors**

John B. Wilson

SciRealm.org

4-Tensor SRQM Interpretation of QM

$V \times V = V^2 \eta_{\mu\nu} V^\nu = ([v^\mu] - v^\mu v^\nu) = (v^\nu v^\lambda)$

$= \text{Lorentz Scalar}$
Lorentz Transforms $\Lambda_{\mu'}^\nu = \partial_\nu [X_{\mu'}]$  
**Discrete (non-continuous)**  
(Parity-Inversion) vs (Time-Reversal) vs (Identity)

**General Lorentz Parity-Inversion Transform:**  
$\Lambda_{\mu'}^\nu \rightarrow P_{\mu'}^\nu$ (Improper, symmetric, discrete)  
$\Lambda_{\mu'}^\nu = \begin{bmatrix} 1 & 0 \\ 0^t & -\delta^t \end{bmatrix}$

**General Lorentz Time-Reversal Transform:**  
$\Lambda_{\mu'}^\nu \rightarrow T_{\mu'}^\nu$ (Improper, symmetric, discrete)  
$\Lambda_{\mu'}^\nu = \begin{bmatrix} 1 & 0 \\ 0^t & \delta^t \end{bmatrix}$

**General Lorentz Identity Transform:**  
$\Lambda_{\mu'}^\nu \rightarrow I_{\mu'}^\nu$ (Proper, symmetric, discrete)  
$\Lambda_{\mu'}^\nu = \begin{bmatrix} 1 & 0 \\ 0^t & \delta^t \end{bmatrix}$

The Trace of Discrete Lorentz Transforms  
$\text{Trace}[P_{\mu'}^\nu] = \eta_{\nu\delta} \delta^\nu_{\mu'} = -1$

$\text{Trace}[T_{\mu'}^\nu] = \eta_{\nu\delta} \delta^\nu_{\mu'} = +1$

$\text{Trace}[I_{\mu'}^\nu] = \eta_{\nu\delta} \delta^\nu_{\mu'} = +1$

Note that the Trace of Discrete Lorentz Transforms goes in steps from $-1, +1$. As we will see in a bit, this is a major hint for SR antimatter.
Lorentz Transforms $\Lambda^\mu_\nu = \partial_\nu [X^\mu]$  

Discrete & Fixed Rotation → Particle Exchange  

Lorentz Coordinate-Flip Transforms 

SR: Lorentz Transform 

$\partial_\nu [X^\mu] = \partial R^\nu_\mu' = \Lambda^\mu_\nu$  

$L^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\alpha_\beta, \Lambda^\nu_\alpha = \eta^\mu_\nu = \delta^\mu_\nu$  

$\eta_{\alpha\beta} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$  

$\text{Det}[\Lambda^\mu_\nu] = \pm 1$  

$\Lambda^\mu_\nu \Lambda^\nu_\mu = 4$  

Any single Lorentz Flip Transform is Improper, with a Determinant of -1. However, pairwise combinations are Proper, with a Determinant of +1.  

The combination of any two Spatial Flips is the equivalent of a Spatial Rotation by (π) about the associated rotational axis. Since this is a Proper transform, it is also the equivalent of a particle location exchange.  

The combination of all three Spatial Flips, Flip-xyz, gives the Lorentz Parity Transform, which is again Improper.  

The Flip-t is the standard Lorentz Time-Reversal, Improper.  

Proof: 

$\text{Tr}[R^\mu_\nu] = 2 + 2 \cos[\theta] = \{0, +1\}$  

$\text{Det}[R^\mu_\nu] = \cos[\theta]^2 + \sin[\theta]^2 = +1$
Lorentz Transforms $\Lambda^\mu_\nu = \partial_\nu [X^\mu]$

Lorentz Transform Connection Map

SR: Lorentz Transform
\[
\delta_i [R^\mu_i] = \partial R^\mu_i / \partial R^\nu_i = \Lambda^\mu_\nu
\]

Identity $I_{(4)}$
\[
\Lambda^\mu_\nu \rightarrow I^\mu_\nu = \delta^\mu_\nu = B^\mu_i [0] = R^\mu_i [0] = R^\mu_i [2\pi] \text{ no mixing unitary}
\]

Rotation $z$
\[
\Lambda^\mu_\nu \rightarrow R^\mu_{\nu[\pi/2]} = \text{Flip-xy} = F_{xy}^\mu_\nu \text{ x:y = unitary}
\]

Rotation $z$
\[
\Lambda^\mu_\nu \rightarrow R^\mu_{\nu[\pi]} = \text{Flip-xy} = F_{xy}^\mu_\nu \text{ x:y = unitary}
\]

Parity-Inversion
\[
\Lambda^\mu_\nu \rightarrow P^\mu_\nu = -\text{space parity unitary}
\]

Neg Identity $-I_{(4)}$
\[
\Lambda^\mu_\nu \rightarrow -I^\mu_\nu = -\delta^\mu_\nu = \eta^\mu_\nu = \delta^\nu_\mu
\]

Charge-Conjugation
\[
\Lambda^\mu_\nu \rightarrow C^\mu_\nu
\]

Separate Set of Boosts & Rotations

By CPT Symmetry, this should be equivalent to the regular Positive Identity $I_{(4)}$

Feynman-Stueckelberg

John B. Wilson
SciRealm.org
Examine all possible combinations of Discrete Lorentz Transformations which are Linear (Determinant of \(\pm 1\)).

A lot of the standard SR texts only mention (P)arity-Inverse and (T)ime-Reversal. However, there are many others, including (F)lips and (R)otations of a fixed amount. However, the (T)imeReversal and Combo(P)arity(T)ime take one into a separate section of the chart. Taking into account all possible discrete Lorentz Transformations fills in the rest of the chart. The resulting interpretation is that there is CPT Symmetry (Charge:Parity:Time) and Dual TimeSpace (with reversed timeflow). In other words, one can go from the Identity Transform (all +1) to the Negative Identity Transform (all -1) by doing a Combo PT Lorentz Transform or by Negating the Charge (Matter→Antimatter). The Feynman-Stueckelberg Interpretation aligns with this as the AntiMatter Side.

This is similar to Dirac’s prediction of AntiMatter, but without the formal need of Quantum Mechanics, or Spin. In fact, it is more general than Dirac’s work, which was about the electron. This is from general Lorentz Transforms for any kind of particle.
Lorentz Transforms \( \Lambda_{\mu' \nu} = \partial_{\nu} [X^{\mu}] \)

Lorentz Transform Connection Map – Trace Identification

CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

All Lorentz Transforms have Tensor Invariants: Determinant of \( \pm 1 \) and Inner Product of 4. However, one can use the Tensor Invariant Trace to identify CPT Symmetry

\[
\text{Trace} : \text{Determinant} \\
\text{Tr} [\text{NM-Rotate}] = \{0...+4\} \quad \text{Tr}[\text{NM-Identity}] = +4 \quad \text{Tr}[\text{NM-Boost}] = \{+4...+\infty\} \\
\text{Tr} [\text{AM-Rotate}] = \{0...-4\} \quad \text{Tr}[\text{AM-Identity}] = -4 \quad \text{Tr}[\text{AM-Boost}] = \{-4...-\infty\}
\]

Two interesting properties of (1,1)-Tensors, of which the Lorentz Transform is an example:

\[
\text{Trace} = \text{Sum} (\Sigma) \text{ of EigenValues} : \text{Determinant} = \text{Product (\Pi) of EigenValues} \\
\Sigma (\text{EV's}) = \Sigma (\text{EV's}) \quad \Pi (\text{EV's}) = \Pi (\text{EV's}) \quad \Sigma (\text{EV's}) = \Sigma (\text{EV's}) \quad \Pi (\text{EV's}) = \Pi (\text{EV's})
\]

As Rank 4 Tensors, each Lorentz Transform has 4 EigenValues (EV’s). Create an Anti-Transform which has all EigenValue Tensor Invariants negated.

\[
\Sigma [-\text{EV's}] = -\Sigma (\text{EV's}) \quad \Pi [-\text{EV's}] = \Pi (\text{EV's}) \quad \Sigma [-\text{EV's}] = \Sigma (\text{EV's}) \quad \Pi [-\text{EV's}] = \Pi (\text{EV's})
\]

The Trace Invariant identifies a “Dual” Negative-Side for all Lorentz Transforms.
Based on the Lorentz Transform properties of the last few pages, here is an interesting observation about Lorentz Transforms: They all have a Determinant of ±1, and an Inner Product of 4, but the Trace varies depending on the particular Transform.

The Trace of the Identity is at 4. Assume this applies to normal matter particles.
The Trace of normal matter particle Rotations varies from (0..4)
The Trace of the normal matter particle Boosts varies from (4..Infinity)
So, one can think of Trace = 4 being the connection point between normal matter Rotations and Boosts.

Now, various Flip Transforms (inc. the Time Reversal and Parity Transforms, and their combination as PT transform), take the Trace in steps from (-4,-2,0,+2,+4). Applying a bit of symmetry:

The Trace of the Negative Identity is at -4. Assume this applies to anti-matter particles.
The Trace of anti-matter particle Rotations varies from (0..-4)
The Trace of the anti-matter particle Boosts varies from (-4..-Infinity)
So, one can think of Trace = -4 being the connection point between anti-matter Rotations and Boosts.

This observation would be in agreement with the CPT Theorem (Feynman-Stueckelberg) idea that normal matter particles moving backward in time are CPT symmetrically equivalent to anti-matter particles moving forward in time.

Now, scale this up to Universe size: The Baryon Asymmetry problem (aka. The Matter-AntiMatter Asymmetry Problem). If the Universe was created as a huge chunk of energy, and matter-creating energy is always transformed into matter-antimatter mirrored pairs, then where is all the antimatter???

Answer: It is temporally on the “Other/Dual side" of the Big-Bang! The antimatter created at the Big-Bang is travelling in the negative time (-t) direction from the Big-Bang creation point, and the normal matter is travelling in the positive time direction (+t).

Universal CPT Symmetry. So, what happened “before" the Big-Bang? It "is" the AntiMatter Dual to our normal matter universe! Pair-production is creation of AM-NM mirrored pairs within SpaceTime. The Big-Bang is the creation of SpaceTime itself.

This also resolves the Arrow-of-Time Problem. If all known physical microscopic processes are time-symmetric, why is the flow of Time experienced as uni-directional??? (see Wikipedia “CPT Symmetry"; "CP Violation"; "Andrei Sakharov")

Answer: Time flow on this side of the Universe is in the (+t) direction, while time flow on the dual side of the Universe is in the (-t) direction. The math all works out. Time flow is bi-directional, but on opposite sides of the Big-Bang! Universal CPT Symmetry.

This gives total CPT Symmetry to all of the possible Lorentz Transforms (AM=Antimatter, NM=Normal Matter):

This solves the:
Baryon (Matter-AntiMatter) Asymmetry Problem & Arrow(s)-of-Time Problem (+ / -)
This idea of Universal CPT Symmetry also gives a Universal Dimensional Symmetry as well.

Consider the well known “balloon” analogy of the universe expansion. The “spatial” coordinates are on the surface of the balloon, and the expansion is in the +t direction. There is symmetry in the +/- directions of the spatial coordinates, but the time flow is always uni-directional, +t, as the balloon gets bigger.

By allowing a “dual side”, it provides a universal dimensional symmetry. One now has +/- symmetry for the temporal directions. The “center” of the Universe is literally, the Big Bang Singularity. It is the “center=zero” point of both time and space directions.

The expansion gives time flow away from Big Bang singularity in both the Normal Side (+) and the Dual “Side (−). The spatial coordinates expand in both the (+/-) directions on both sides. Note that this gives an unusual interpretation of what came “before” the Big Bang. The “past” on either side extends only to the BB singularity, not beyond. Time flow is always away from this creation singularity.

This is also in accord with known black hole physics, in that all matter entering a BH ends at the BH singularity. Time and space coordinates both come to a stop at either type of singularity, from the point of view of an observer that is in the spacetime but not at the singularity.

So, the Big Bang is a “starting” singularity, and black holes are “ending” singularities. Also provides for idea of “white holes” actually just being black holes on the alternate side. White hole=time-reversed black hole. This way, the mass is still attractive. Time flow is simply reversed on the alternate side so stuff still goes into the hole...

So, Universal CPT Symmetry = Universal Dimensional Symmetry.

And, going even further, I suspect this is the reason there is a duality in Metric conventions. In other words, physicists have wondered why one can use {+,−,−,−} or {−,+,+,−}. I submit that one of these metrics applies to the Normal Matter side, while the other complementarily applies to the Dual side. This would allow correct causality conditions to apply on either side.

Again, this is similar to the Dirac prediction of antimatter based on a duality of possible solutions.

This gives total CPT Symmetry to all of the possible Lorentz Transforms (AM=Antimatter, NM=Normal Matter):

- Various (AM_Flips) : Various (NM_Flips)
- Infinity... (AM_Boosts) ... (AM_Identity=−4)... (AM_Rotations)... 0... (NM_Rotations)... (NM_Identity) ... (NM_Boosts)... +Infinity

This solves the: Baryon (Matter-AntiMatter) Asymmetry Problem & Arrow(s)-of-Time Problem (+ / −)
SRQM Study: Model SpaceTimes

### Lie Groups

**de Sitter Group SO(1,4)**
- de Sitter invariant relativity (maybe?)

**Poincaré Group ISO(1,3)**
- \( \{ r \ll r_{dS} = \text{de Sitter Radius}\} \)
- \( r_{dS} = \sqrt{3/\Lambda} = L_{\mu}/\sqrt{\Omega_{\Lambda}} \)

**SR & GR Physics**
- (** currently thought correct **) 

\( \Lambda^\mu_{\nu} \rightarrow B^\nu_{\nu} = \text{Boost} \)

\( \Lambda^\mu_{\nu} \rightarrow S^\nu_{\nu} = \text{Motion:Shear} \)

### Model SpaceTimes

<table>
<thead>
<tr>
<th>Model SpaceTimes</th>
<th>( \Lambda &lt; 0 )</th>
<th>( \Lambda = 0 )</th>
<th>( \Lambda &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Klein Geometry G/H</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lorentzian pseudo-Riemann</td>
<td>Anti de Sitter SO(3,2)/SO(3,1)</td>
<td>Minkowski ISO(3,1)/SO(3,1)</td>
<td>De Sitter SO(4,1)/SO(3,1)</td>
</tr>
<tr>
<td></td>
<td>( ds^2 = (cdt)^2 - dx \cdot dx )</td>
<td>( ds^2 = (cdt)^2 + dx \cdot dx )</td>
<td></td>
</tr>
<tr>
<td>Riemannian</td>
<td>Hyperbolic SO(4,1)/SO(4)</td>
<td>Euclidean ISO(4)/SO(4)</td>
<td>Spherical SO(5)/SO(4)</td>
</tr>
</tbody>
</table>

\( ds^2 = (cdt)^2 - dx \cdot dx \)

\( ds^2 = (cdt)^2 + dx \cdot dx \)

\( ds^2 = (cdt)^2 + dx \cdot dx \)
SRQM Transforms: Venn Diagram
Poincaré = Lorentz + Translations

(10) (6) (4)

Transformations
(# of independent parameters = # continuous symmetries = # Lie Dimensions)

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation
Lie group of all affine isometries of SR:Minkowski TimeSpace (preserve quadratic form $\eta_{\mu\nu}$)
General Linear,Affine Transform $X' = \Lambda^\mu_\nu X^\nu + \Delta X^\mu$ with $\text{Det}[\Lambda^\mu_\nu] = \pm 1$

(6+4=10)

SRQM Transforms: Venn Diagram

4-Vectors SRQM Interpretation of QM
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Lorentz Transform $\Lambda^\mu_\nu$
4-Tensor {mixed type-(1,1)}

<table>
<thead>
<tr>
<th>Discrete</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time-reversal $T^\mu_v$</td>
<td>$T^\mu_v$</td>
</tr>
<tr>
<td>SpatialFlipCombos $F^\mu_v$</td>
<td>$F^\mu_v$</td>
</tr>
<tr>
<td>Charge-Conjugation $C^\mu_v$</td>
<td>$C^\mu_v$</td>
</tr>
<tr>
<td>Parity-Inversion $P^\mu_v$</td>
<td>$P^\mu_v$</td>
</tr>
<tr>
<td>Identity $I_{(4)}$</td>
<td>$I_{(4)}$</td>
</tr>
<tr>
<td>Rotation $R^\mu_v$</td>
<td>$R^\mu_v$</td>
</tr>
<tr>
<td>Boost $B^\mu_v$</td>
<td>$B^\mu_v$</td>
</tr>
<tr>
<td>Isotropy {same all directions}</td>
<td>Isotropy {same all directions}</td>
</tr>
</tbody>
</table>

Translation Transform $\Delta X^\mu$
4-Vectors {same all directions}

<table>
<thead>
<tr>
<th>Discrete</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temporal $\Delta^\mu$</td>
<td>$\Delta^\mu$</td>
</tr>
<tr>
<td>Spatial $\Delta X^\mu$</td>
<td>$\Delta X^\mu$</td>
</tr>
</tbody>
</table>

Amusingly, Inhomogeneous Lorentz adds homogeneity.

Lorentz Matrices can be generated by a matrix $M$ with $\text{Tr}[M]=0$ which gives:

$\Lambda = e^M$

Rotations $J_i$ are

Feynman-Stueckelberg Interpretation

Minkowski TimeSpace (Minkowski TimeSpace)
Classical Transforms: Venn Diagram

Full Galilean = Galilean + Translations

**Lie Groups**

- **de Sitter Group** $SO(1,4)$
  - de Sitter invariant relativity (maybe?)

- **Poincaré Group** $ISO(1,3)$
  - $r << r_{ds} = \text{de Sitter Radius}$
  - $r_{ds} = \sqrt{\frac{3}{\Lambda}} = L_H/\sqrt{\Omega_\Lambda}$

**SR & GR Physics**

- (** currently thought correct **)
Review of SR Transforms

10 Poincaré Symmetries, 10 Conservation Laws
10 Generators: Noether’s Theorem

A Tensor Study of Physical 4-Vectors

4-Displacement
\[ \Delta X = (c \Delta t, \Delta x) \]

4-Velocity
\[ U = \gamma (c, \mathbf{u}) = \frac{dX}{d\tau} \]

4-AngMomentum Tensor
\[ M^{\mu\nu} = X^i P^\mu - X^\mu P^i = X^\mu P^i \]

4-Gradient
\[ \partial = (\partial_i, \nabla) \]

Minkowski Metric
\[ \partial [X] = \partial [X'] = \eta_{\mu\nu} \]

Conservation of linear 3-momentum (spatial)
\[ \partial_\mu \mathbf{P} = 0 \]

Generated by angular-momentum 3-vector

Lorentz General Time-Space Transform
\[ \Lambda^\nu_\nu \rightarrow B^\nu_\nu = \exp[\gamma \omega_\nu - \frac{1}{2} \beta_\nu \beta^\nu \delta^\nu_\nu + \frac{1}{8} \gamma^3 \beta_\nu \beta^\nu \beta^\mu \beta^\mu] \]

Generated by relativistic 3-mass-momentum 3-vector

Conservation of relativistic 3-mass-momentum (temporal-spatial)

Lagrangian “Shift Operator” version of Taylor’s Theorem: \( e^{\delta x \omega}(x) = f(x + \delta x) \)

Bloch Theorem: Translation Operator: \( e^{i k \cdot \nu}(X) = \psi(X + R) \), with \( R \) as reciprocal lattice

Conservation of Angular 3-momentum (spatial-spatial)

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation
The group of all isometries of SR: Minkowski Spacetime (6 + 4 = 10)
(preserve quadratic form)

General Linear, Affine Transform
\[ X'^\nu = \Lambda^\nu_\nu X_\nu + \Delta X^\nu \]

with \( \det[\Lambda^\nu_\nu] = \pm 1 \)

4-AngularMomentum Linear
\[ M^{\mu\nu}[X] = X^i P^\mu - X^\mu P^i \]

4-LinearMomentum
\[ P^\mu = \mathbf{P} = mc \cdot p = (E/c, p) \]

Angular \( M^\mu + \text{Linear} P^\mu \)
\( \left( \begin{array}{c} 3 + 3 \\ 1 + 3 \end{array} \right) = \left( \begin{array}{c} 6 \\ 4 \end{array} \right) \)

= 10 Symmetries = 10 Generators = 10 Conservation Laws: Noether’s Theorem

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Review of SR Transforms

Poincaré Algebra & Generators

Casimir Invariants

The (10) one-parameter groups can be expressed directly as exponentials of the generators:

- Hamiltonian = Energy = Temporal Momentum H
- Linear Momentum p
- Angular Momentum j
- Lorentz Boost k

The Poincaré Algebra is the Lie Algebra of the Poincaré Group:

- \( U[\Lambda(\lambda,0)] = e^{(i\lambda\hat{\boldsymbol{\eta}} \cdot \hat{\boldsymbol{p}})} \) (3) Lorentz Boost k

Component form:

These are the commutators of the Poincaré Algebra:

- Covariant form:
- Total of \((1+3+3+3 = 4+6 = 10)\) Invariances from Poincaré Symmetry

Covariant form:

These are the commutators of the the Poincaré Algebra:

- \( [X^\mu, X^\nu] = 0 \) for free particles
- \( [P^\mu, P^\nu] = -i\hbar (F^\mu\nu) \) if interacting with EM field; otherwise = 0 for free particles
- \( [M^\mu, P^\nu] = i\hbar (\eta^\mu\nu P^\rho - \eta^\nu\rho P^\mu) \)
- \( [M^\mu, M^\nu] = i\hbar (\eta^\mu\nu M^\rho + \eta^\nu\rho M^\mu + \eta^\mu\rho M^\nu + \eta^\nu\mu M^\rho) \)

Component form: Rotations J = -\( \epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} / 2 \), Boosts K = M_0

\[
\begin{align*}
J_{[m,P]} &= i\epsilon_{\mu\nu\rho\sigma} P^\rho \\
J_{[m,P]} &= 0 \\
K_{[m,P]} &= i\epsilon_{\mu\nu\rho\sigma} P^\rho \\
K_{[m,P]} &= -iP^\rho \\
J_{[m,J]} &= i\epsilon_{\mu\nu\rho\sigma} J^\rho \\
J_{[m,K]} &= i\epsilon_{\mu\nu\rho\sigma} K^\rho \\
K_{[m,J]} &= -i\epsilon_{\mu\nu\rho\sigma} J^\rho, \text{ a Wigner Rotation resulting from consecutive boosts} \\
J_{[m,K]} &= i\epsilon_{\mu\nu\rho\sigma} K^\rho \\
K_{[m,K]} &= i\epsilon_{\mu\nu\rho\sigma} K^\rho, \text{ a Pauli-Lubanski Pseudovector}
\end{align*}
\]

Poincaré algebra has 2 Casimir Invariants = Operators that commute with all of the Poincaré Generators

These are \( \{P^2, P\} = (m_0 c)^2, \) \( W^2 = W_\rho W^\rho = -(m_0 c)^2 \) for free particles, with \( W^\rho = (-1/2)\epsilon_{\mu\nu\rho\sigma} J^\rho P^\sigma \) as the Pauli-Lubanski Pseudovector

These Casimir Invariants are {Mass m, Spin j}, with Mass *and* Spin are purely SR phenomena, no QM axioms required!

This Representation of the Poincaré Group or Representation of the Lorentz Group is known as Wigner's Classification in Representation Theory of Particle Physics

Rotations J = -\( \epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} / 2 \), Boosts K = M_0

The set of all Lorentz Generators V = \{\( \zeta \cdot K \) \+ \( \theta \cdot J \)\} forms a vector space over the real numbers. The components of the axis-angle vector and rapidity vector \( \{\theta, \xi, \zeta, \zeta_\theta, \zeta_\xi, \zeta_\zeta\} \) are the coordinates of a Lorentz generator w.r.t. this basis.

Very importantly, the Poincaré group has Casimir Invariant Eigenvalues = \{ Mass m, Spin j \}, hence Mass *and* Spin are purely SR phenomena, no QM axioms required!
10 Poincaré Symmetry Invariances

Noether’s Theorem: 10 SR Conservation Laws

A Tensor Study of Physical 4-Vectors

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**SR → QM**

4-Vector SRQM Interpretation of QM

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**4-Gradient**

\[ \partial=(\partial/c, -\nabla) \]

\[ = (\partial/c, -\partial_x, -\partial_y, -\partial_z) \]

**Invariant d’Alembertian Wave Equation**

\[ \partial=(\partial/c)^2 - \nabla \cdot \nabla \]
SR 4-Vector Magnitudes

Dot Product, Lorentz Scalar Product

Einstein Summation Convention

An example of the magnitude of a 3-vector is the length of a 3-displacement \( \Delta r = (r_1 - r_0) \).

Examine 3-position \( r \rightarrow r = (x,y,z) \), which is a 3-displacement with the base at the origin \( r_0 \rightarrow 0 = (0,0,0) \).

The Dot Product of \( r \), \( \{ r \cdot r = r_0^2 = r_1^2 = (x^2 + y^2 + z^2) = r^2 \} \) is the Pythagorean Theorem.

The Kronecker Delta \( \delta_{jk} \) is \( \text{Diag} [1,1,1] = I_{3\times 3} \).

The magnitude is \( \sqrt{r \cdot r} = \sqrt{r^2} = |r| \). 3D magnitudes are always positive.

The magnitude of a 4-Vector is very similar to the magnitude of a 3-vector, but there are some interesting differences.

One uses the Lorentz Scalar Product, a 4D Dot Product, which includes a time component, and is based on the SR:Minkowski Metric Tensor. I typically use the “Particle Physics” convention of the Minkowski Metric \( \eta_{\mu\nu} \rightarrow \text{Diag} [1,-1,1,-1] \) (Cartesian form), with the other entries zero.

\[ A' \cdot A = A^\mu \eta_{\mu\nu} A'_{\nu} = (a_0^2 + a_1^2 + a_2^2 + a_3^2) = \sum_{\mu=0..3} [a_\mu a_\mu] \]

using the Einstein summation convention where upper-lower paired indices are summed over.

\[ R \cdot R = (ct)^2 \cdot r \cdot r = (x^2 + y^2 + z^2) = (c \Delta t)^2 \]

4D magnitudes can be negative, zero, positive if \( R = (ct) \).

The 4-Vector version has the Pythagorean element in the spatial components, the temporal component is of opposite sign.

This gives a “causality condition”, where SpaceTime intervals (in the [+,-,-,-] metric) can be:

- Time-like: Temporal
- Light-like: Null: Photonic
- Space-like: Spatial

\[ \Delta R \cdot \Delta R = [(c\Delta t)^2 - \Delta r \cdot \Delta r] = \begin{cases} (c\Delta t)^2 & \text{Time-like: Temporal} \\ 0 & \text{Light-like: Null: Photonic} \\ -(\Delta r)^2 & \text{Space-like: Spatial} \end{cases} \]

\( + \) {causal = temporally-ordered}

\( 0 \) {causal, maximum signal speed (\( |\Delta r/\Delta t| = c \))}

\( - \) {non-causal, spatially-extended}

SR 4-Vector Magnitudes

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\( + \) {causal = temporally-ordered}

\( 0 \) {causal, maximum signal speed (\( |\Delta r/\Delta t| = c \))}

\( - \) {non-causal, spatially-extended}
Lorentz Scalar Product $A \cdot B = A_\mu B^\mu$

Exterior Product $A^\wedge B = A^\mu B^\nu - A^\nu B^\mu$

4-Gradient $\partial = (\partial / c, \nabla)$

Minkowski Metric $\partial[R] = \delta[R^\mu] = \eta^{\mu\nu}$

Lorentz Transform $\partial[R^\mu] = A^\mu$, $\eta^{\mu\nu} = 4$ Dimension

4-Position $R = (ct, \mathbf{r})$

4-Velocity $U = \gamma(c, \mathbf{u})$

4-Momentum $P = (mc, p) = (E/c, \mathbf{p})$

4-Gradient $\partial = (\partial / c, \nabla)$

There are at least three 4-Vector relations which use the Exterior Product.

$\partial^A = \partial^\mu A_\nu - \partial^\nu A_\mu = F^{\mu\nu}$: the Faraday EM 4-Tensor

$R^\mu P = R^\mu P^\nu - R^\nu P^\mu = M^{\mu\nu}$: the 4-Angular Momentum

$R^\mu F = R^\mu F^\nu - R^\nu F^\mu = \Gamma^{\mu\nu}$: the 4-Angular Torque

This gives the components of each remarkably similar properties.

Likewise, each of these has a physical Dot Product relation as well.

$\partial \cdot A = \partial_\mu A^\mu = 0$: the Lorenz Gauge, a conservation of 4-EMVectorPotential

$R \cdot P = R^\mu P^\mu = -S_{\text{action,free}}$: the Action Scalar

$R \cdot F = R^\mu F^\mu = ???$: probably something important

4-ChargeFlux $J = (pc, j) = \rho(c, \mathbf{u})$

4-EMVectorPotential $A = (\varphi / c, \mathbf{a})$

Maxwell EM Wave Eqn

$\varepsilon_0 c^2 / 1 / \mu_0$

Electric: Magnetic

Lorentz Scalar

SR 4-Tensor $(2,0)$-Tensor $T^{\mu\nu}$

SR 4-CoVector $(1,0)$-Tensor $V_\mu$ = $V = (\mathbf{v}, v^0)$

SR 4-Scalar $(0,0)$-Tensor $S$

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SRQM Study: 4-Momentum, 4-Force, 4-AngularMomentum, 4-Torque

Linear:
4-Force is the ProperTime Derivative of 4-Momentum.

Angular:
4-Torque is the ProperTime Derivative of 4-AngularMomentum.

d/dτ[ M^μν ]
= d/dτ[ X^μP_ν − X^νP_μ ]
= [ U^μP_ν + X^μF_ν − U^νP_μ − X^νF_μ ]
= [ U^μm_0U_ν + X^μF_ν − U^νm_0U_μ − X^νF_μ ]
= [ U^μO_νU_μ − U^νO_μU_μ + X^μF_ν − X^νF_μ ]
= [ m_0(U^μO_νU_μ − U^νO_μU_μ) + X^μF_ν − X^νF_μ ]
= [ X^μF_ν − X^νF_μ ]
d/dτ[ M^μν ] = Γ^μν = [ X^μF_ν − X^νF_μ ] = X ^ F

4-Force
P^μ = P = (mc, p) = (E/c, p)

4-Torque Tensor
Γ^μν = X^μF_ν − X^νF_μ = X ^ F
= dM^μν/dτ

Trace[ T^μν ] = η_μνT^μν = T^μ_μ = T
V^2 = V^μV_μ = (v^μ)^2 − v · v = (v^0)^2
= Lorentz Scalar
A Tensor Study

Invariant Lorentz Scalar Product

Einstein & Lorentz “saw” the physics of SR, Minkowski & Poincaré “saw” the mathematics of SR. We are indebted to all of them for the simplicity, beauty, and power of how SR and 4-vectors work...

4-Vectors are actually tensorial entities of Minkowski SpaceTime, (1,0)-Tensors, which maintain covariance for inertial observers, meaning that they may have different components for different observers, but describe the same physical object. (like viewing a sculpture from different angles – snapshots look different but it’s actually the same object)

There are also 4-CoVectors, or One-Forms, which are (0,1)-Tensors and dual to 4-Vectors.

4-Vectors are actually tensorial entities of Minkowski SpaceTime, (1,0)-Tensors, which maintain covariance for inertial observers.

\[ A = A^\mu = (a^\mu, a^\nu) = (a^\mu, a^\nu, a^\rho, a^\sigma) \]

\[ B = B^\nu = (b^0, b^1, b^2, b^3) \]

4-CoVectors = (0,1)-Tensors

\[ A_\mu = (a_\mu, a_\nu, a_\rho, a_\sigma) \]

\[ B_\nu = (b_0, b_1, b_2, b_3) \]

4-Vector Product

\[ A \cdot B = a^\mu b_\mu = (a^0 b^0 - a^1 b^1 + a^2 b^2 - a^3 b^3) \]

\[ A_\mu B^\nu = \eta_{\mu\nu} \sigma_{\mu\nu} \] using the Einstein summation convention where upper-lower paired indices are summed over

\[ \Lambda_{\mu\nu} = \eta_{\mu\nu} \sigma_{\mu\nu} \]

Lorentz Scalars Product → Lorentz Invariant Scalar = Same value for all inertial observers

Lorentz Invariants are also tensorial entities: (0,0)-Tensors

Def \[ \Lambda_{\mu\nu} = \pm 1 \]

\[ \text{Trace}[\Lambda_{\mu\nu}] = \eta_{\mu\nu} \sigma_{\mu\nu} = T_{\mu\nu} = T \]

\[ V \cdot V = V^\nu \Lambda_{\mu\nu} \sigma_{\mu\nu} = (v^0)^2 - v \cdot v = (v^0)^2 \]

\[ A = (a^0, a^1, a^2, a^3) \]

\[ B = (b^0, b^1, b^2, b^3) \]

\[ A' = (a'^0, a'^1, a'^2, a'^3) \]

\[ B' = (b'^0, b'^1, b'^2, b'^3) \]

\[ \Lambda_{\mu\nu} \sigma_{\mu\nu} = 4 \]
SR 4-Vectors & Lorentz Scalars

Rest Values ("naughts" = ₀) are Lorentz Scalars

A Tensor Study
of Physical 4-Vectors

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A · A = (a⁰a⁰ - a·a) = (a⁰, a²)₂, where (a⁰, a) is the rest-value, the value of the temporal coordinate when the spatial coordinate is zero. The “rest-values” of several physical properties are all Lorentz scalars.

They are usually related via a Lorentz Factor: \( m = \frac{\text{relativistic component values which can vary}, \ (m), \text{ versus Rest Value Invariant Scalars, like }}{}} \)

This property of SR equations is a very good reason to use the “naught” convention for specifying the difference between relativistic component values which can vary, like \((m)\), versus Rest Value Invariant Scalars, like \((m₀)\), which do not vary. They are usually related via a Lorentz Factor: \{ m = \gamma m₀ \} and \{ E = \gamma E₀ \}, as seen in the relation of \( P \) and \( U \).

This leads to simple relations between 4-Vectors.

\[
P = (m₀)U = (E₀/c²)U \quad K = (\omega/c²)U
\]

And gives nice Scalar Product relations between 4-Vectors as well.

\[
P·U = (m₀)U·U = (m₀)c² = (E₀) \quad K·U = (\omega/c²)U·U = (\omega/c²)c² = (\omega)
\]

Choose a frame in which the spatial component is zero. This is known as the “rest-frame” of the 4-Vector. It is not moving spatially.

\[
P·P = (mc²)² - p·p = (m₀c²)² \quad K·K = (\omega/c²)² - k·k
\]

The resulting simpler expressions then give the “rest values”, indicated by \( _₀ \).

RestMass \((m₀)\) and RestAngularFrequency \((ω₀)\) are Lorentz Scalars. We can choose a frame that may simplify the expressions.

SR 4-Vector

\[
A = (a⁰,a) = (a⁰,a¹,a²,a³)
\]

→ \( (a⁰, 0) \) [in spatial rest frame]

Notation:
"₀" for rest values (naughts)
"₀" for temporal components (₀ index)

4-Vector

\[
P = (mc,p) = (E/c,p)
\]

4-Momentum

\[
P·U = m₀c² = E₀
\]

K·U = ω₀

4-WaveVector

\[
K = (ω/c,k) = (ω/c,ω̂ν/νphase)
\]

\[
K·K = (ω₀/c²)
\]

Trace\[T^\nu\] = ηₜₜ T^ₜₜ = T^νν = T

\[V·V = V^νV^ν = [(v^ν)² - V^νV] = (v^ν)²\]

= Lorentz Scalar
SR 4-Vectors & 4-Tensors

Lorentz Scalar Product & Tensor Trace

Invariants: Similarities

All {4-Vectors:4-Tensors} have an associated {Lorentz Scalar Product:Trace}

Each 4-Vector has a “magnitude” given by taking the Lorentz Scalar Product of itself.

\[ V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = (v_0^2 + v_1^2 + v_2^2 + v_3^2) = (v_0^2 - v \cdot v) = (v_0^2) \]

The absolute magnitude of \( V \) is \( \sqrt{|V \cdot V|} \)

Each 4-Tensor has a “magnitude” given by taking the Tensor Trace of itself.

\[ \text{Trace}[T^{\mu\nu}] = T^\mu_\mu = \eta_{\mu\nu} T^{\mu\nu} = (T_{00} - T_{11} - T_{22} - T_{33}) = T \]

Note that the Trace runs down the diagonal of the 4-Tensor.

Notice the similarities. In both cases there is a tensor contraction with the Minkowski Metric Tensor \( \eta_{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1] \) {Cartesian basis}

ex. \( P \cdot P = (E/c)^2 - p \cdot p = (E_0/c)^2 = (m_0 c)^2 \)

which says that the “magnitude” of the 4-Momentum is the RestEnergy/c = RestMass*c

ex. \( \text{Trace}[\eta^{\mu\nu}] = (\eta^{00} - \eta^{11} - \eta^{22} - \eta^{33}) = 1 + 1 + 1 + 1 = 4 \)

which says that the “magnitude” of the Minkowski Metric = SpaceTime Dimension = 4
Some 4-Vectors have an alternate form of Tensor Invariant: \( \frac{dv'}{\sqrt{v'}} = \frac{dv}{\sqrt{v}} \)

, in addition to the standard Lorentz Invariant \( V \cdot V = V^\mu V_\mu = (v^0)^2 \).

If \( V \cdot V = (\text{constant}) \), with \( V = (v^0, v) \)
then \( d(V \cdot V) = 2v \cdot dv = d(\text{constant}) = 0 \)

hence \( (v \cdot dv) = 0 = v^0 dv^0 - v \cdot dv \)
\( dv^0 = v \cdot dv/v^0 \)

Generally: with \( \Lambda = \Lambda^\nu_\mu \), Lorentz Boost Transform in the \( \beta \)-direction
\( V' = \Lambda V \) : from which the temporal component \( v^0' = (\gamma v^0 - \gamma \beta v) \)
\( dV' = \Lambda dv \) : from which the spatial component \( dv' = (\gamma dv - \gamma \beta dv^2) \)

Combining:
\( dv' = (\gamma dv - \gamma \beta(v^0 dv/v^0)) \)
\( dv' = (1/\sqrt{v^0})[(\gamma v^0 dv - \gamma \beta(v \cdot dv)] \)
\( dv' = (1/\sqrt{v^0})[(\gamma v^0 - \gamma \beta v)v] dv \)
\( dv' = (\gamma v^0 - \gamma \beta v)^2 (1/\sqrt{v^0}) dv \)
\( dv' = (\sqrt{v'/v^0}) dv \)
\( dv'/\sqrt{v'} = dv/v^0 = \text{Invariant of } V = (v^0, v) \) for \( V \cdot V = (\text{constant}) \)

So, for example:
\( P \cdot P = (m_c)^2 = (\text{constant}) \)

Thus, \( dp'/E' = dp/E \) Invariant
Or: \( dp'/\sqrt{E'} = dp/E \rightarrow d^3p/E = dp^2 dp^3 p/E = \text{Invariant, usually seen as } \int F(\text{various invariants}) \cdot d^3p/E = \text{Invariant} \)
The 4D Position coords that are integrated to give a 4D volume: SI units $[m^4]$.

$$d^4X = -(V_o) dT \cdot dX = -(dT) \cdot (dX) = cdt \, d^3x = cdt \, dx \, dy \, dz$$

The 4D Position coords that are integrated to give a 4D volume: SI units $[m^4]$. 

4-Differential $dX = (cdt, dx)$; $dR = (cdt, dr)$.

$$V = \int dV = \int dx \int dy \int dz = \int \int \int dx \, dy \, dz$$

$$V = V_o / \gamma = 3D \text{ Spatial Volume}$$

$$dV = d^3x = 3D \text{ Spatial Volume Element}$$

$$\gamma = V_o / V$$

$$\gamma dV = -(V_o / V^2) dV cdt$$

$$-V_o \eta_{\mu\nu} \cdot \int F[\text{various Invariants}] d^4X$$

Phase Space Tensor Invariant

$$d^4X = cdt \cdot dx \cdot dy \cdot dz = cdt \cdot d^3x$$

$$\gamma dV = \gamma d^3x$$

Phase Space Integration

4-UnitTemporalDifferential $dT = (d[\gamma], d[\gamma \beta])$

4-Differential $dR = dR^\gamma = (cdt, dr)$

And, this makes sense.

$T$ is a temporal 4-Vector with fixed magnitude: $T \cdot T = 1$

Therefore, $dT$ must be a spatial 4-Vector.

If $dX$ is also spatial, then the Lorentz scalar product $(dT \cdot dX) = -\text{magnitude}$ will be negative with this choice of Minkowski Metric.

Thus, multiplying by $-V_o$ gives a positive volume element $(cdt \, dx \, dy \, dz = d^4X)$

It is sort of quirky though, that the temporal $(cdt)$ comes from the $dX$ part, and the spatial $(d^3x)$ comes from the $dT$ part.
SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

Phase Space Integration

\[ \rho \, d^3x = \rho' \, d^3x' = (-V/c) dT \cdot J = \text{Lorentz Scalar Invariant} \]

\[ n \, d^3x = n' \, d^3x' = (-V/c) dT \cdot N = \text{Lorentz Scalar Invariant} \]

4-Current Density \( J = (pc, j) \)

4-(Dust) Number Flux \( N = (cn, nu) = n(c, u) \)

4-Unit Temporal \( T = \gamma (1, \beta) = (\gamma, \gamma \beta) \)

4-Unit Temporal Differential \( dT = (d[\gamma], d[\gamma \beta]) \)

\( V = V_o/\gamma \)

\( d^3x = -\left( V_o/\gamma^2 \right) dV \)

Total Charge \( Q = \int n_o \rho_o d^3x = \text{Lorentz Scalar Invariant} \)

Total Particle # \( N = \int n_o d^3x = \text{Lorentz Scalar Invariant} \)

Total Rest Volume \( V_o = \int \gamma d^3x = \text{Lorentz Scalar Invariant} \)

This also gives an alternate way to define the Rest Volume Invariant \( V_o \).

\( (-V/c) dT \cdot N = nd^3x = \int n_o \rho_o d^3x = \text{Lorentz Scalar Invariant} \)

\( cN/V_o = -\int dT \cdot N \)

\( \int \gamma d^3x = \int n_o \rho_o d^3x = V_o \)

\( \int (V/c) dT \cdot J = \text{Invaraint, because (Rest Scalar * Lorentz Scalar Product) = Invariant} \)

\( \int (-V/c) dT \cdot N = \int (V_o/c) dT \cdot J = \int (V_o/c) dT \cdot J = \int \gamma d^3x = \int n_o \rho_o d^3x = -cN/V_o = -cQ/V_o = -cQ/\int dT \cdot J \)

Phase Space Tensor Invariants
The 4D Momentum coords that are integrated to give a 4D Momentum Volume: SI Units $[(kg\cdot m/s)^4]$

The 4D WaveVector coords that are integrated to give a 4D WaveVector Volume: SI Units $[(1/m)^4]$

$d^4P = (dE/c, dp) = (dE/c, dp_x dp_y dp_z)$

$d^4K = (d\omega/c, dk) = (d\omega/c, dk_x dk_y dk_z)$

The 4D Momentum coords that are integrated to give a 4D Momentum Volume: SI Units $[(kg\cdot m/s)^4]$

The 4D WaveVector coords that are integrated to give a 4D WaveVector Volume: SI Units $[(1/m)^4]$

$d^4P = (dE/c, dp_x dp_y dp_z)$

$d^4K = (d\omega/c, dk_x dk_y dk_z)$

$V_P = \int dV_P = \int dp_x \int dp_y \int dp_z = \int \int \int dp_x dp_y dp_z = \int d^3p$

$V_P = \gamma(V_{Po}) = 3D Volume in Momentum Space: SI Units [(kg\cdot m/s)^3]$

$dV_P = d\gamma(V_{Po}) = 3D Volume Element in Momentum Space$

$d\gamma = (dV_P/V_{Po})$

$\gamma = (V_{Po})/(V_{Po})$

$\gamma = \gamma(1,\beta) = (\gamma, \gamma\beta) = (d[\gamma], d[\gamma\beta])$

$d\gamma = (dV_P/V_{Po})$

$d^4P = (dE/c, dp_x dp_y dp_z)$

$V_P = \int dV_P = \int dp_x \int dp_y \int dp_z = \int \int \int dp_x dp_y dp_z = \int d^3p$

$\gamma = (V_{Po})/(V_{Po})$

$\gamma = \gamma(1,\beta) = (\gamma, \gamma\beta) = (d[\gamma], d[\gamma\beta])$

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$d^4P = (dE/c, dp_x dp_y dp_z)$

$V_P = \int dV_P = \int dp_x \int dp_y \int dp_z = \int \int \int dp_x dp_y dp_z = \int d^3p$
SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

Phase Space Integration
SRQM Study: SR 4-Tensors

General → Symmetric & Anti-Symmetric

Any SR Tensor $T^{\mu\nu} = (S^{\mu\nu} + A^{\mu\nu})$ can be decomposed into parts:

Symmetric $S^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2$ with $S^{\mu\nu} = +S^{\nu\mu}$

Anti-Symmetric $A^{\mu\nu} = (T^{\mu\nu} - T^{\nu\mu})/2$ with $A^{\mu\nu} = -A^{\nu\mu}$

$S^{\mu\nu} + A^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2 + (T^{\mu\nu} - T^{\nu\mu})/2 = T^{\mu\nu} + 0 = T^{\mu\nu}$

Independent components: $\{4^2 = 16 = 10 + 6\}$

Max 16 possible
Max 10 possible
Max 6 possible

Symmetric 4-Tensor
$S^{\mu\nu} = \begin{bmatrix}
S^{00}, S^{01}, S^{02}, S^{03} \\
S^{10}, S^{11}, S^{12}, S^{13} \\
S^{20}, S^{21}, S^{22}, S^{23} \\
S^{30}, S^{31}, S^{32}, S^{33}
\end{bmatrix}$

Anti-Symmetric 4-Tensor
$A^{\mu\nu} = \begin{bmatrix}
A^{00}, A^{10}, A^{20}, A^{30} \\
A^{10}, A^{11}, A^{21}, A^{31} \\
A^{20}, A^{21}, A^{22}, A^{23} \\
A^{30}, A^{31}, A^{32}, A^{33}
\end{bmatrix}$

Proof:
$S^{\mu\nu} A_{\mu\nu} = 0$

Physically, the anti-symmetric part contains rotational information and the symmetric part contains information about isotropic scaling and anisotropic shear.

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

*Note* These don’t have to be composed from a single general tensor.

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $V^\mu = (V^0, V^\mu)$
(0,2)-Tensor $T_{\mu\nu}$
SR 4-Vector
(1,0)-Tensor $V^\mu$
SR 4-CoVector
(0,1)-Tensor $V_\mu = (V_0, V^\mu)$
SR 4-Scalar
Lorentz Scalar

Trace $[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T_{\mu\nu} = T$

$V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = (V^0)^2 - V^\mu V^\mu = (V_0)^2$

Lorentz Scalar
SRQM Study: SR 4-Tensors

Symmetric → Isotropic & Anisotropic

Any Symmetric SR Tensor $S^{\mu\nu} = (T_{\text{iso}}^{\mu\nu} + T_{\text{aniso}}^{\mu\nu})$ can be decomposed into parts:

Isotropic $T_{\text{iso}}^{\mu\nu} = (1/4)\text{Trace}[S^{\mu\nu}] \eta^{\mu\nu} = (T) \eta^{\mu\nu}$

Anisotropic $T_{\text{aniso}}^{\mu\nu} = S^{\mu\nu} - T_{\text{iso}}^{\mu\nu}$

The Anisotropic part is Traceless by construction, and the Isotropic part has the same Trace as the original Symmetric Tensor. The Minkowski Metric is a symmetric, isotropic 4-tensor with $T=1$.

**Independent components:**

- Max 10 possible
- Max 1 possible
- Max 9 possible

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

*Note* These don’t have to be composed from a single general tensor.

Proof:

$S^{\mu\nu} A_{\mu\nu} = 0$

$= S^{\mu\nu} A_{\mu\nu}$ because we can switch dummy indices

$= (+S^{\mu\nu}) A_{\mu\nu}$ because of symmetry

$= S^{\mu\nu} (-A_{\mu\nu})$ because of anti-symmetry

$= -S^{\mu\nu} A_{\mu\nu}$

$= 0$ because the only solution of $\{c = -c\}$ is 0

Physically, the isotropic part represents a direction independent transformation (e.g., a uniform scaling or uniform pressure); the deviatoric part represents the distortion.

Trinvariant Variables:

- Lorentz Scalar $\langle \nu, \nu \rangle = \langle (\nu^0)^2 - \nu \cdot \nu \rangle = (\nu^0)^2$ Lorentz Scalar

SR 4-Vector $V^\mu = (V^0, V^1, V^2, V^3)$

SR 4-CoVector $\nu^\mu = (\nu^0, -\nu^1, -\nu^2, -\nu^3)$

SR 4-Scalar $S$ Lorentz Scalar

$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T_{\mu\nu} = T$
SR Tensor Invariants

(0,0)-Tensor = Lorentz Scalar $S$: Has either (0) or (1) Tensor Invariant, depending on exact meaning (S) itself is Invariant

V-V = (v^0,v^1,v^2,v^3) = (v^0,v^0 - v \cdot v) = (v_0)^2

(1,0)-Tensor = 4-Vector $V^\mu$: Has (1) Tensor Invariant = The Lorentz Scalar Product $V \cdot V = V_\mu V^\mu = \eta_{\mu \nu} V^\mu V^\nu = \text{Tr}[V^\mu V^\nu] = (V^\mu)_0 + (V^1)_0 + (V^2)_0 + (V^3)_0 = (v_0)^2$

(2,0)-Tensor = 4-Tensor $T^\mu_\nu$: Has (4+) Tensor Invariants (though not all independent)

\[ T^\mu_\nu = \text{Trace} = \Sigma \text{EigenValues for (1,1)-Tensors (mixed)} \]

a) $T^\mu_\nu = \text{Anti} = \Sigma \text{Anti-Symmetric}$

b) $T^\mu_\nu = \text{Inner Product}$

c) $T^\mu_\nu = \text{Symmetric}$

(d) Determinant $D\text{et}[T^\mu_\nu] = \Sigma \text{Symmetric}$

\[ D\text{et}[T^\mu_\nu] = (1/2) \epsilon_{\mu \nu \lambda \rho} T^\lambda_\nu T^\rho_\nu \]

for anti-symmetric: $D\text{et}[T^\mu_\nu] = \text{Pfaffian}[T^\mu_\nu]^2$ (The Pfaffian is a special polynomial of the matrix entries)
The Faraday EM Tensor $F^\mu_\nu = \partial^\mu A^\nu - \partial^\nu A^\mu$ is an anti-symmetric tensor that contains the Electric and Magnetic Fields. The 3-electric components ($\epsilon = e$) are in the temporal-spatial sections. The 3-magnetic components ($b = b'$) are in the only-spatial section.

The 4-Gradient and 4-EMVector have (4) independent components each, for total of (8).

4-Gradient $\partial = \partial^\mu = (\partial/c, \nabla)$

\[ \text{Tr}[F^\mu_\nu] = F_{\nu}^\nu = 0 \]

Trace Tensor Invariant

\[ F_{\mu \nu} F^{\mu \nu} = 2((b \cdot b) - (e/e/c)^2) \]

Inner Product Tensor Invariant

\[ \text{AsymmTri}[F^\mu_\nu] = 0 \]

Asymmetric Tri-Product Tensor Invariant

\[ \text{Det}[F^\mu_\nu] = ((e \cdot b)/c)^2 \]

Determinant Tensor Invariant

The Faraday EM Tensor has only (2) linearly-independent invariants:

1. $2((b \cdot b) - (e/e/c)^2)$
2. $((e \cdot b)/c)^2$

a) $2((b \cdot b) - (e/e/c)^2)$ gives 0, and do not provide additional constraints

The 4-Gradient and 4-EMVector have (4) independent components each, for total of (8). Subtract the (2) invariants which provide constraints to get a total of (6) independent components

$= (6)$ independent components of a 4x4 anti-symmetric tensor

$= (3)$ 3-electric $e = e'$ and $b = b'$ have (6) independent EM field components

$\partial \ ^4 A$ is the exterior product of the 4-Gradient with the 4-EMVector Potential.

$\epsilon_l$ is the Levi-Civita symbol, the fully anti-symmetric tensor.

with Latin indices it ranges from (1..3), with Greek indices it ranges from (0..3)

SRQM Study: SR 4-Tensors

SR Tensor Invariants for Faraday EM Tensor

- The Faraday EM Tensor $F^\mu_\nu = \partial^\mu A^\nu - \partial^\nu A^\mu$ is an anti-symmetric tensor that contains the Electric and Magnetic Fields.
- The 3-electric components ($\epsilon = e$) are in the temporal-spatial sections.
- The 3-magnetic components ($b = b'$) are in the only-spatial section.
- Subtract the (2) invariants which provide constraints to get a total of (6) independent components.
- The 4-Gradient and 4-EMVector have (4) independent components each, for total of (8).
- The Faraday EM Tensor has only (2) linearly-independent invariants.
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- The 4-Grad...
SRQM Study: SR 4-Tensors

SR Tensor Invariants
for 4-AngularMomentum Tensor

The 4-AngularMomentum Tensor $M^{ii}$ is an anti-symmetric tensor
The 3-mass-moment components ($n = n$) are in the temporal-spatial sections.
The 3-angular-momentum components ($I = I$) are in the only-spatial section.

(2,0)-Tensor = 4-Tensor $T_{ii}$: Has (4+) Tensor Invariants (though not all independent)

a) $T_{ii} = \text{Trace} = \Sigma$ of EigenValues for (1,1)-Tensors (mixed)
b) $T_{i}T_{ij} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product}$
c) $T_{i}T_{ij}T_{ij} = \text{Asymm Tri-Product} \rightarrow ?$Name?
d) $T_{i}T_{ij}T_{ij}T_{ij} = \text{Asymm Quad-Product}$

Importantly, the 4-AngularMomentum Tensor has only (2) linearly-independent invariants:

- $\text{Trace} [M^{ii}] = M_{ii}$
- $\text{Determinant} [M_{ii}] = \text{Asymm Quad-Product}$

- $\text{Inner Product Tensor Invariant}$
- $\text{Asymm Tri-Product Tensor Invariant}$
- $\text{Determinant Tensor Invariant}$

The 4-AngularMomentum Tensor $M^{ii}$ is related to the LRL = Laplace-Runge-Lenz 3-vector:

$V^{\nu} = V_{\nu} \quad \eta_{\nu} V^{\nu} = (V^{2}) \quad \omega \times V = (V^{2}) \quad \text{Lorentz Scalar}$

The 4-Position $X = (ct, x)$

$4$-Vector $P^{\mu} = (mc, p) = (E/c, \text{p})$

4-Momentum $P = P^{\mu} = (mc, p) = (E/c, \text{p})$

Trace $T^{\mu\nu} = \eta_{\mu} T^{\nu\sigma} = T^{\nu}_{\sigma} = T$

$V^{\nu} V^{\nu} = (V^{2})^{2} = (V^{2})^{2}$

$\text{SR 4-Tensor}$

$\text{SR 4-Vector}$

$\text{SR 4-Scalar}$

$\text{SR 4-CoVector}$

$\text{SR 4-Tensor}$

$\text{SR 4-Vector}$

$\text{SR 4-Scalar}$

$\text{SR 4-CoVector}$

$\text{SR 4-Tensor}$

$\text{SR 4-Vector}$

$\text{SR 4-Scalar}$

$\text{SR 4-CoVector}$
The Minkowski Metric Tensor $\eta_{\mu\nu}$ is the tensor all SR 4-Vectors are measured by.

(2,0)-Tensor = 4-Tensor $T_{\mu\nu}$, Has (4+) Tensor Invariants (though not all independent)

a) $T_{\mu\nu}^\alpha = \text{Trace} = \text{Sum of EigenValues for (1,1)-Tensors (mixed)}$

b) $T_{\mu\nu\alpha\beta} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product}$

c) $T_{\mu\nu\alpha\beta\gamma\delta} = \text{Asymm Tri-Product} \rightarrow \text{?Name?}$

d) $T_{\mu\nu\alpha\beta\gamma\delta\epsilon\zeta} = \text{Asymm Quad-Product} \rightarrow 4D \text{Determinant} = \text{Product of EigenValues for (1,1)-Tensors}$

a): Minkowski Trace

\[ \eta_{\mu\nu} \eta_{\nu\mu} = 4 \]

b): Minkowski Inner Product

\[ \eta_{\mu\nu} \eta_{\mu\nu} = 4 \]

c): Minkowski Asymmetric Tri

\[ \eta_{\mu\nu} \eta_{\mu\nu} = 24 = 4! \]

, if I did the math right...

d): Minkowski Determinant

\[ \det(\eta_{\mu\nu}) = -1 \]

EigenValues not defined for the standard Minkowski Metric Tensor since it is a type (2,0)-Tensor, all upper indices. However, they are defined for the mixed form (1,1)-Tensor, mixed indices

\[ \Lambda_\mu^\nu \Lambda_\nu^\beta \eta_{\mu\beta} = \eta_{\mu\nu} \]

Det(Exp[A])=Exp(Trace[A])

Det(A)=$(tr A)^4 - 6 (tr A^2)^2$ + $(3 (tr A^3))^2$ + $8 (tr A^4) tr A$ - $6 (tr A^4)^2$/$24$

SR 4-Tensor

SR 4-Vector

SR 4-CoVector

SR 4-Scalar

SR 4-Vector

(1,0)-Tensor $V_\mu = V = (\vec{v}, 0)$

SR 4-CoVector

(0,1)-Tensor $V_\mu = (0, -\vec{v})$

SR 4-Scalar

(0,0)-Tensor $S$ Lorentz Scalar

\[ \det[T_{\alpha\beta}] = \Pi_{k}(\lambda_k) ; \text{with } \{\lambda_k\} = \text{Eigenvalues} \]

Characteristic Eqns: $\det[T_{\alpha\beta} - \lambda_k I_{(4)}] = 0$
SRQM Study: SR 4-Tensors

SR Tensor Invariants for Continuous Lorentz Transform Tensors

The Lorentz Transform Tensor \( \Lambda^\mu_{\nu} \) is the tensor all SR 4-Vectors must transform by.

(2,0)-Tensor = 4-Tensor \( \Lambda^\mu_{\nu} \); Has (4+) Tensor Invariants (though not all independent)

a) Lorentz Trace \( \Lambda^\mu_{\nu} \) = Sum of EigenValues for (1,1)-Tensors (mixed)

b) Lorentz Inner Product \( \Delta^\mu_{\nu} = \Lambda^\mu_{\nu} \Lambda^\nu_{\mu} = \eta_{ab} \) and \( \eta^a_{1} \eta_{ab} = 4 \)

c) Lorentz Asymmetric Tri-Product Invariant

d) Lorentz Determinant Invariant

An even more general version would be with \( a \) & \( b \) as arbitrary complex values:

could be 2 boosts, 2 rotations, or a boost/rotation combo

SR Lorentz Transform

\[ \partial[R^\mu_{\nu}] = \delta R^\nu_{\alpha} / R^\nu_{\alpha} = \Lambda^\mu_{\nu} \]

Eigenvalues of \( \Lambda^\mu_{\nu} \) = \[ e^a, e^b, e^c, e^d \]

Sum of Eigenvalues of \( \Lambda^\mu_{\nu} \) = \[ e^a + e^b + e^c + e^d \]

Product of Eigenvalues of \( \Lambda^\mu_{\nu} \) = \[ e^a e^b e^c e^d \]

Determinant of \( \Lambda^\mu_{\nu} \) = Det[\( \Lambda^\mu_{\nu} \)] = \[ e^0 = e^a e^b e^c e^d \]

Trace of \( \Lambda^\mu_{\nu} \) = \[ 0 \ldots 0 \ldots \ldots \ldots 0 \]

Trace Product of \( \Lambda^\mu_{\nu} \) = \[ 0 \ldots 0 \ldots \ldots \ldots 0 \]

Trace Sum of \( \Lambda^\mu_{\nu} \) = \[ 0 \ldots 0 \ldots \ldots \ldots 0 \]

Characteristic Eqns: Det[T^\alpha_{\nu} - \lambda \Lambda^\alpha_{\nu}] = 0

SR 4-Vector

(1,0)-Tensor \( V^\mu = V \) = \( (v^0, v) \)

SR 4-Scalar

(0,0)-Tensor \( S \) = Lorentz Scalar

SR 4-CoVector

(0,1)-Tensor \( V_\nu = V^0 \) = \( (v^0, -v) \)

\( V^0 V^\mu = (v^0)^2 - v^2 \) = Lorentz Scalar

John B. Wilson

SciRealm.org
**SRQM Study: SR 4-Tensors**

**SR Tensor Invariants for Discrete Lorentz Transform Tensors**

<table>
<thead>
<tr>
<th>Invariant Type</th>
<th>Tensor Descriptions</th>
<th>Determinants and Products</th>
<th>Eigenvalues</th>
<th>Trace</th>
<th>Characteristic Eqn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lorentz SR</td>
<td>TP Combo Tensor $\Lambda^\nu_\mu \rightarrow T^\nu_\mu$</td>
<td>Determinant $\Lambda^\mu_\nu = \det \Lambda = -1$</td>
<td>${ -1, -1, -1, -1 }$</td>
<td>$0$</td>
<td>$\det[T^\nu_\mu] = 0$</td>
</tr>
<tr>
<td>Lorentz SR</td>
<td>Parity-Inversion Tensor $\Lambda^\nu_\mu \rightarrow P^\nu_\mu$</td>
<td>Determinant $\Lambda^\mu_\nu = \det \Lambda = -1$</td>
<td>${1, 1, 1, 1}$</td>
<td>$0$</td>
<td>$\det[P^\nu_\mu] = 0$</td>
</tr>
<tr>
<td>Lorentz SR</td>
<td>Flip-xy-Combo Tensor $\Lambda^\nu_\mu \rightarrow F_{xy}^\nu_\mu$</td>
<td>Determinant $\Lambda^\mu_\nu = \det \Lambda = -1$</td>
<td>${1, 1, 1, 1}$</td>
<td>$0$</td>
<td>$\det[F_{xy}^\nu_\mu] = 0$</td>
</tr>
<tr>
<td>Lorentz SR</td>
<td>Time-Reversal Tensor $\Lambda^\nu_\mu \rightarrow T^\nu_\mu$</td>
<td>Determinant $\Lambda^\mu_\nu = \det \Lambda = -1$</td>
<td>${1, 1, 1, 1}$</td>
<td>$0$</td>
<td>$\det[T^\nu_\mu] = 0$</td>
</tr>
<tr>
<td>Lorentz SR</td>
<td>Identity Tensor $\Lambda^\nu_\mu \rightarrow \eta^\nu_\mu$, $\eta^\nu_\mu = \delta^\nu_\mu$</td>
<td>Determinant $\Lambda^\mu_\nu = \det \Lambda = 1$</td>
<td>${1, 1, 1, 1}$</td>
<td>$0$</td>
<td>$\det[\eta^\nu_\mu] = 0$</td>
</tr>
</tbody>
</table>

The Trace of various discrete Lorentz transforms varies in steps from $\{-4, -2, 0, 2, 4\}$.

This includes Mirror Flips, Time Reversal, and Parity Inverse – essentially taking all combinations of $\pm 1$ on the diagonal of the transform.
More SR Tensor Invariants for Discrete Lorentz Transform Tensors

### Lorentz SR Transform
\[
\delta[R^\mu] = \partial R^\mu / \partial R^\nu = \Lambda^\mu_\nu \quad \Lambda^\mu_\nu = (\Lambda^\nu_\alpha)^\mu - \Lambda^\nu_\nu \quad \eta^\mu_\nu = \delta^\mu_\nu \quad \eta^\nu_\alpha \Lambda^\alpha_\beta = \eta^\gamma_\beta \quad \Delta^\gamma_\delta \Lambda^\delta_\mu = \delta^\gamma_\mu
\]
\[
\text{Det}[\Lambda^\gamma_\delta] = \pm 1, \quad \Lambda^\mu_\nu, \Lambda^{\mu\nu} = 4
\]

**Note:**
The Flip-xy-Combo is the equivalent of a \(\pi\)-Rotation-z.

I suspect that this may be related to exchange symmetry and the Spin-Statistics idea that a particle-exchange is the equivalent of a spin-rotation.

A single Flip would not be an exchange because it leaves a mirror-inversion of \(<\text{right}-\text{left}>\).

But the extra Flip along an orthogonal axis corrects the mirror-inversion, and would be an overall exchange because the particle is in a different location.

### Lorentz SR Identity Tensor \( \Lambda^\nu_\nu \rightarrow \eta^\nu_\nu = \delta^\nu_\nu \)
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\( \text{Minkowski Delta} \)

### Lorentz SR Flip-x Tensor \( \Lambda^\nu_\nu \rightarrow \eta^\nu_\nu = \delta^\nu_\nu \)
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\( \text{Rotation-z} \)

### Lorentz SR Flip-y Tensor \( \Lambda^\nu_\nu \rightarrow \eta^\nu_\nu = \delta^\nu_\nu \)
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\( \text{Rotation-z} \)

### Lorentz SR Flip-xy-Combo Tensor \( \Lambda^\nu_\nu \rightarrow \eta^\nu_\nu = \delta^\nu_\nu \)
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\( \text{Rotation-z} \)

---

**SR 4-Tensor**
- Tensor \( T^{\mu\nu} \)
- Tensor \( V = (\nu^\alpha, \nu) \)

**SR 4-Vector**
- Tensor \( V = (\nu^\alpha, \nu) \)

**SR 4-Scalar**
- Tensor \( S \) Lorentz Scalar

\[ \text{Det}[T^{\mu\nu}] = \Pi_k \{ \lambda_k \}; \text{ with } \{ \lambda_k \} = \text{Eigenvalues} \]

**Characteristic Eqns:**
\[ \text{Det}[T^{\mu\nu} - \lambda_k I_{2x2}] = 0 \]

**Trace**
\[ \text{Trace}[T^{\nu\mu}] = \eta^{\nu\gamma} T^{\gamma\mu} = T^{\nu\mu} = T \]

**V-V**
\[ V \cdot V = V^\nu [\nu^\alpha, \nu^\beta] = (\nu^\alpha \nu^\beta - \nu^\nu \nu^\gamma) = (\nu^\gamma) \]

\[ = \text{Lorentz Scalar} \]
SR 4-Scalars, 4-Vectors, 4-Tensors beautifully and elegantly display the relations between lots of different physical properties and relations. Their notation makes navigation through the physics very simple.

They also devolve very nicely into the limiting/approximate Newtonian cases of \{ |v| \ll c \} by letting \( \gamma \rightarrow 1 \) and \( \gamma' = \frac{dv}{dt} \rightarrow 0 \).

SR tells us that several different physical properties are actually dual aspects of the same thing, with the only real difference being one's point of view, or reference frame.

Examples of 4-Vectors = (1,0)-Tensors include:
- \((\text{Time }, \text{Space})\), \((\text{Energy }, \text{Momentum})\), \((\text{Power }, \text{Force})\), \((\text{Frequency }, \text{WaveNumber})\),
- \((\text{Time Differential }, \text{Spatial Gradient})\),
- \((\text{ChargeDensity }, \text{CurrentDensity})\), \((\text{EM-ScalarPotential }, \text{EM-VectorPotential})\), etc.

One can also examine 4-Tensors, which are type (2,0)-Tensors. The Faraday EM Tensor similarly combines EM fields:
- Electric \( \mathbf{e} = (e_x, e_y, e_z) \) and Magnetic \( \mathbf{b} = (b_x, b_y, b_z) \)

\[
\mathbf{F}^{\text{EM}} = \begin{bmatrix}
0 & -e_x/c & -e_y/c \\
+e_x/c & 0 & -e_z/c \\
+e_y/c & +e_z/c & 0
\end{bmatrix}
\]

Also, things are even more related than that. The 4-Momentum is just a constant times 4-Velocity. The 4-WaveVector is just a constant times 4-Velocity.

In addition, the very important conservation/continuity equations seem to just fall out of the notation. The universe apparently has some simple laws which can be easy to write down by using a little math and a super notation.
SR Gradient 4-Vectors = (1,0)-Tensors
SR Gradient One-Forms = (0,1)-Tensors

4-Vector = Type (1,0)-Tensor
4-Position \( \mathbf{R} = R^\mu = (ct,r) \)
4-Gradient \( \partial_\mathbf{R} = \partial \partial^\mu = \partial/\partial R_\mu = (\partial / c, - \nabla) \)

Standard 4-Vector
4-Position \( \mathbf{R} = R^\mu = (ct,r) \)
4-Velocity \( \mathbf{U} = U^\mu = \gamma(c,u) \)
4-Momentum \( \mathbf{P} = P^\mu = (E/c,p) \)
4-WaveVector \( \mathbf{K} = K^\mu = (\omega/c,k) \)

[Temporal : Spatial] components
[Time (t) : Space (r)]
[Time Differential (\partial_1) : Spatial Gradient(\nabla)]

Related Gradient 4-Vector (from index-raised Gradient One-Form)
4-PositionGradient \( \partial_\mathbf{R} = \partial R^\mu = \partial/\partial R_\mu = (\partial / c, - \nabla_R) = \partial = \partial^\mu = 4-Gradient \)
4-VelocityGradient \( \partial_\mathbf{U} = \partial U^\mu = \partial/\partial U_\mu = (\partial / c, - \nabla_u) \)
4-MomentumGradient \( \partial_\mathbf{P} = \partial P^\mu = \partial/\partial P_\mu = (\partial / c, - \nabla_p) \)
4-WaveGradient \( \partial_\mathbf{K} = \partial K^\mu = \partial/\partial K_\mu = (\partial / c, - \nabla_k) \)

In each case, the (Whichever)Gradient 4-Vector is derived from an SR One-Form or 4-CoVector, which is a type (0,1)-Tensor
ex. One-Form PositionGradient \( \partial_\mathbf{R} = \partial R^\nu = \partial/\partial R_\nu = (\partial / c, - \nabla_R) \)

The (Whichever)Gradient 4-Vector is the index-raised version of the SR One-Form (Whichever)Gradient
ex. 4-PositionGradient \( \partial_\mathbf{R} = \partial R^\nu = \partial/\partial R_\nu = (\partial / c, - \nabla_R) = \eta^{\mu\nu}(\partial / c, - \nabla_R)_\nu = \eta^{\mu\nu}(One-Form PositionGradient) \)

This is why the 4-Gradient is commonly seen with a minus sign in the spatial component, unlike the other regular 4-Vectors, which have all positive components.

4-Tensors can be constructed from the Tensor Outer Product of 4-Vectors
Some Basic 4-Vectors

Minkowski SpaceTime Diagram
Events & Dimensions

past
future
elsewhere

now · here

"Stack of Motion Picture Photos"

“LightCone

Classical
Mechanics

Δt time-like interval
Δr space-like interval

1/c
time displacement
Δt

3-displacement
Δr = Δr → (Δx,Δy,Δz)

Note the separate dimensional units: (time + 3D space)
Δt is [time], |Δr| is [length]

Δt time-like interval (+)
c light-like interval (0) = null
Δr space-like interval (-)

Note the matching dimensional units: (4D SpaceTime)
(cΔt) is [length/time]*[time] = [length], |Δr| is [length], |ΔR| is [length]
τ is the Proper Time = “rest-time”, time as measured by something not moving spatially
The Minkowski Diagram provides a great visual representation of SpaceTime

SR → QM

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

SR 4-Tensor
(2,0)-Tensor T^μ\nu
(1,1)-Tensor T^μ, or T^ν\nu
(0,2)-Tensor T^μ\nu

SR 4-Vector
(2,0)-Tensor V^μ = V = (v^0, v)
(1,1)-Tensor V^μ = (v^0, v)
(0,2)-Tensor V^μ = (v^0, -v)

SR 4-CoVector
(0,1)-Tensor S Lorentz Scalar

Classical (scalar) 3-vector
Galilean Invariant Not Lorentz Invariant

Trace[T^μν] = η_μνT^μν = T^μ_μ = T
V·V = V^μη_μνV^ν = (v^0)^2 - V·V = (v^0)^2 = Lorentz Scalar

Classical Mechanics

4-Displacement
ΔR = (cΔt, Δr)

4-Position
R = (ct, r)

Special Relativity

(cΔt)^2 Time-Like (+)
(Δr)^2 Space-like (-)

ΔR·ΔR = [(cΔt)^2 - Δr·Δr] = 0

Light-like:Null (0)
Some Basic 4-Vectors
Minkowski SpaceTime Diagram, WorldLines, LightSpeed to the Future!

The 4-Position is a particular type of 4-Displacement, for which the vector base is at the origin (0,0,0,0) = 4-Zero.

4-Position is Lorentz Invariant, but not Poincaré Invariant. A standard 4-Displacement is both.

\[ \Delta R \cdot \Delta R = [[(c\Delta t)^2 - \Delta \mathbf{r} \cdot \Delta \mathbf{r}]] = 0 \]

\[ \text{for time-like (+)} \]

\[ -(\Delta r_0)^2 \]

\[ \text{for light-like (0)} \]

\[ -((\Delta r_0)^2) \]

\[ \text{for space-like (−)} \]

An Event (*) is a point in SpaceTime The 4-Position points to an Event.

A WorldLine is a series of connected Events which trace out a path in SpaceTime, such as the track of a moving particle.

Massive particles move temporally into future at the speed-of-light (c) in their own rest-frame.

Massless particles (photonic) move nully into the future at the speed-of-light (c), and have no rest-frame.
SR Invariant Intervals

Minkowski Diagram: Lorentz Transform

Since the SpaceTime magnitude of $U$ is a constant ($c$), changes in the components of $U$ are like rotating the 4-Vector without changing its length. It keeps the same magnitude. Rotations, purely spatial changes, (eg. along x,y) result in circular displacements. Boosts, or temporal-spatial changes, (eg. along x,t) result in hyperbolic displacements. The interval between the origin and a given topograph-line is a Lorentz Invariant Constant.

$U \cdot U = \gamma(c,u) - \gamma(c,u) = \gamma^2(c^2 - u \cdot u) = (c^2)$

**Rotation (x,y): Purely Spatial**

**Boost (x,t): Spatial-Temporal**

The Minkowski Diagram provides a great visual representation of SpaceTime.
Since the SpaceTime magnitude of $U$ is a constant ($c$), changes in the components of $U$ are like rotating the 4-Vector without changing its length. It keeps the same magnitude. Rotations, purely spatial changes, {eg. along $x,y$} result in circular displacements. Boosts, or temporal-spatial changes, {eg. along $x,t$} result in hyperbolic displacements. The interval between the origin and a given topograph-line is a Lorentz Invariant Constant.

The Minkowski Diagram provides a great visual representation of SpaceTime.
SRQM: Some Basic 4-Vectors

4-Position, 4-Velocity, 4-Acceleration

SpaceTime Kinematics

**Special Relativity**
- \(|v| = |u| = 0 \rightarrow c\)
- \(\gamma = 1/\sqrt{1-(v/c)^2}\)

**Classical Mechanics**
- \(|v| = |u| \ll c\)
- \(\gamma \rightarrow 1 + O[(v/c)^2]\)
- \(\gamma' \rightarrow 0\)

- The relativistic Gamma factor \(\gamma = 1/\sqrt{1-(v/c)^2}\)
- The 1st order Newtonian Limit gives \(\gamma \sim 1 + O((v/c)^2)\)
- The 2nd order Newtonian Limit gives \(\gamma \approx 1 + (v/c)^2/2 + O((v/c)^4)\)

**SR 4-Tensor**
- (2,0)-Tensor \(T_{\mu \nu}\)
- (1,1)-Tensor \(T_{\mu \nu}^\sigma\) or \(T_{\mu \nu}\)
- (0,2)-Tensor \(T_{\mu \nu \rho \sigma}\)

- **SR 4-Vector**
  - (1,0)-Tensor \(V_{\mu} = (v^\mu, v)\)
  - (0,1)-Tensor \(V_\nu = (v_\nu, -v)\)

- **SR 4-Scalar**
  - Galilean Invariant
  - Lorentz Invariant

**SRQM Interpretation of QM**

**4-Vectors**
- \(R = <\text{Event}>\)
- \(U = dR/d\tau\)
- \(A = dU/d\tau\)

- **SR 4-Vector**
  - (1,0)-Tensor \(V_\nu = (v_\nu, -v)\)
  - **SR 4-CoVector**
    - (0,1)-Tensor \(V_\nu = (v_\nu, -v)\)

**4-Vector SRQM**
- **Interpretation of QM**
- **SciRealm.org**
- **John B. Wilson**

**Physics Symbols**
- **4-Gradient**
  - \(\partial = \partial_t/c - \nabla\)
  - \(\partial' = \partial_t/c - \nabla\)
- **ProperTime**
  - \(R \cdot U / U \cdot U = (ct, r \cdot \gamma(c, u) / c^2 = (c^2 t - r \cdot u) / c^2 = t_\gamma = \tau\)

**4-Vectors**
- **4-Position**
  - \(R = (ct, r)\)
- **4-Velocity**
  - \(U = \gamma(c, u)\)
- **4-Acceleration**
  - \(A = \gamma(c, \gamma' u + \gamma a)\)

**4-Position**
- \(R = (ct, r)\)

**4-Velocity**
- \(U = \gamma(c, u)\)

**4-Acceleration**
- \(A = \gamma(c, \gamma' u + \gamma a)\)

**Trace**
- \(\eta_{\mu \nu} T^{\mu \nu} = T_\mu^{\nu} = T\)
- \(V \cdot V = V_{\mu} V^{\mu} = (v^\mu v_\mu) = (v^2) = Lorentz Scalar\)
SRQM: Some Basic 4-Vectors

4-Position, 4-Velocity, 4-Acceleration, 4-Momentum, 4-Force

SpaceTime Dynamics

SR 4-Tensor
(2,0)-Tensor $T^{iv}$
(1,1)-Tensor $T^v$, or $T_v^\mu$
(0,2)-Tensor $T_{iv}$

SR 4-Vector
(1,0)-Tensor $V^\mu = V = (\gamma v, \mathbf{v})$

SR 4-CoVector
(0,1)-Tensor $V_\mu = (\gamma v_0, -\mathbf{v})$

SR 4-Scalar
(0,0)-Tensor $S$
Lorentz Scalar

ProperTime Derivative
$U \cdot \partial = \gamma (c, u) \cdot (\partial_t/c, -\nabla) = \gamma (\partial_t + u \cdot \nabla) = \gamma \frac{d}{d\tau}$

4-Gradient
$\partial = (\partial_t/c, -\nabla) \rightarrow (\partial_t/c, -\partial_x, -\partial_y, -\partial_z)$

This group of 4-Vectors are the main ones that are connected by the ProperTime Derivative.

$U \cdot \partial = d/d\tau = \gamma d/dt = \gamma (c \partial_t/c + u \cdot \nabla) = \gamma (\partial_t + u \cdot \nabla)$

The classical part of it, the convective derivative, $(\partial_t + u \cdot \nabla)$, is known by many different names:
The convective derivative is a derivative taken with respect to a moving coordinate system. It is also called the advective derivative, derivative following the motion, hydrodynamic derivative, Lagrangian derivative, material derivative, particle derivative, substantial derivative, substantive derivative, Stokes derivative, or total derivative.
SRQM: Some Basic 4-Vectors

4-Velocity, 4-Momentum, $E=mc^2$

**4-Velocity**

$U = \gamma(c, u)$

**Euler-Lagrange**

$E = mc^2$

**4-Momentum**

$P = (E/c, p) = (mc, p)$

Temporal part: $E = \gamma m c^2 = mc^2$

Spatial part: $p = \gamma m u = m u$

**Newtonian/Classical Limit**

Since time:space don’t mix in CM, Typically use energy $E$ & 3-momentum $p$ separately

The relativistic Gamma factor $\gamma = 1/\sqrt{1-(v/c)^2}$

The 1st order Newtonian Limit gives $\gamma \sim 1 + O((v/c)^2)$

The 2nd order Newtonian Limit gives $\gamma \sim 1 + (v/c)^2/2 + O((v/c)^4)$

For historical reasons, velocity can be represented by either $(v)$ or $(u)$

Temporal part: $E \sim (1+(v/c)^2/2)mc^2$ + $E_0$ + $O((v/c)^4)$

Spatial part: $p \sim (1)m_0 u = m_0 u \rightarrow m u$
SRQM: Some Basic 4-Vectors

4-Velocity, 4-Acceleration, SpaceTime Orthogonality

4-Velocity
\[ U = \gamma(c, u) \]

4-Acceleration
\[ A = \gamma(c' \gamma', u' + \gamma a) \]

ProperTime Derivative
\[ \frac{d}{d \tau} U = \gamma(c, u) \]

ProperTime Derivative
\[ \frac{d}{d \tau} \frac{d}{d \tau} U = A \]

4-Gradient
\[ \frac{d}{d \tau} \frac{d}{d \tau} U = \gamma \sqrt{1 - \frac{u^2}{c^2}} \]

ProperTime Derivative
\[ \frac{d}{d \tau} \frac{d}{d \tau} U = \gamma \sqrt{1 - \frac{u^2}{c^2}} \]

ProperTime Derivative
\[ \frac{d}{d \tau} \frac{d}{d \tau} U = \gamma \sqrt{1 - \frac{u^2}{c^2}} \]

The Lorentz Scalar Product can be used to show SpaceTime orthogonality when the result is zero.

\[ U \cdot U = c^2 \]

\[ d/d\tau [U \cdot U] = d/d\tau [c^2] = 0 \]

\[ d/d\tau [U \cdot U] = d/d\tau [U] \cdot U + U \cdot d/d\tau [U] = A \cdot U + U \cdot A = 2(U \cdot A) = 0 \]

\[ U \cdot A = U \cdot U' = 0 \]

SpaceTime Orthogonality

4-Velocity is the direction along a WorldLine.

4-Acceleration is the thing which causes a WorldLine to bend/curve.
SRQM: Some Basic 4-Vectors

4-Displacement, 4-Velocity, Relativity of Simultaneity

If Lorentz Scalar \( (\mathbf{U} \cdot \Delta \mathbf{X}) = 0 = c^2 \Delta t \), then the ProperTime displacement (\( \Delta \tau \)) is zero, and the event separation (\( \Delta \mathbf{X} = \mathbf{X}_2 - \mathbf{X}_1 \)) is orthogonal to the worldline \( \mathbf{U} \).

\( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are therefore simultaneous for the observer on this worldline \( \mathbf{U} \).

Examining the equation we get \( \gamma(c^2 \Delta t - \mathbf{u} \cdot \Delta \mathbf{x}) = 0 \). The coordinate time difference between the events is \( (\Delta t = \mathbf{u} \cdot \Delta \mathbf{x}/c^2) \).

The condition for simultaneity in an alternate frame (moving at 3-velocity \( \mathbf{u} \) wrt. the worldline \( \mathbf{U} \)) is \( \Delta t = 0 \), which implies \( (\mathbf{u} \cdot \Delta \mathbf{x}) = 0 \).

This can be met by:

(\( |\mathbf{u}| = 0 \)), the alternate observer is not moving wrt. the events, i.e. is on worldline \( \mathbf{U} \) or on a worldline parallel to \( \mathbf{U} \).

(\( ||\Delta \mathbf{x}|| = 0 \)), the events are at the same spatial location (co-local).

(\( \mathbf{u} \cdot \Delta \mathbf{x} = 0 \)), the alternate observer's motion is perpendicular (orthogonal) to the spatial separation \( \Delta \mathbf{x} \) of the events in that frame.

If none of these conditions is met, then the events will not be simultaneous in the alternate reference frame.

This is the mathematics behind the concept of Relativity of Simultaneity.
SR Diagram:
SR Motion * Lorentz Scalar
= Interesting Physical 4-Vector

4-Displacement
\[ \Delta R = (c \Delta t, \Delta r) \]

4-Motion
\[ \mathbf{U} \cdot \mathbf{\partial} = \gamma \frac{d}{d\tau} \mathbf{U} \]

4-Velocity
\[ \mathbf{U} = \gamma (c, \mathbf{u}) \]

4-Momentum
\[ \mathbf{P} = m(c, \mathbf{u}) = \frac{\mathbf{E}}{c} + \mathbf{p} = \frac{mc}{c^2} \mathbf{u} + \mathbf{p} \]

4-Wave Vector
\[ \mathbf{K} = \left( \frac{\omega}{c}, \mathbf{k} \right) = \left( \frac{\omega}{c}, \omega \mathbf{v}_{\text{phase}} \right) = \frac{c^2}{\omega} \]

4-Charge Flux
\[ \mathbf{J} = (\rho c, \mathbf{j}) = \rho(c, \mathbf{u}) \]

4-Current Density
\[ \mathbf{J} = (\rho c, \mathbf{j}) = \rho(c, \mathbf{u}) \]

4-Number Flux
\[ \mathbf{N} = (n c, n) = n(c, \mathbf{u}) \]

4-Position
\[ \mathbf{R} = (ct, \mathbf{r}) \]

Trace\[ T_{\mu \nu} = \eta_{\mu \nu} T = T_{\mu \nu} = T \]

Interesting note:
Most 4-Vectors have 4 independent components.
(1 temporal, 3 spatial)
The 4-Velocity has only the 3 spatial however, due to its invariant magnitude \( \mathbf{U} \cdot \mathbf{U} = c^2 \).

This fact allows one to multiply it by a Lorentz Scalar to make a new 4-Vector with 4 independent components, as shown in the diagram.

Proof of non-varying (c).

SR 4-Tensor
\[ (2,0)-\text{Tensor} \mathbf{T}^{\mu \nu} \]

SR 4-Vector
\[ (1,1)-\text{Tensor} \mathbf{T}^{\mu} = (v^\mu, v) \]

SR 4-CoVector
\[ (0,1)-\text{Tensor} \mathbf{V}_\mu = (v_\mu, -v) \]

SR 4-Scalar
\[ (0,0)-\text{Tensor} S \]

Lorentz Scalar
\[ \mathbf{\eta} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{\eta} \]

\[ (v^\mu, v) \cdot (v_\mu, -v) = (v^\mu v_\mu)^2 \]
SRQM Diagram:  
SRQM Motion * Lorentz Scalar  
= Interesting Physical 4-Vector

4-Displacement \( \Delta R = (\Delta c \Delta t, \Delta r) \)
4-Position \( R = (c t, r) \)

- **4-Position** \( R = (c t, r) \)
- **4-Displacement** \( \Delta R = (\Delta c \Delta t, \Delta r) \)

**4-Number Flux** \( N = (n c, n) = n (c, u) \)

**4-Probability Flux** \( J_{\text{prob}} = (p_{\text{prob}} c, j_{\text{prob}}) \)

**4-Charge Flux** \( J = (p c, j) = \rho (c, u) \)

**4-Current Density** \( J = (p c, j) = \rho (c, u) \)

**4-Probability Density** \( \rho = \chi \psi \)

**Born Rule** \( \rho_{\text{prob}} = \chi \psi \)

**EM Charge** \( q \)

**Electric-Magnetic** \( \frac{\varepsilon_0 c^2}{1/\mu_0} = c^2 \)

**Maxwell EM Wave Eqn**

\[ \varepsilon_0 c^2 (\varepsilon_0 c^2 - \mu_0) = \mu_0 \]

**4-Momentum** \( P = m (c, p) = (E/c, p) \)

**Rest Mass-Energy** \( E = mc^2 \)

**EM Frequency** \( \omega = c \sqrt{K} \)

**SRQM Diagram:**

- **4-Gradient** \( \vec{\partial} = (\partial/c, \vec{\nabla}) \)
- **4-Position Probability** \( \rho_0 \)
- **EM Charge** \( q \)
- **Born Rule** \( \rho_{\text{prob}} = \chi \psi \)
- **EM Frequency** \( \omega = c \sqrt{K} \)
- **Maxwell EM Wave Eqn**

**Proof of non-varying (c)**

- **Most 4-Vectors have 4 independent components.**
- **The 4-Velocity has only 3 spatial components.**

**Interesting note:**

- **The 4-Velocity** has only 3 independent components, as shown in the diagram.

**Existing SR Rules**

- **Quantum Principles**

**SR 4-Tensor**

- **T**\(4\times4\) (0,0)-Tensor \( S \)
- **T**\(4\times4\) (0,1)-Tensor \( V_\nu = (v_\nu, v) \)
- **T**\(4\times4\) (1,0)-Tensor \( V^\nu = (v^\nu, v) \)
- **T**\(4\times4\) (1,1)-Tensor \( T_{\nu \nu} = (v^\nu v, v) \)
- **T**\(4\times4\) (0,2)-Tensor \( V_{\mu \nu} = (v_\mu, v_\nu) \)

**SR 4-Vector**

- **T**\(4\times1\) (0,0)-Tensor \( S \)
- **T**\(4\times1\) (0,1)-Tensor \( V_{\nu} = (v_\nu, 0) \)

**SR 4-Scalar**

- **T**\(4\times4\) (0,0)-Tensor \( S \)
- **T**\(4\times1\) (0,0)-Tensor \( V = (v) \)

**Trace**

\[ T_{\nu \nu} = \eta_{\mu \nu} T_{\mu \nu} = T_{\nu \nu} = T \]

**Proof:**

\[ V \cdot V = V^\nu V_\nu = (v^\nu v_\nu) = (v^2) \]

**Lorentz Scalar**
SRQM Diagram: ProperTime Derivative

Very Fundamental Results

Conservation of the 4-Velocity Flow (4-Velocity Flow-Field)

\[ d/dτ (\gamma c U) = 0 \]

4-Acceleration of an Event is the thing which causes a WorldLine to bend.
4-Velocity is the direction of an Event along a WorldLine.

\[ \frac{d}{dτ} (c U) = 0 \]

\[ \text{ProperTime Derivative} \]

\[ U \cdot \partial [ \ldots ] \]

\[ \gamma d/dτ [ \ldots ] \]

\[ 4-Momentum \]

\[ P = (E/c, p) = (mc, p) \]

\[ 4-Force \]

\[ F = \gamma (E/c, f) \]

\[ \text{4-Vectors:} \]

\[ R = <\text{Event}> \]

\[ U = dR/dτ \]

\[ A = dU/dτ \]

\[ P = m_o U \]

\[ F = dP/dτ \]

\[ \text{Trace}[T^\mu_\nu] = \eta^\mu_\nu T^\mu_\nu = T_{\mu}^\mu = T \]

\[ V \cdot V = V^\mu \eta^\mu_\nu V_\nu = (V^\mu) \eta^\mu_\nu (V^\nu) = (V^\mu)^2 \]

\[ = \text{Lorentz Scalar} \]

4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]
\[ dR = (c dt, dr) \]

4-Position
\[ R = (ct, r) \]

\[ \partial \cdot R = 4 \text{ SpaceTime Dimension} \]

4-Vectors:

\[ \partial \cdot R = 4 \text{ SpaceTime Dimension} = 4 \]

\[ d/dτ (\partial \cdot R) = d/dτ (4) = 0 \]

\[ (U \cdot \partial) (\partial \cdot R) = (U \cdot \partial) (4) = 0 \]

\[ \gamma d/dτ [\ldots] \]

\[ 4-Velocity \]

\[ U = \gamma (c, u) \]

\[ d/dτ \]

\[ 4-Momentum \]

\[ P = (E/c, p) = (mc, p) \]

\[ 4-Force \]

\[ F = \gamma (E/c, f) \]

\[ \partial \cdot R = 0 \text{ Conservation of the 4-Velocity Flow (4-Velocity Flow-Field)} \]
SRQM Diagram: Local Continuity of 4-Velocity leads to all the Conservation Laws

Conservation Laws:
All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local spatial region.

Conservation of Charge:
\[ \partial \cdot J = 0 \]

Example:
\[ \partial \cdot (\rho_c) U = 0 \]
\[ \partial \cdot J = 0 \]
\[ (\partial / c \cdot p c + \nabla \cdot j) = 0 \]
\[ (\partial p + \nabla \cdot j) = 0 \]

= Conservation of Charge
= A Continuity Equation

\[ \Delta R = (c \Delta t, \Delta r) \]
\[ \Delta R = (c dt, dr) \]

SR 4-Tensor
(2,0)-Tensor \( T^{\mu}_{\nu} \)
(1,1)-Tensor \( T^\nu \), or \( T_\nu^\nu \)
(0,2)-Tensor \( T_{\mu \nu} \)

SR 4-Vector
(1,0)-Tensor \( V^\nu = (\vec{v}, v) \)

SR 4-CoVector
(0,1)-Tensor \( V_\nu = (v_0, \vec{v}) \)

SR 4-Scalar
(0,0)-Tensor \( S \)

Lorentz Scalar

Trace[\( T^{\nu}_{\nu} \)] = \( \eta_{\mu \nu} T^{\mu}_{\nu} = T^\nu = T \)
\[ V \cdot V = V^\nu \eta_{\nu \nu} V^\mu = (v_0^2 - \vec{v} \cdot \vec{v}) = (v_0)^2 \]

= Lorentz Scalar
SRQM Diagram: SRQM Motion * Lorentz Scalar

Conservation Laws, Continuity Eqns

Conservation Laws:

All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local spatial region.

Conservation of Charge: $\partial \cdot J = (\partial / c \cdot \nabla) \cdot J = 0$

Conservation of Mass: $\partial \cdot \rho = 0$

Conservation of Energy: $\partial \cdot (\rho c^2 v) = 0$

Conservation of Momentum: $\partial \cdot (\rho c v^2) = 0$

Conservation of Angular Momentum: $\partial \cdot (\rho c^2 \omega) = 0$

Conservation of ProperTime: $\partial \cdot (\rho c) = 0$

These are Fluid or Density-type Conservation/Continuity Laws

These are Individual Particle/Wave/Delta-function Conservation/Continuity Laws

SR 4-Tensor

(2,0)-Tensor $T^{\mu\nu}_{\alpha\beta}$

(1,1)-Tensor $T^m$ or $T^m_{\alpha\beta}

(0,2)-Tensor $T_{\mu\nu\rho}$

SR 4-Vector

(1,0)-Tensor $V^m = (v^m, v_0)$

SR 4-CoVector

(0,1)-Tensor $V_m = (v_0, v^m)$

SR 4-Scalar

(0,0)-Tensor $S$ Lorentz Scalar

Existing SR Rules

Quantum Principles

SR → QM

4-Vector SRQM Interpretation of QM

SciRealm.org

John B. Wilson

4-Gradient $\partial = (\partial / c \cdot \nabla)$

4-Divergence $\partial \cdot J = (\partial / c \cdot \nabla) \cdot J$

4-Density $\rho$

4-Momentum $P = (m, p) = (mc, c \rho)$

4-WaveVector $k = (\omega, \kappa) = (\omega, \kappa / c)$

4-Scalar Field $\psi$

4-Probability $\rho$

4-Probability Density $p_{\rho}$

4-Displacement $\Delta R = (c \Delta t, \Delta r)$

4-Position $R = (ct, r)$

4-Momentum Density $j = (p, c, \rho)$

4-Mass Density $n = (n, c, u)$

4-Curvature Density $\kappa = (\kappa, c, j)$

4-Gradient $\nabla$

4-Scalar Field $\phi$

4-Vector Field $A = (\phi, \psi)$

4-Probability Flux $\phi = (\phi, \psi)$

4-Mass Flux $\rho = (\rho, c)$

4-Charge Flux $\rho = (\rho, c, u)$

4-Current Density $J = (pc, \rho)$

4-Displacement Gradient $\partial R = (\partial / c \cdot \nabla)$

Minkowski Metric $\eta^{\mu\nu}$

SRQM Diagram:

SRQM Motion * Lorentz Scalar
SRQM: Some Basic 4-Vectors

4-Velocity, 4-Gradient, Time Dilation

The Minkowski Diagram provides a great visual representation of SpaceTime.

at-rest worldline \( U_0 \)
(\( u=0 \))
fully temporal

const inertial motion worldline \( U \)
(\( 0<u<c \))
trades some time for space

Since the SpaceTime magnitude of \( U \) is a constant, changes in the components of \( U \) are like “rotating” the 4-Vector without changing its length. However, as \( U \) gains some spatial velocity, it loses some “relative” temporal velocity. Objects that move in some reference frame “age” more slowly relative to those at rest in the same reference frame.

Time Dilation!

\[
\Delta t = \gamma \Delta \tau = \gamma \Delta t_0,
\]
\[
dt = \gamma d\tau
\]
\[
d/d\tau = \gamma d/dt
\]

Each observer will see the other as aging more slowly; similarly to two people moving oppositely along a train track, seeing the other as appearing smaller in the distance.

\[
\mathbf{V} \cdot \mathbf{V} = \eta_{\nu\mu} V^\nu V^\mu = \left( v^0 \right)^2 - \mathbf{v} \cdot \mathbf{v} = \left( v^0 \right)^2
\]

is Lorentz Scalar

\[
\text{Trace}[T^\nu] = \eta_{\nu\mu} T^\mu = T^\nu_\nu = T
\]
SRQM: Some Basic 4-Vectors

SR 4-WaveVector $K$

A Tensor Study of Physical 4-Vectors

4-WaveVector, aka. Wave 4-Vector, solution of d'Alembertian Wave Eqn.

$$K = (ω/c, k) = (ω/c, ω/ν = \text{phase}) = (ω/c, ων/c^2) = (ω/c^2)(c, ω) = (ω/c)(1, β) = (1/cT, \hat{n}/λ) = -\partial[Φ_{\text{phase, plane}}]$$

There are multiple ways of writing out the components of the 4-WaveVector, with each one giving an interesting take on what the 4-WaveVector means.

An SR wave $Ψ$ is actually composed of two tensors:

1) 4-Vector propagation part = $K^\mu$, (the engine)
2) Variable amplitude part = $A$ (the load), depends on what is waving...

4-Scalar $A$: $Ψ = A e^{-i(ω/c)\hat{n}/λ}$

ex. KG Quantum Wave

4-Vector $A^\mu$: $Ψ^\mu = A^\mu e^{-i(ω/c)\hat{x}_n}$

ex. Maxwell Photon Wave

4-Tensor $A^{μν}$: $Ψ_{μν} = A^{μν} e^{-i(ω/c)\hat{x}_n}$

ex. Gravitational Wave Approx.

The $Ψ$ tensor-type will match the tensor-type as the propagation part $e^{-i(ω/c)\hat{x}_n}$ is overall dimensionless.

One comparison I find very interesting is:

$$R \cdot R = (ct)^2 - r^2 = (ct)^2$$

$K \cdot K = (1/cT)^2 = (c^2/c^2)^2$ (for photon)

$$\partial \partial = (∂/c\partial τ)^2 = (c/\partial τ)^2$$

I believe the last one is correct: $(∂/c\partial τ)^2 R = 0 = (∂/c\partial τ)^2 = A_{μν}/c = 0$: The 4-Acceleration seen in the ProperTime Frame = RestFrame = 0

Normally $(∂/c\partial τ)^2 R = A_{μν}/c$, which could be non-zero. But that is for the total derivative, not the partial derivative.

SR 4-Tensor

(2,0)-Tensor $T^{μν}$
(0,2)-Tensor $T^{νμ}$
(1,1)-Tensor $T^{ν} = V = (ν^2, ν)$
(0,1)-Tensor $V = (ν_0)$

SR 4-Vector

$V^{ν} = v = (ν^2, ν)$

SR 4-Scalar

Lorentz Scalar

In Kaluza-Klein Theory, the 4-Dimension $n$ = 4 is a compactification of $n = 5$. The 5-Dimension $S$ is a Lorentz Scalar.
SRQM: Some Basic 4-Vectors

4-Velocity, 4-WaveVector

Wave Properties, Relativistic Doppler Effect

4-Velocity, 4-WaveVector

\[ \mathbf{U} = \gamma(c, u) \]
\[ \mathbf{K} = \gamma(c, k) \]
\[ \mathbf{V} = \mathbf{V} = (\mathbf{v}, \mathbf{v}) \]

\[ \mathbf{K} \cdot \mathbf{U} = \gamma(\omega - k \cdot u) = \omega_0 \]
\[ \mathbf{K} \cdot \mathbf{K} = (\omega_0/c)^2 \]

\[ \omega_0/c^2 \]

\[ \mathbf{4-Velocity} \]
\[ \mathbf{4-WaveVector} \]

Relativistic SR Doppler Effect

\( (\mathbf{n}) \) here is the unit-directional 3-vector of the photon

Choose an observer frame for which:

\[ \mathbf{K} = (\omega/c, k), \text{ with } k, \mathbf{n} \text{ pointing toward observer} \]

\[ \mathbf{U}_{\text{obs}} = (c, 0) \]
\[ \mathbf{K} \cdot \mathbf{U}_{\text{obs}} = (\omega/c, k) \cdot (c, 0) = \omega = \omega_{\text{obs}} \]

\[ \mathbf{U}_{\text{emit}} = \gamma(c, u) \]
\[ \mathbf{K} \cdot \mathbf{U}_{\text{emit}} = (\omega/c, k) \cdot \gamma(c, u) = \gamma(\omega - k \cdot u) = \omega_{\text{emit}} \]

\[ \mathbf{K} \cdot \mathbf{U}_{\text{obs}} / \mathbf{K} \cdot \mathbf{U}_{\text{emit}} = \omega_{\text{obs}} / \omega_{\text{emit}} = \omega / [\gamma(\omega - k \cdot u)] \]

For photons, \( \mathbf{K} \) is null \( \rightarrow \mathbf{K} \cdot \mathbf{K} = 0 \rightarrow k = (\omega/c)\mathbf{n} \)

\[ \omega_{\text{obs}} / \omega_{\text{emit}} = \omega / [\gamma(\omega - (\omega/c)\mathbf{n} \cdot u)] = 1/[\gamma(1 - \mathbf{n} \cdot \beta)] = 1/[\gamma(1 - |\beta| \cos(\theta_{\text{obs}}))] \]

\[ \omega_{\text{obs}} / \omega_{\text{emit}} = \gamma \omega_{\text{obs}} / \omega_{\text{emit}} = \omega_{\text{obs}} / \omega_{\text{emit}}^\circ \]

\[ \omega_{\text{obs}} = \omega_{\text{emit}} / [\gamma(1 - \mathbf{n} \cdot \beta)] = \omega_{\text{emit}} * [1 + |\beta|] * [1 - |\beta|] / (1 - \mathbf{n} \cdot \beta) \]

with \( \gamma = 1/\sqrt{1 - \beta^2} = 1 / (\sqrt{1 + |\beta|^2} * \sqrt{1 - |\beta|^2}) \)

For motion of emitter \( \beta \): (in observer frame of reference)

Away from obs, \( (\mathbf{n} \cdot \beta) = -\beta \), \( \omega_{\text{obs}} = \omega_{\text{emit}} * [1 - |\beta|] / \sqrt{1 + |\beta|} = \text{Red Shift} \)

Toward obs, \( (\mathbf{n} \cdot \beta) = +\beta \), \( \omega_{\text{obs}} = \omega_{\text{emit}} * [1 + |\beta|] / \sqrt{1 - |\beta|} = \text{Blue Shift} \)

Transverse, \( (\mathbf{n} \cdot \beta) = 0 \), \( \omega_{\text{obs}} = \omega_{\text{emit}} * \gamma = \text{Transverse Doppler Shift} \)

The Phase Velocity of a Photon \( \{v_{\text{phase}} = c\} \) equals the Particle Velocity of a Photon \( \{u = c\} \)

The Phase Velocity of a Massive Particle \( \{v_{\text{phase}} > c\} \) is greater than the Velocity of a Massive Particle \( \{u < c\} \)
SRQM: Some Basic 4-Vectors

4-Velocity, 4-WaveVector

Wave Properties, Relativistic Aberration

Relativistic SR Doppler Effect

\( \omega_{obs} = \omega_{emit}/[\gamma(1 - \hat{n} \cdot \beta)] = \omega_{emit}/[\gamma(1 - |\beta|\cos[\theta_{emit}])] \)

Change reference frames with \{obs→emit\} & \{β → -β \}

\( \omega_{emit} = \omega_{obs}/[\gamma(1 + \hat{n} \cdot \beta)] = \omega_{obs}/[\gamma(1 + |\beta|\cos[\theta_{emit}])] \)

\((\omega_{obs})^*(\omega_{emit}) = (\omega_{obs}/[\gamma(1 - |\beta|\cos[\theta_{obs}]))^*(\omega_{emit}/[\gamma(1 + |\beta|\cos[\theta_{emit}])]) \)

1 = \((1/[\gamma(1 - |\beta|\cos[\theta_{obs}]))^*(1/[\gamma(1 + |\beta|\cos[\theta_{emit}])]) \)

1 = \(\gamma(1 - |\beta|\cos[\theta_{obs}])^*(\gamma(1 + |\beta|\cos[\theta_{emit}]) \)

1 = \(\gamma^2(1 - |\beta|\cos[\theta_{obs}])^*(1 + |\beta|\cos[\theta_{emit}]) \)

Solve for \(|\beta|\cos[\theta_{obs}]) and use \((\gamma^2-1) = \beta^2\gamma^2 \)

Relativistic SR Aberration Effect

\( \cos[\theta_{obs}] = (\cos[\theta_{emit} + |\beta|]) / (1 + |\beta|\cos[\theta_{emit}] ) \)

The Phase Velocity of a Massive Particle \( \{v_{phase} > c\} \) is greater than the Velocity of a Massive Particle \( \{u < c\} \)
SRQM: Some Basic 4-Vectors

4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

\[
\{ P = (E/c, p) = -\partial[S] = (-\partial/c\partial[S], V[S]) \}
\]
{temporal component} $E = -\partial t[S]$ = $-\partial t[S]$
{spatial component} $p = V[S]$

**Note** This is the Action ($S_{\text{action}}$) for a free particle.
Generally Action is for the 4-TotalMomentum $P_T$ of a system.

4-Momentum
\[
P = (mc, p) = (E/c, p)
\]
\[
P = -\partial[S_{\text{action,free}}]
\]

4-Gradient
\[
\partial = (\partial/c, -\nabla) \rightarrow (\partial/c, -\partial_x, -\partial_y, -\partial_z)
\]
d'Alembertian
\[
\partial^2 = (\partial/c)^2 - \nabla \cdot \nabla = (\partial/c)^2
\]

Treating motion like a particle
Moving particles have a 4-Velocity
4-Momentum is the negative 4-Gradient of the SR Action ($S$)

\[
\partial R = 4 \text{ Dimension}
\]

\[
\partial[R] = \eta^{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1]
\]

Minkowski Metric

ProperTime

Derivative

SpaceTime

\[
\gamma d\tau \rightarrow \gamma d\tau / c^2
\]

\[
U = \gamma(c, u)
\]

4-Vector SRQM Interpretation of QM

SR 4-Tensor
(2,0)-Tensor $T^{iv}$
(1,1)-Tensor $T^{iv}$, or $T^{i}$
(0,2)-Tensor $T_{iv}$

SR 4-Vector
(1,0)-Tensor $V^i = (\vec{V}, v)$

SR 4-CoVector
(0,1)-Tensor $V_i = (v^i, -\vec{V})$

SR 4-Scalar
(0,0)-Tensor $S$

Lorentz Scalar

Trace[$T^{iv}$] = $\eta_{iv} T^{iv} = T^{i}$

$V \cdot V = V^i \eta_{iv} V^v = (v^i)^2 - \vec{V} \cdot \vec{V} = (v^i)^2$

= Lorentz Scalar

See SR Wave Definition
for info on the Lorentz Scalar Invariant SR WavePhase.
\[
\{ K = (\omega/c, k) = -\partial[\Phi] = (-\partial/c\partial[S], V[S]) \}
\]
{temporal component} $\omega = -\partial t[S]$ = $-\partial t[S]$
{spatial component} $k = V[S]$

**Note** This is the Phase ($\Phi$) for a single plane-wave.
Generally WavePhase is for the 4-TotalWaveVector $K_T$ of a system.
Some Cool Minkowski Metric Tensor Tricks

4-Gradient, 4-Position, 4-Velocity

SpaceTime is 4D

U·∂=γ(c, u) (∂/c, -∇)=γ(∂/c, -u·∇)

= d/dτ = γd/dt

δ[R] = ημν

→Diag[1,1,1,1]

Minkowski Metric

δ·R=4

Space Time Dimension

ημν

→Diag[1,1,1,1]

Index-Lowered Minkowski Metric

δμν

=Diag[1,1,1,1]

Kronecker Delta

{ημμ}

= 1/{ημμ}

Trace[Minkowski Metric] = Tr[ημν] = ημμ = T = 4

Thus

The Divergence of 4-Position (δ·R) = “Magnitude” of the Minkowski Metric Tr[ημν] = the Dimension of SpaceTime (4)

(U·∂)[R] = (Uα∂α)[R] = (Uαηαβ∂β)[R] = (Uα∂α)[R] = (Uα)∂α[R] = (Uα)ηαβ = Uβ = U = (d/dτ)[R]

Thus

Lorentz Scalar Product (U·∂) = Derivative wrt. ProperTime (d/dτ) = Relativistic Factor * Derivative wrt. CoordinateTime γ(d/dt):

SR 4-Tensor

(2,0)-Tensor Tμν

SR 4-Vector

(1,0)-Tensor Vμ = V = (v0, v)

SR 4-CoVector

(0,1)-Tensor Vμ = (v0, -v)

SR 4-Scalar

(0,0)-Tensor S

Lorentz Scalar

Trace[Tμν] = ημνTμν = Tμ = T

V·V = VμVμ = [(v0)² - V·V] = (v0)²

= Lorentz Scalar
SRQM+EM Diagram: 4-Vectors

4-Displacement
$$ \Delta R = (c \Delta t, \Delta r) $$

4-Position
$$ R = (ct, r) $$

4-UnitTemporal
$$ T = \gamma(1, \beta) $$

4-NumberFlux
$$ N = (nc, n) = n(c, u) $$

4-ChargeFlux
$$ J = (pc, j) = p(c, u) $$

4-Gradient
$$ \partial = (\partial_t/c, -\nabla) $$

4-Acceleration
$$ A = \gamma(c', \gamma' u + \gamma a) $$

4-Displacement
$$ dR = (cdt, dr) $$

4-ProbabilityFlux
$$ J_{prob} = (\rho_{prob} c, j_{prob}) $$

4-MassFlux
$$ N = (nc, n) = n(c, u) $$

4-ProbCurrDensity
$$ J_{prob} = (\rho_{prob} c, j_{prob}) $$

4-Polarization: Spin
$$ E = (e^0, e) = (\epsilon \cdot \beta, \epsilon) $$

4-ProbabilityFlux
$$ J = (pc, j) = p(c, u) $$

4-CurrentDensity
$$ J = (pc, j) = p(c, u) $$

4-ForceDensity
$$ F_{den} = \gamma(E_{den}', c, f_{den}) $$

4-MassFlux
$$ G = (\rho_{mass} c, g) = (\rho_{mass} c, g) $$

4-MomentumDensity
$$ G = (\rho_{mass} c, g) = (\rho_{mass} c, g) $$

4-Force
$$ F = \gamma(E/c, f) $$

4-ProbabilityFlux
$$ J = (pc, j) = p(c, u) $$

4-Momentum
$$ P = (mc, p) = (E/c, p) $$

4-TotalMomentum
$$ P_T = (E/c, p_T) = (H/c, p_T) $$

4-TotalWaveVector
$$ K_T = (\omega/c, k_T) $$

4-WaveVector
$$ K = (\omega/c, k) $$

4-TotalWaveVector
$$ K_T = (\omega/c, k_T) $$

4-Gradient
$$ \partial = (\partial/c, -V) $$

4-UnitSpatial
$$ S = \gamma_{\beta n}(\hat{n} \cdot \beta, \hat{n}) $$

4-Displacement
$$ dR = (cdt, dr) $$

4-Force
$$ F = \gamma(E/c, f) $$

4-Momentum
$$ P = (mc, p) = (E/c, p) $$

4-EMVectorPotential
$$ A = (\phi/c, a) $$

4-Displacement
$$ \Delta R = (c \Delta t, \Delta r) $$

4-MassFlux
$$ N = (nc, n) = n(c, u) $$

4-MomentumField
$$ P_f = (E/f, p_f) $$

4-4-MomentumField
$$ P_f = (E_f/c, p_f) $$

4-WeakFlux
$$ Q = (U/c, q) = qA $$

SR → QM

Existing SR Rules

Quantum Principles
SRQM+EM Diagram: 4-Vectors, 4-Tensors
Lorentz Scalars / Physical Constants

- 4-Displacement: $\Delta R=(\Delta x, \Delta y, \Delta z)$
- 4-Acceleration: $\mathbf{a} = \frac{d^2 \mathbf{r}}{dt^2}$
- 4-Polarization: Spin
  $\mathbf{S} = s\mathbf{\beta}$
- 4-Position: $\mathbf{r} = (x, y, z)$
- 4-Momentum: $\mathbf{p} = (\text{mc}, \text{E})$
- 4-Force: $\mathbf{F} = \nabla \mathbf{U}$
- 4-Probability Flux: $\mathbf{J} = \nabla \rho$
- 4-EM Vector Potential: $\mathbf{A} = (\text{E}, \text{B})$
- 4-Mass Flux: $\mathbf{\rho} = \nabla \text{mc}$
- 4-Charge Flux: $\mathbf{J} = (\text{p}, \mathbf{c})$
- 4-Current Density: $\mathbf{J} = \mathbf{p}_\text{c}$
- 4-Probability Density: $\rho = \eta \mathbf{\beta}$
- 4-Number Flux: $\mathbf{N} = \text{nc}$
- 4-Unit Temporal: $T = \gamma(1, 0)$
- 4-Unit Spatial: $S = \gamma(0, \mathbf{1})$
- 4-Gradient: $\nabla$
- 4-Trace: $\text{Trace}[T_{\mu\nu}] = T_{\mu\mu}$

SR Perfect Fluid
$T^\mu_{\nu} = (\rho_0 + p_0/c^2)U^\mu U^\nu - (p_0)\eta^\mu_{\nu}$

SR Conservation of Energy-Momentum:
$\partial_v T^\mu_{\nu} = 0$

Maxwell EM Wave Eqn:
$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}$
$\nabla \cdot \mathbf{B} = 0$

EM Faraday:
$F_{\mu\nu} = 2\text{mc} (\mathbf{E}, \mathbf{B})$

Conservation of Electric Charge:
$\rho = \nabla \cdot \mathbf{J}$

Conservation of Momentum:
$\mathbf{P} = \nabla \times \mathbf{F}$

Trace of Stress Tensor:
$\text{Trace}[T_{\mu\nu}] = T_{\mu\mu}$

Maxwell Equations:
$\nabla \cdot \mathbf{E} = \rho_\text{den}$
$\nabla \times \mathbf{B} = \mathbf{J}$

SR Conservation of Energy-Momentum:
$\partial_v T_{\mu\nu} = 0$

Conservation of Momentum:
$\mathbf{P} = (\text{mc}, \text{E})$

Conservation of Energy:
$\rho = \nabla \cdot \mathbf{J}$

SR Conservation of Charge:
$\mathbf{P}_{\text{em}} = 0$

Conservation of Momentum:
$\mathbf{P} = (\text{mc}, \text{E})$

Maxwell Equations:
$\nabla \cdot \mathbf{E} = \rho_\text{den}$
$\nabla \times \mathbf{B} = \mathbf{J}$

Trace of Stress Tensor:
$\text{Trace}[T_{\mu\nu}] = T_{\mu\mu}$

Maxwell Equations:
$\nabla \cdot \mathbf{E} = \rho_\text{den}$
$\nabla \times \mathbf{B} = \mathbf{J}$

SR Conservation of Charge:
$\mathbf{P}_{\text{em}} = 0$

Conservation of Momentum:
$\mathbf{P} = (\text{mc}, \text{E})$

Maxwell Equations:
$\nabla \cdot \mathbf{E} = \rho_\text{den}$
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$\mathbf{P}_{\text{em}} = 0$

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$\nabla \cdot \mathbf{E} = \rho_\text{den}$
$\nabla \times \mathbf{B} = \mathbf{J}$

Trace of Stress Tensor:
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Maxwell Equations:
$\nabla \cdot \mathbf{E} = \rho_\text{den}$
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SR Conservation of Charge:
$\mathbf{P}_{\text{em}} = 0$

Conservation of Momentum:
$\mathbf{P} = (\text{mc}, \text{E})$

Maxwell Equations:
$\nabla \cdot \mathbf{E} = \rho_\text{den}$
$\nabla \times \mathbf{B} = \mathbf{J}$

Trace of Stress Tensor:
$\text{Trace}[T_{\mu\nu}] = T_{\mu\mu}$

Maxwell Equations:
$\nabla \cdot \mathbf{E} = \rho_\text{den}$
$\nabla \times \mathbf{B} = \mathbf{J}$

SR Conservation of Charge:
$\mathbf{P}_{\text{em}} = 0$

Conservation of Momentum:
$\mathbf{P} = (\text{mc}, \text{E})$
SRQM+EM Diagram: 4-Vectors, 4-Tensors

Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

4-UnitSpatial

T-S=0

4-UnitTemporal

T=γ(1,β)

A Tensor Study of Physical 4-Vectors

4-Displacement

ΔR=(Δt,Δr)

4-Position

R=(ct,r)

4-Displacement

ΔR=(Δt,Δr)

dR=(cdt,dr)

4-Position

R=(ct,r)

Derivative

\( \partial \bullet \)

\( \Delta \)

\( \rho \)

\( p \)

\( \mu \)

\( \nu \)

\( \varepsilon \)

\( \gamma \)

\( \omega \)

\( \varphi \)

\( \beta \)

\( \theta \)

\( \eta \)

\( \lambda \)

\( \kappa \)

\( \lambda \)

\( \zeta \)

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\( \psi \)

\( \phi \)

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\( \delta \)

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SRQM Diagram:
Physical Constants Emphasized

\[ \delta \mathbf{R} = \eta^{\mu \nu} \mathbf{R} \]  
\text{SpaceTime Dimension}

\[ K \cdot R = - \Phi_{\text{phase}}(i) \]  
\text{Minkowski Metric}

\[ P \cdot T \cdot R = -S_{\text{action}} \]  
\text{Derivative}

\[ q \eta_{\mu \nu} \partial \partial R = 4 \]  
\text{SpaceTime Dimension}

\[ \partial \partial R = \eta_{\mu \nu} \]  
\text{Diag[1,-1,-1,-1]}

\[ \text{Minkowski Metric} \]

\[ -p_o \]  
\text{ProperTime}

\[ \omega_o / c^2 \]  
\text{ProperTime}

\[ n_o \mathbf{E}_o / c^2 \]  
\text{ProperTime}

\[ \varphi_o / c^2 \]  
\text{ProperTime}

\[ m_o \mathbf{E}_o / c^2 \]  
\text{ProperTime}

Notice that all the main “Universal” or “Fundamental” Physical Constants are here: \( G, c, \hbar, \varepsilon_o, \mu_o \).

Some depend on the actual particle type: \( q, m_o, \omega_o \)

Some depend on regional conditions: \( \tau, \rho_{\text{eo}}, \rho_o, \psi^* \psi \)

Some depend on interaction: \( \Phi_{\text{phase}} \)

Some are mathematical: \( 0, 4, \pi, i, \text{Diag[1,-1,-1,-1]}, d/dt \)

Conservation Laws are also a type of “zero” constant in this regard.

The majority of the constants are Lorentz Scalars, but some are 4-Vector or 4-Tensor, and all are valid for all inertial observers.

Fundamental Physical Constants are SR Lorentz Scalars

The fact that these “tie together” a network of 4-Vectors is a good argument for why their values are constant. Changing even one would change the relationship properties among all of the 4-Vectors.
SRQM Diagram: Projection Tensors
Temporal, Spatial, Null, SpaceTime

Projection Tensors act as follows:
Generic 4-Vector:
\( A^\nu = (a^0, a^1, a^2, a^3) \)

Temporal Projection:
\( V^\mu_{\nu} = \eta_{\omega\nu} V^\mu_{\omega} \rightarrow \text{Diag}[1, 0, 0, 0] \)
\( V^\mu_{\nu} A^\nu = (a^0, 0, 0, 0) = (a^3, 0) \)

Spatial Projection:
\( H^\mu_{\nu} = \eta_{\omega\nu} H^\mu_{\omega} \rightarrow \text{Diag}[0, 1, 1, 1] \)
\( H^\mu_{\nu} A^\nu = (0, a^1, a^2, a^3) = (0, a) \)

SpaceTime Projection:
\( V^\mu_{\nu} A^\nu + H^\mu_{\nu} A^\nu = \eta^\nu_{\nu} A^\nu = \delta^\nu_{\nu} A^\nu = (a^0, a) \)

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.

\[ \text{Trace}[V^\mu_{\nu}] = \eta_{\mu\nu} V^\mu_{\nu} = T^\nu_{\nu} = T \]
\[ V \cdot V = V^\mu_{\nu} \eta_{\nu\nu} V^\mu = (v^0)^2 - \mathbf{v} \cdot \mathbf{v} = (v^0)^2 \]

Lorentz Scalar
SRQM Diagram: Projection Tensors & Perfect-Fluid Stress-Energy Tensor

Projection Tensors act as follows:

\[ A^i = (a^0, a) = (a^0, a^1, a^2, a^3) \]

\[ V^i = \eta_{\mu \nu} V^{\mu \nu} \rightarrow \text{Diag}[1,0,0,0] \]

\[ V^i, A^i = (a^0, a^0, 0, 0) = (a^0, 0) \]

\[ H^i, A^i = \eta_{\mu \nu} H^{\mu \nu} \rightarrow \text{Diag}[0,1,1,1] \]

\[ H^i, A^i = (0, a^1, a^2, a^3) = (0, a) \]

\[ V^i, A^i + H^i, A^i = \eta_{\mu \nu} A^\mu A^\nu = (a^0, a) \]

\[ V^i + H^i, V^i = \eta_{\mu \nu} V^\mu V^\nu = \eta_{\mu \nu} V^\mu V^\nu \]

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.

The rest-energy-density \( p_{eo} \) is the Temporal Projection.

The neg rest-pressure \(-p_o\) is the Spatial Projection.

\[ T^{\mu \nu}_{\text{rest}} \rightarrow \text{Diag}[p_{eo}, p_o, p_o, p_o] \]

Perfect-Fluid StressEnergy 4-Tensor:

\[ T_{\text{rest}}^{\mu \nu} = (p_{eo} V^{\mu \nu} + (-p_o) H^{\mu \nu}) \]

4-Unit Temporal Tensor \( T = \gamma(1, \beta) \)

4-Vector \( R = (ct, r) \)

4-Gradient \( \partial \cdot R = 4 \) SpaceTime Dimension

Projection Tensors act as follows:

\[ \partial \Delta \tau = (\partial/c, -\nabla) \]

Perfect-Fluid StressEnergy 4-Tensor:

\[ T^{\mu \nu} = ((p_{eo} + p_o)/c^2) U^\mu U^\nu - (p_o) \eta^{\mu \nu} \]

T^{\mu \nu} = (p_{eo}) V^{\mu \nu} - (p_o) H^{\mu \nu}

can be written in much simpler form using Projection Tensors:

\[ (p_{eo}) V^{\mu \nu} - (p_o) H^{\mu \nu} \]

\[ \text{Diag}[1,0,0,0] \rightarrow \text{Diag}[1,1,1,1] \]

Temporal "Vertical" Projection Tensor

\[ \text{Diag}[0,-1,-1,-1] \rightarrow \text{Diag}[0,1,1,1] \]

Spatial "Horizontal" Projection Tensor

4-Position \( R = (ct, r) \)

4-Metric \( U = \gamma(c, u) \)

4-Velocity \( U \cdot \partial = d/d\tau = \gamma d/dt \)

Derivative

ProperTime

Diag\[\eta\] = Lorentz Scalar

\[ \Delta t \]

Time-like Interval \( (-) \)

\( V^\mu V^\nu \)

"Vertical" Temporal Projection

\[ \Delta r \]

Light-like Interval \( (0) \)

\( (V^\mu)^2 \)

"Null" Projection

Space-like Interval \( (+) \)

\( (H^\mu)^2 \)

"Horizontal" Spatial Projection

LightCone
SRQM+EM Diagram: Projection Tensors & Stress-Energy Tensors: Special Cases

A few interesting special cases:

- For Perfect Fluid (no viscosity):
  - $T^{\mu\nu} = \rho_0 V^{\mu\nu} - (p_0) H^{\mu\nu}$
  - $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
  - $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

- If $(p_0) = (\rho_0) / 3$:
  - then {Null Dust = Photon Gas = Radiation}
  - $T^{\mu\nu}_{\text{Null Dust}} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
  - $T^{\mu\nu}_{\text{Photon Gas}} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
  - $T^{\mu\nu}_{\text{Null Gas}} = 0$: Null (Light-Like) Projection

- If $(\rho_0) = 0$:
  - then {Cold Matter Dust (pressureless)}
  - $T^{\mu\nu}_{\text{Matter Dust}} = P^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
  - $T^{\mu\nu}_{\text{Matter Dust}} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

- If $(p_0) = (\rho_0) / 3$:
  - then {Lambda Vacuum Energy}
  - $T^{\mu\nu}_{\text{Vacuum Energy}} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu} + \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
  - $T^{\mu\nu}_{\text{Vacuum Energy}} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

- If $(\rho_0) = 0$:
  - then {Zero Vacuum Energy}
  - $T^{\mu\nu}_{\text{Vacuum Energy}} = 0$
  - $T^{\mu\nu}_{\text{Vacuum Energy}} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

- For Perfect Fluid (no viscosity):
  - $T^{\mu\nu} = \rho_0 V^{\mu\nu} - (p_0) H^{\mu\nu}$
  - $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
  - $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

Special cases of a Perfect Fluid:

- $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
- $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
- $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

Traces:

- $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
- $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
- $T^{\mu\nu} = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

$V V = V^\mu V^\nu = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

$V^\mu V^\nu = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$

$V^\mu V^\nu = \left(\frac{\rho_0}{c^4}\right) U^{\mu\nu}$
Gauss' Theorem in SR:
\[ \int_{\Omega} d^4X \left( \partial_{\mu}V^{\mu} \right) = \oint_{\partial\Omega} dS \left( V^{\mu}N_{\mu} \right) \]
\[ \int_{\Omega} d^4X \left( \partial \cdot V \right) = \oint_{\partial\Omega} dS \left( V \cdot N \right) \]

where:
- \( V = V^{\mu} \) is a 4-Vector field defined in \( \Omega \)
- \( (\partial \cdot V) = (\partial_{\mu}V^{\mu}) \) is the 4-Divergence of \( V \)
- \( (V \cdot N) = (V^{\mu}N_{\mu}) \) is the component of \( V \) along the \( N \)-direction
- \( \Omega \) is a 4D simply-connected region of Minkowski SpaceTime
- \( \partial\Omega = S \) is its 3D boundary with its own 3D Volume element \( dS \) and outward pointing normal \( N \).
- \( N = N^{\mu} \) is the outward-pointing normal
- \( d^4X = (c\, dt)(d^3x) = (c\, dt)(dx\, dy\, dz) \) is the 4D differential volume element

In vector calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem, is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface. More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.

In vector calculus, and more generally in differential geometry, the generalized Stokes' theorem is a statement about the integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus.
SRQM Diagram:

**Minimal Coupling = Potential Interaction**

**Conservation of 4-TotalMomentum**

\[ P = (E/c, p) \text{ : 4-Momentum} \]
\[ Q = (V/c, q) \text{ : 4-PotentialMomentum} \]
\[ A = (q/c, a) \text{ : 4-VectorPotential} \]
\[ P_f = (E/c, p_f) \text{ : 4-MomentumIncPotentialField} \]
\[ P_T = (E/c, p_T) = (H/c, p_T) \text{ : 4-TotalMomentum} \]

\[ P = P_f - qA = (E/c-q\varphi/c, p_f-q\varphi) \text{ : Minimal Coupling Relation} \]
\[ P_f = P + Q = P + qA \text{ : Conservation of 4-MomentumIncPotentialField} \]
\[ P_f = (m_0)U + (q\varphi/c^2)U \]
\[ P_f = ((E+q\varphi)/c^2)(c,u) \]
\[ P_f = ((E+q\varphi)/c^2)p + qa \]

4-MomentumIncPotentialField has a contribution from a Mass “charge” \( m \), an EM charge \( q \) interacting with a potential \( \varphi \).

\[ P_T = \sum_n P_f \text{ : Conservation of 4-TotalMomentum} \]

4-TotalMomentum is the Sum over all such 4-Momenta.

**SRQM Diagram:**

- \[ \varphi/c^2 \text{ : Rest Scalar Potential} \]
- \[ U = \gamma(c,u) \text{ : 4-Velocity} \]
- \[ \varphi = \varphi(c,a) \text{ : 4-EMVectorPotential} \]
- \[ Q = (U/c, q) = qA \text{ : 4-EMPotentialMomentum} \]
- \[ P_f = (E/c, p_f) = P + Q = P + qA \text{ : 4-MomentumIncField} \]
- \[ \frac{\partial R}{\partial R} = \text{4-Gradient} \]
- \[ \varphi/c^2 = \text{Rest Mass:Energy} \]
- \[ E = mc^2 \]
- \[ \varphi/c^2 = \text{Minimal Coupling Relation} \]
- \[ \sum_n [ \ldots ] \]
- \[ [P_T R] = \text{Hamilton-Jacobi} \]
- \[ [P_T dR] = \text{4-TotalMomentum} \]
- \[ H = -\varphi[S] \]
- \[ p = \nabla[S] \]
- \[ \text{Conservation of 4-TotalMomentum} \]
- \[ P_T = \sum_n P_f \]
- \[ \sum_n [ \ldots ] \]
- \[ \varphi/c^2 \text{ : Rest Scalar Potential} \]
- \[ U = \gamma(c,u) \text{ : 4-Velocity} \]
- \[ \varphi = \varphi(c,a) \text{ : 4-EMVectorPotential} \]
- \[ Q = (U/c, q) = qA \text{ : 4-EMPotentialMomentum} \]
- \[ P_f = (E/c, p_f) = P + Q = P + qA \text{ : 4-MomentumIncField} \]
- \[ \frac{\partial R}{\partial R} = \text{4-Gradient} \]
- \[ \varphi/c^2 = \text{Rest Mass:Energy} \]
- \[ E = mc^2 \]
- \[ \varphi/c^2 = \text{Minimal Coupling Relation} \]
- \[ \sum_n [ \ldots ] \]
- \[ [P_T R] = \text{Hamilton-Jacobi} \]
- \[ [P_T dR] = \text{4-TotalMomentum} \]
- \[ H = -\varphi[S] \]
- \[ p = \nabla[S] \]
- \[ \text{Conservation of 4-TotalMomentum} \]
- \[ P_T = \sum_n P_f \]
- \[ \sum_n [ \ldots ] \]
- \[ \varphi/c^2 = \text{Rest Scalar Potential} \]
- \[ U = \gamma(c,u) \text{ : 4-Velocity} \]
- \[ \varphi = \varphi(c,a) \text{ : 4-EMVectorPotential} \]
- \[ Q = (U/c, q) = qA \text{ : 4-EMPotentialMomentum} \]
- \[ P_f = (E/c, p_f) = P + Q = P + qA \text{ : 4-MomentumIncField} \]
- \[ \frac{\partial R}{\partial R} = \text{4-Gradient} \]
- \[ \varphi/c^2 = \text{Rest Mass:Energy} \]
- \[ E = mc^2 \]
- \[ \varphi/c^2 = \text{Minimal Coupling Relation} \]
- \[ \sum_n [ \ldots ] \]
- \[ [P_T R] = \text{Hamilton-Jacobi} \]
- \[ [P_T dR] = \text{4-TotalMomentum} \]
- \[ H = -\varphi[S] \]
- \[ p = \nabla[S] \]
- \[ \text{Conservation of 4-TotalMomentum} \]
- \[ P_T = \sum_n P_f \]
- \[ \sum_n [ \ldots ] \]
SRQM Hamiltonian: Lagrangian Connection

\[ H + L = (p_T \cdot u) = \gamma(P_T \cdot U) + -(P_T \cdot U)/\gamma \]

**4-Momentum** \( P = m_0 U = (E_0/c^2)U \); 4-Vector Potential \( A = (\phi_0/c^2)U \)

**4-Total Momentum** \( P_T = (P + qA) = (H/c, p_T) \)

\[ P \cdot U = \gamma(E - p \cdot u) = E_0 = m_0 c^2 \; ; \; A \cdot U = \gamma(\phi - a \cdot u) = \phi_0 \]

\[ P_T \cdot U = (P \cdot U + qA \cdot U) = E_0 + q\phi_0 = m_0 c^2 + q\phi_0 \]

\[ \gamma = 1/\sqrt{1 - \gamma^2} :\text{Relativistic Gamma Identity} \]

(\( \gamma - 1/\gamma \) = \( \gamma\beta\beta \)): Manipulate into this form... still an identity

(\( \gamma - 1/\gamma \))\( (P_T \cdot U) = (\gamma\beta\beta \))\( (P \cdot U) \)

Still covariant with Lorentz Scalar

\[ \gamma(P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma\beta\beta \)\( (P \cdot U) \]

\[ \gamma(P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma\beta\beta \)\( (E_0 + q\phi_0) \]

\[ \gamma(P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma\beta\beta \)\( (u \cdot u) \]

\[ \gamma(P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma\beta\beta \)\( (E_0/c^2 + q\phi_0/c^2) \]

\[ \gamma(P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma\beta\beta \)\( (p + qa) \]

\[ \gamma(P_T \cdot U) + -(P_T \cdot U)/\gamma = (p_T \cdot u) \]

\{ H \} + \{ L \} = (p_T \cdot u):\text{The Hamiltonian/Lagrangian connection} \]

\[ H = \gamma(P_T \cdot U) = \gamma((P + qA) \cdot U) = \text{The Hamiltonian with minimal coupling} \]

\[ L = -(P_T \cdot U)/\gamma = -(P + qA) \cdot U)/\gamma = \text{The Lagrangian with minimal coupling} \]

**H:L Connection in Density Format**

\[ H + L = (p_T \cdot u) \]

\[ nH + nL = n(p_T \cdot u), \text{with number density } n = \gamma n_o \]

\[ \mathcal{H} + \mathcal{L} = (g_T \cdot u), \text{with} \]

**momentum density** \( \{ g_T \cdot n p_T \} \)

**Hamiltonian density** \( \{ H = nH \} \)

**Lagrangian Density** \( \{ L = nL = (\gamma n_o)(L_o/\gamma) = n_o L_o \} \)

**Lagrangian Density is Lorentz Scalar**

for an EM field (photonic):

\[ \mathcal{H} = (1/2)(\varepsilon_0 e \cdot e + b \cdot b/\mu_o) \]

\[ \mathcal{L} = (1/2)(\varepsilon_0 e \cdot e - b \cdot b/\mu_o) = (1/4 \mu_o)F_{\mu\nu}F^{\mu\nu} \]

\[ \mathcal{H} + \mathcal{L} = \varepsilon_0 e \cdot e = (g_T \cdot u) \]

\[ |u| = c \]

\[ |g_T| = \varepsilon_0 e \cdot e/c \]

**Poynting Vector** \( |s| = |g|c^2 \rightarrow c\varepsilon_0 e \cdot e \)

\[ \text{Ho + Lo = 0 Calculating the Rest Values} \]

4-Vector notation gives a very nice way to find the Hamiltonian/Lagrangian connection:

\( (H) + (L) = (p_T \cdot u) \), where \( H = \gamma(P_T \cdot U) \) & \( L = -(P_T \cdot U)/\gamma \)
SRQM Study:
SR Lagrangian, Lagrangian Density, and Relativistic Action (S)

Relativistic Action (S) is Lorentz Scalar Invariant
\[ S = \int \mathcal{L} \, dt = \int \left( \frac{\rho \mathcal{L}}{(n/c)^2} \right) \, (d^4x) = \int \left( \frac{\rho \mathcal{L}}{(n/c)^2} \right) \, (d^4x) \]

Explicitly-Covariant Relativistic Action (S)

Particle Form
\[ S = \int \mathcal{L}(n) \, dt = \int \mathcal{L}(n) \, (d^4x) dt = \int \mathcal{L}(n) \, (d^4x) (c dt) = \int \mathcal{L}(c) \, (d^4x) \]

Density Form \( n \cdot \text{*Particle} \)
\[ S = \int n \mathcal{L}(n) \, dt = \int n \mathcal{L}(n) \, (d^4x) dt = \int n \mathcal{L}(n) \, (d^4x) (c dt) = \int n \mathcal{L}(c) \, (d^4x) \]

Lagrangian \( L = (\mathbf{p} \cdot \mathbf{u}) - H \) is *not* Lorentz Scalar Invariant

Rest Lagrangian \( L = \gamma L = \gamma (\mathbf{p} \cdot \mathbf{u}) \) is Lorentz Scalar Invariant

Lagrangian Density \( \mathcal{L} = nL = (\gamma n)(L/c) = nL \) is Lorentz Scalar Invariant

\[ n = \gamma n_o = \#/(dx)(dy)(dz) = \text{number density} \]
\[ dt = \gamma dt \]
\[ cdt = n_o(cdt)(dx)(dy)(dz) = n_o(d^4x) \]
\[ d\tau = (n_o/c)(d^4x) \]

\[ H + L = (\mathbf{p} \cdot \mathbf{u}) \]
\[ nH + nL = n(n\mathbf{p} \cdot \mathbf{u}), \text{ with number density } n = n_o \]

\[ \mathcal{H} + \mathcal{L} = (g \cdot \mathbf{u}), \text{ with momentum density } \{ g \tau \} \]

Hamiltonian density \( \mathcal{H} = nH \)

Lagrangian Density \( \mathcal{L} = nL = (\gamma n_o)(L_o/c) = n_oL_o \)

Lagrangian Density is Lorentz Scalar

for an EM field (photonic):
\[ \mathcal{H} = (1/2)\epsilon \mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B}/\mu_o \]
\[ \mathcal{L} = (1/2)\epsilon \mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B}/\mu_o = (-1/4\mu_o)F_{\mu\nu}F^{\mu\nu} = (-1/4\mu_o)\text{Faraday EM Tensor Inner Product} \]
\[ |\mathbf{u}| = c \]
\[ |\mathbf{g}| = \epsilon \mathbf{e}/c \]

Poynting Vector \( |\mathbf{s}| = |\mathbf{g}|c^2 \rightarrow c\epsilon \mathbf{e} \cdot \mathbf{e} \]
\[ \epsilon_o\mu_o = 1/c^2 : \text{Electric} \cdot \text{Magnetic} \text{ Constant Eqn} \]

The Relativistic Action Equation is seen in many different formats
SRQM Study:
SR Hamilton-Jacobi Equation and Relativistic Action (S)

Lagrangian \{L = \langle p_T \cdot u \rangle - H\} is *not* a Lorentz Scalar
Rest Lagrangian \{L_o = \gamma L = -\langle p_T \cdot U \rangle\} is a Lorentz Scalar

Relativistic Action (S) is Lorentz Scalar
S = \int L dt
S = \int (L_o/\gamma)(\gamma d\tau)
S = \int (L_o)(d\tau)

Explicitly Covariant
Relativistic Action (S)
S = \int L_o d\tau = -\int H_o d\tau
S = -\int (p_T \cdot dR/d\tau)d\tau
S = -\int (p_T \cdot dR) d\tau
S = -\int ((P + qA) \cdot U)d\tau
S = -\int (P \cdot U + qA \cdot U)d\tau
S = -\int (E_o + q\phi_o)d\tau
S = -\int (E_o + V)d\tau \quad \text{with} \quad V = q\phi_o
S = -\int (m_o c^2 + V)d\tau
S = -\int (H_o)d\tau

4-TotalMomentum
P_T = (E/c, p_T) = (H/c, p_T)
\text{verified!}
\text{Hamilton-Jacobi Equation}
\vartheta[-S] = -\vartheta[S] = P_T
S = \int (E_o + q\phi_o) d\tau
S = -\int (E_o + q\phi_o)(\tau + \text{const})

SRQM → QM
A Tensor Study of Physical 4-Vectors

4-Vector SRQM Interpretation of QM

The Hamilton-Jacobi Equation is incredibly simple in 4-Vector form
SRQM Diagram:
Relativistic Hamilton-Jacobi Equation
\( P_T = -\partial [S] \) Differential Format : 4-Vectors

- **4-Displacement**: \( \Delta R = (c\Delta t, \Delta r) \)
- **4-Position**: \( R = (c t, r) \)
- **4-Vector SRQM**: 4-Tensor Study
- **SR 4-Tensor**: (2,0)-Tensor \( T^{\mu\nu} \), (1,1)-Tensor \( T^{\nu\mu} \), or \( T^{\nu\mu}_s \)
- **SR 4-Vector**: (1,0)-Tensor \( V = (\nu, v) \)
- **SR 4-Scalar**: (0,0)-Tensor \( S \) Lorentz Scalar

Relativistic Action (S) is Lorentz Scalar Invariant
\( S = J = \int (L_m^o - L_m^c) dt = \int (L_x^o - L_x^c) dt \)

Explicitly-Covariant Relativistic Action (S):
\( d\tau = (1/c)\sqrt{dR\cdot dR} \)

Proper Time Derivative
\( d\tau = (1/c)\sqrt{dR\cdot dR} \)

Proper Time
\( U\cdot d\tau = \gamma d\tau = \gamma d\nu/dt \)

4-Velocity
\( U = \gamma (c, u) \)

4-Force
\( F = \gamma (E/c, f) \)

4-Momentum
\( P = (mc, p) = (E/c, p) \)

4-EMVectorPotential
\( A = (\varphi/c, a) \)

4-EMPotentialMomentum
\( Q = (U/c, q) = qA \)

4-ChargeFlux
\( N = (nc, n) = n(c, u) \)

4-NumberFlux
\( J = (pc, j) = p(c, u) \)

4-CurrentDensity
\( J = \varphi/c^2 \)

4-Gradient
\( \mathbf{P}_T = \mathbf{P}^R = \mathbf{H} \) Hamilton-Jacobi Equation

4-EMVectorPotential
\( A = (\varphi/c, a) \)

4-Force
\( F = \gamma (E/c, f) \)

4-Momentum
\( P = (mc, p) = (E/c, p) \)

4-EMPotentialMomentum
\( Q = (U/c, q) = qA \)

Trace \( [\mathbf{T}] = \eta_{\mu\nu} T^{\mu\nu} = T^\nu_{-\nu} = T \)

\( V\cdot V = V\cdot c = (v^\mu, v) \) Lorentz Scalar
SRQM Diagram: Relativistic Action Equation

\[ S = -\int (P_T \cdot dR) \]

Integral Format: 4-Scalars
SRQM Diagram:
Relativistic Euler-Lagrange Equation
The Easy Derivation (\(U=(d/d\tau)R\) → (\(\partial_R=(d/d\tau)\partial_U\))

Note Similarity:
4-Velocity is ProperTime Derivative of 4-Position
\(U = (d/d\tau)R\) \([\text{m/s}] = [1/\text{s}]^*[\text{m}]\)

Relativistic Euler-Lagrange Eqn
\(\partial x = (d/d\tau)\partial u\) \([1/\text{m}] = [1/\text{s}]^*[\text{s/m}]\)
The differential form just inverses the dimensional units, so the placement of the \(R\) and \(U\) switch.

That is it: so simple! Much, much easier than how I was taught in Grad School.

To complete the process and create the Equations of Motion, one just applies the base form to a Lagrangian.

This can be:
a classical Lagrangian
a relativistic Lagrangian
a Lorentz scalar Lagrangian
a quantum Lagrangian

SR 4-Tensor
(2,0)-Tensor \(T^{\mu\nu}_{\tau}\)
(1,1)-Tensor \(T^{\mu\nu}_\tau\), or \(T_\tau^{\mu\nu}\)
(0,2)-Tensor \(T^\nu_{\tau\mu}\)
SR 4-Vector
(1,0)-Tensor \(V^\nu = V = (v^\nu, v)\)
SR 4-CoVector
(0,1)-Tensor \(V_\nu = (v_\nu, -v)\)
SR 4-Scalar
(0,0)-Tensor \(S\)
Lorentz Scalar
SRQM Diagram:

Relativistic Euler-Lagrange Equation
Alternate Forms: Particle vs. Density

4-Velocity $U$ is ProperTime Derivative of 4-Position $R$. The Euler-Lagrange Eqn can be generated by taking the differential form of the same equation.

Relativistic 4-Vector Kinematical Eqn
$U = (d/d\tau)R$
$U \cdot K = (d/d\tau)R \cdot K$

Relativistic Euler-Lagrange Eqns
{uses gradient-type 4-Vectors}

$\partial_R = (d/d\tau) \partial_U$: {particle format}
$\partial_{(\Phi)} = (d/d\tau) \partial_{U}$
$\partial_{K} = (U \cdot \partial_R)$
$\partial_{(-\Phi)} = (U \cdot \partial_R) \partial_{(-U)}$
$\partial_{(\Phi)} = (U \cdot \partial_R)$
$\partial_{[\partial_{(-\Phi)}]} = (\partial_R)$
$\partial_{(\Phi)} = (\partial_R)$
$\partial_{[\partial_{(\Phi)}]} = (\partial_R)$

$L = (1/2)\{ \partial_R [\Phi] \cdot \partial_R [\Phi] - (m_c/c)^2 \Phi^2 \}$: KG Lagrangian Density

$\partial_{(\Phi)} L = (\partial_{U}) \partial_{[\partial_{(-\Phi)}]} L$: Euler-Lagrange Eqn {density format}

$-(m_c/c)^2 \Phi = (\partial_R) \cdot \partial_R [\Phi]$
$\partial_{(\Phi)} [\partial_R [\Phi]] = - (m_c/c)^2 \Phi$
$(\partial_{(\Phi)} \partial_R) [\Phi] = - (m_c/c)^2 \Phi$

Klein-Gordon Relativistic Quantum Wave Eqn

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^\nu$, or $T^\nu_\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\nu = (\hat{V}^\nu, V)$

SR 4-CoVector
(0,1)-Tensor $V_\nu = (V_\nu, V)$

SR 4-Scalar
(0,0)-Tensor $S$
Lorentz Scalar

4-Vector SRQM Interpretation of QM
SciRealm.org
John B. Wilson

John B. Wilson
SciRealm.org
SRQM Diagram:

Relativistic Euler-Lagrange Equation

Equation of Motion (EoM) for EM particle

4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]

4-Position
\[ R = (ct, r) \]

\[ \delta_R[R] = n^{ix} \rightarrow \text{Diag}[1, -1, -1, -1] \]

Minkowski Metric

\[ L_0 = -(P_1, U) \]

\[ \delta_0[U_0] = -P_0 = -(P + qA) \]

(d/dτ)[δU[0]] = (d/dτ)[P + qA] = -(F + qU)(d/dτ)[A] = -(F + qU, δA)[A]

δR[L0] = δP - P1[U] - δQ[(P + qA)U] = (0) + qδQ(U, A) - δQ(U, A) = qU, δQ(U, A)

assuming the 4-Gradient δR of the 4-Velocity U is zero.

Euler-Lagrange Eqn:
\[ (d/dτ)[P_τ U] = -δU[\ell] \]

\[ (F + qU, δA)[A] \]

\[ F = qU, δA[A'] - qU, δA[A] \]

\[ F' = qU, δA[A'] - δA[A'] \]

\[ F'' = qU, (δA[A'] - δA[A']) \]

\[ \ell = T_\mu \cdot U \cdot U \]

\[ \ell = \gamma(T_\mu\cdot U) \]

\[ \ell = U \cdot \delta U = \gamma d/dτ \]

Derivative of 4-Position
\[ U = (d/dτ)[R] \quad [m/s] = [1/s][m] \]

Relativistic Euler-Lagrange Eqn
\[ \delta_\alpha = (d/d\tau)[\alpha'] \]

The differential form just inverses the dimensional units

4-NumberFlux
\[ N = (nc, n) = n(c, U) \]

4-ChargeFlux
\[ J = (pc, j) = p(c, U) \]

4-CurrentDensity
\[ J = (pc, j) = p(c, U) \]

4-Force
\[ F = \gamma(E/c, F) \]

4-Momentum
\[ P = (mc, p) = (E/c, p) \]

4-TotalMomentum
\[ P_T = (E/c, p_T) = (H/c, p_T) \]

4-EMVectorPotential
\[ A = (\phi/c, a) \]

4-TotalMomentum
\[ \Sigma_i[\ldots] \]

4-PositionGradient
\[ \delta_R = \delta = (\ell/c, -V) \]

Relativistic Euler-Lagrange Eqn
\[ \delta_R = \delta = (\ell/c, -V) \]

Hamilton-Jacobi
\[ P_T = (S, p_T) \]

Conservation of TotalMomentum
\[ (E/c, p_T) = (H/c, p_T) \]

EM Faraday
\[ F^{\gamma, \delta} = [\gamma, \delta/c] \]

4-Tensor
\[ (\gamma, \delta/c) \]

\[ \gamma \]

\[ q \]

\[ x \]

\[ 0, -\gamma/c \]

\[ +[e/c, \epsilon b] \]

\[ 4-Vector \]

SR 4-Scalar
\[ S = (0, 1)-Tensor \]

SR 4-Vector
\[ V = (\gamma, V) \]

SR 4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]

SR 4-Position
\[ R = (ct, r) \]

SR 4-Vector
\[ V = (\gamma, V) \]

SR 4-Dispacement
\[ \Delta R = (c\Delta t, \Delta r) \]

SR 4-Position
\[ R = (ct, r) \]

SR 4-Vector
\[ V = (\gamma, V) \]

SR 4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]

SR 4-Position
\[ R = (ct, r) \]

SR 4-Vector
\[ V = (\gamma, V) \]
\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \text{; Relativistic Gamma Identity} \]
\[ \gamma - 1 \gamma = (\gamma - \beta \cdot \beta) \text{; Manipulate into this form... still an identity} \]
\[ \gamma \eta = (\gamma \beta \cdot \beta) \text{; } (P - U) \gamma = (\gamma \beta \cdot \beta) \gamma \]
\[ \gamma \gamma \gamma = (\gamma \beta \cdot \beta) \gamma = (P - U) \gamma = -(P + qA) \eta \gamma = \text{The Lagrangian with minimal coupling} \]

\[ H = \gamma H_0 = \gamma (P - U) = \gamma (P + qA) \eta \]

\[ L = L_{\gamma} = -(P - U) / \gamma = -(P + qA) / \gamma \text{; The Lagrangian with minimal coupling} \]

\[ H_0 = (P - U) = -L_0 = (U \cdot P) ; \text{Rest Hamiltonian} = \text{Total Rest Energy} \]
\[ L_0 = -(P - U) = -H_0 \]

\[ \frac{d}{d\tau} \partial_\eta [L_0] = \partial_{\eta} [L_0] \]

4-Velocity is ProperTime Derivative of 4-Momentum
\[ U = \frac{d}{d\tau} R \text{ } [m/s] = [1/s][m] \]

Relativistic Euler-Lagrange Eqn
\[ \partial \partial \partial \partial = \frac{d}{d\tau} \partial_\eta [L_0] \]
\[ \eta = \frac{d}{d\tau} \partial_\eta \]
\[ \partial \partial \partial \partial = \frac{d}{d\tau} \partial_\eta [L_0] \]

Classical limit, spatial component
\[ \partial \partial \eta = \frac{d}{d\tau} \partial_\eta [L_0] \]
\[ \partial \partial \partial \partial = \frac{d}{d\tau} \partial_\eta [L_0] \]
\[ \partial \partial \partial \partial = \frac{d}{d\tau} \partial_\eta [L_0] \]

\[ F_{\text{EM}} = q \left( \bar{u} \times \vec{B} \right) \text{, } (e) \text{, } u = [-\nabla \phi - \partial a] \text{ and } \bf{b} = [\nabla \times \vec{a}] \]

If \( \partial a = 0 \text{, then } f = -q \nabla \phi = -\nabla U, \text{ the force is neg grad of a potential} \]
SRQM Diagram: Relativistic Hamilton’s Equations of Motion (EqM) for EM particle
**SRQM Diagram:**

**EM Lorentz Force Eqn**

→ Force = - Grad[Potential]

---

**Lorentz EM Force Equation:**

\[ F^\alpha = q(F^{\alpha \beta})U_\beta \]

\[ F^\alpha = q(\partial^\alpha A^\beta - \partial^\beta A^\alpha)U_\beta \]

Examine just the spatial components of 4-Force \( F^\alpha \):\n
\[ F^i = q(\partial^i A^\beta - \partial^\beta A^i)U_\beta \]

\[ \gamma f = q(-\nabla[\varphi/c] - (\partial^i/c)a)^i + q(-\nabla[a\cdot u] - u\cdot \nabla[a]) \gamma \]

\[ f = q(-\nabla[\varphi/c] - (\partial^i/c)a)^i + q(u\cdot \nabla[a]) \]

\[ f = q(-\nabla[\varphi] - \partial^i a + u\cdot \nabla[a] - \nabla[a\cdot u]) \]

\[ f = q(-\nabla[\varphi] - \partial^i a + u \times b) \]

Take the limit of \{ \nabla[\varphi] \} >> \{ \partial^i a \} + \{ u \times b \}

\[ f \sim q(-\nabla[\varphi]) = -\nabla[\varphi] = -\nabla[U] \]

The Classical Force = -Grad[Potential]

when \{ \nabla[\varphi] \} >> \{ \partial^i a \} + \{ u \times b \} or when \{ a = 0 \}

The majority of non-gravity, non-nuclear potentials dealt with in CM are those mediated by the EM potential.

ex. Spring Potential \{ U = kx^2/2 \}, then \{ f = -\nabla[kx^2/2] = -kx \} Hooke’s Law
SRQM: The Speed-of-Light (c) Invariant Relations (part 1)

The Speed-of-Light (c) is THE connection between Time and Space: \( dR = (cdt,dr) \)

This physical constant appears in several seemingly unrelated places. You don’t notice these cool relations when you set \( c \rightarrow 1 \). Also notice that the set of all these relations definitely rules out a variable speed-of-light. (c) is an Invariant Lorentz Scalar constant.

\[ U \cdot U = \gamma^2 (c^2 \cdot u \cdot u) = c^2 \]

Speed of all things into the Future

\[ (E/m_c) = (\gamma E / m) = (E/m) = c^2 \] Mass is concentrated Energy, \( E = mc^2 \)

\[ |u \cdot v| = [v_{\text{group}} \cdot v_{\text{phase}}] = c^2 \]

Particle-Wave “Duality” Correlation

\[ \lambda^2 (\omega^2 - \omega_0^2) = \lambda^2 (f^2 - f_0^2) = c^2 \]

Wavelength-Frequency Relation: \( \lambda f = c \) for photons

\[ (1/e_\text{e}, \mu_\text{e}) = c^2 \]

Electric (\( e_\text{e} \)) and Magnetic (\( \mu_\text{e} \)) EM Field Constants

\[ -h \cdot (m_c)^2 (\partial \partial) = c^2 \]

Relativistic Quantum Wave Equation

\[ \lambda_C (\omega^2 - \omega_0^2) = \lambda_C (f^2 - f_0^2) = c^2 \]

Reduced Compton Wavelength: \( \lambda_C = (h/m_c) \)

\[ 2GM/R_s = c^2 \]

GR Black Hole Equation

\[ 8\pi G / c^2 = \frac{\rho}{c^4} \]

GR Einstein Curvature Constant: \( \kappa = 8\pi G/c^2 \)

\[ (c^2 \cdot \text{scalar}, 3\text{-vector}) = 4\text{-Vector} \]

Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector

\[ c^2 \text{Invariant Relations} \]

4-Vector SRQM:

- \( \partial \cdot R = 4 \) SpaceTime Dimension
- \( \partial^\mu [R^\nu] = \eta^{\mu\nu} \) Minkowski Metric
- \( 4\text{-Position} \ R = (ct, r) \)
- \( \partial \cdot \varphi = \varphi \) Lorentz Gauge
- \( c^2 \) EM Faraday 4-Tensor

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \]

Maxwell EM Wave Eqn

\[ \varphi / c^2 \]

4-EMVectorPotential \( A = (\varphi / c, a) \)

\[ \Delta (\partial \partial) A - \partial (\partial A) = \mu \cdot J \]

\[ 4\text{-CurrentDensity} \ J = (pc, j) = (\rho, c, u) \]

\[ 4\text{-ChargeFlux} \]

\[ \text{Trace}[T] = \eta_{\mu\nu} T^{\mu\nu} = T^0 \]

\[ V \cdot V = V^\mu V_\mu = (V^0)^2 - (V^i)^2 = (V_0)^2 \]

Lorentz Scalar

\[ 1/\mu_0 = c^2 \]

Electric: Magnetic

\[ \epsilon_0 c^2 \]

\[ 1/\epsilon_\text{e} = c^2 \]

\[ \text{SR 4-Tensor} \]

(2,0) Tensor \( T^{\mu} \)

(1,1) Tensor \( T^\mu_\nu \) or \( T^\nu_\mu \)

(0,2) Tensor \( T_{\mu\nu} \)

\[ \text{SR 4-Vector} \]

(1,0) Tensor \( V^\mu = V = (\sqrt{V}, V) \)

(0,1) Tensor \( V_\mu = (V_0, -V) \)

\[ \text{SR 4-Scalar} \]

Lorentz Scalar
SRQM: The Speed-of-Light (c)

**c^2 Invariant Relations (part 2)**

The Speed-of-Light (c) is THE connection between Time and Space: \( dR = (c dt, dr) \)

This physical constant appears in several seemingly unrelated places. You don't notice these cool relations when you set \( c \to 1 \). Also notice that the set of all these relations definitely rules out a variable speed-of-light. (c) is an Invariant Lorentz Scalar constant.

\[ U \cdot U = c^2 \]

- **Energy-Mass**: \( E = mc^2 \)
- **Electric/Magnetic**: \( 1/(\epsilon_0 \mu_0) = c^2 \)
- **Invariant 4-Velocity Magnitude**: \( U \cdot U = c^2 \)
- **Perfect Invariant**: \( U_\text{photon}^2 = U_\text{EMwave}^2 \)

\[ \delta^\nu [R^\nu] = \eta^\mu_\nu \]

Minkowski Metric

4-Vector Scalar Product

**Electric**: \( E \)

**Magnetic**: \( B \)

**Energy**: \( E = mc^2 \)

**Wavelength**: \( \lambda \)

**Frequency**: \( \omega \)

\[ \lambda^2 (\omega^2 - \omega_o^2) = \lambda_c^2 \frac{\omega_o^2}{c^2} \]

\[ \lambda_c^2 (\omega^2 - \omega_o^2) = \lambda_c^2 \omega_o^2 \]

\[ \lambda_c^2 (\omega^2 - \omega_o^2) = \lambda_c^2 \omega_o^2 \]

Waves

**GR Black Hole Equation**

\( R_s = \frac{2GM}{c^2} \)

**GR Einstein Curvature Constant (mass density form)**

\( \kappa = \frac{8\pi G}{c^2} \)

\[ \gamma (\cdot, \cdot) \rightarrow c^2 \]

Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector

**SR 4-Tensor**

\[ (2,0)-Tensor T^{\mu\nu} \]

**SR 4-Vector**

\[ (1,0)-Tensor V^\mu = (v^0, \mathbf{v}) \]

**SR 4-CoVector**

\[ (0,1)-Tensor V_\mu = (v_0, -\mathbf{v}) \]

**SR 4-Scalar**

\[ (0,0)-Tensor S \]

Lorentz Scalar

**SRQM**

\[ \text{Trace}[T^{\nu\mu}] = \eta_{\mu\nu} T^{\nu\mu} = T^{\nu\nu} = T \]

\[ V \cdot V = V_\mu V^\mu = (v^0)^2 - \mathbf{v} \cdot \mathbf{v} = (v_0)^2 \]

Lorentz Scalar
The 4-ThermalVector is used in Relativistic Thermodynamics.

My prime motivation for the form of this 4-Vector is that the probability distributions calculated by statistical mechanics ought to be covariant functions since they are based on counting arguments.

F(state) ~ e^-((E/k_B T) = e^-((βE), with this β = 1/k_B T, (not v/c)

A covariant way to get this is the Lorentz Scalar Product of the 4-Momentum P with the 4-ThermalVector Θ.

F(state) ~ e^-((P·Θ) = e^-((E/κ_B T_o)

This also gets Boltzmann’s constant (κ_B) out there with the other Lorentz Scalars like (c) and (h)

see (Relativistic) Maxwell-Jüttner distribution

f [P] = N_o/(2c(m_o c)^2 K_{(d=1/2)[m_o c_o 2π]}(d-1/2) e^-(P·Θ)

f [P] = N_o/(2c(m_o c)^2 K_{(d=1/2)[m_o c_o 2π]}(d-1/2) e^-(P·Θ)

It is possible to find this distribution written in multiple ways because many authors don’t show constants, which is quite annoying. Show the damn constants people! (κ_B),(c),(h) deserve at least that much respect.

See John B. Wilson SciRealm.org for QM Interpretation of Physical 4-Vectors

Be careful not to confuse (unfortunate symbol clash):

Thermal β = 1/k_B T

Relativistic β = v/c

These are totally separate uses of (β)
The 4-ThermalVector is used in Relativistic Thermodynamics. It can be used in a partial derivation of Unruh-Hawking Radiation (up to a numerical constant).

Let a “Unruh-DeWitt thermal detector” be in the Momentarily-Comoving-Rest-Frame (MCRF) of a constant spatial acceleration (a), in which |u|→0, γ→1, γ→0.

\[T\] further methods give the constant of proportionality \(1/2\) \(\frac{\text{Temperature}}{\text{T}}\) → \(\approx \frac{\text{Invariant}}{\text{Units}} = \frac{\text{[m/s]}}{\text{[kg⋅m/s²]}} = \frac{\text{[kg⋅m/s]}}{\text{[kJ]}}\)

Use \(\text{Dimensional Analysis}\) to find appropriate Lorentz Scalar Invariant with same Units:

\[\text{Invariant Units} = \text{[m/s]} / \text{[kg⋅m²/s²]} = [1/\text{kg⋅s}] \sim c^2/\hbar\]

\[\text{Temperature} T \sim h\omega/k_B\text{c}, \{\text{from EM radiation, only from the dir. of acceleration}\}\]

Further methods give the constant of proportionality (1/2m):

\[T_{\text{Unruh}} = h\omega/2\pi\text{c}\text{c} \{\text{due to constant Minkowski-hyperbolic acceleration}\}\]

\[T_{\text{Hawking}} = h\omega/2\pi\text{c} \{\text{due to gravitational acceleration } a=g\}\]

\[T_{\text{SR}} = -h(a-u)/2\pi\text{c}^2 \{\text{correct version from 4-Vector derivation } A_{\text{MCRF}} = 2\pi\text{c}^2/\hbar\}\]

\[4-\text{Acceleration}\]

\[A = A_\mu = c(y', \gamma' u + y)\]

\[= dU/dτ = dR/dτ^2\]

\[4-\text{Acceleration}_{\text{MCRF}} = A_{\text{MCRF}} = (0, a)_{\text{MCRF}}\]

\[A\cdot A = -(a)^2 = -(a_c)^2\]

\[4-\text{Velocity} U = (c, u)\]

\[\theta/c = 1/k_B T_0\]

\[\frac{u\cdot u}{c^2} = 1\]

\[\frac{m}{E/c^2} = \frac{1}{E/c}\]

\[\text{Trace}[T^{\mu\nu}] = n_\mu T_\nu = T_\nu = T\]

\[V\cdot V = V_\mu n_\nu V = (V_\mu - v_\nu) = (V_\mu + v_\nu) = (V_\mu \cdot V) = \text{Lorentz Scalar}\]

Note that the temperature here is relativistically direction-specific, unlike in the classical use of temperature.
The 4-EntropyVector is used in Relativistic Thermodynamics.

Pure Entropy is a Lorentz Scalar in all frames.

not finished yet…

Page under construction
Up to this point, we have basically been exploring the SR aspects of 4-Vectors.

It is now time to show how RQM and QM fit into the works...

This is SRQM, [ SR → QM ]

RQM & QM are derivable from SR

SRQM: A treatise by John B. Wilson (SciRealm@aol.com)
The basic idea is to show that Special Relativity plus a few empirical facts lead to Relativistic Wave Equations, and thus RQM, without using any assumptions or axioms from Quantum Mechanics.

Start only with the concepts of SR, no concepts from QM

(1) SR provides the ideas of Invariant Intervals and \( c \) as a Physical Constant, as well as:
- Poincaré Invariance
- Minkowski 4D SpaceTime
- ProperTime
- and Physical SR 4-Vectors

Note empirical facts which can relate the SR 4-Vectors from the following:
(2a) Elementary matter particles each have RestMass, \( m_0 \), which can be measured by experiment: eg. collision, cyclotrons, Compton Scattering, etc.

(2b) There is a constant, \( \hbar \), which can be measured by classical experiment – eg. the Photoelectric Effect, the inverse Photoelectric Effect, LED's=Injection Electroluminescence, Duane-Hunt Law in Bremsstrahlung, the Watt/Kibble-Balance, etc. All known particles obey this constant.

(2c) The use of complex numbers \( i \) and differential operators \( \{ \partial_t, \nabla = (\partial_x, \partial_y, \partial_z) \} \) in wave-type equations comes from pure mathematics: not necessary to assume any QM Axioms

These few things are enough to derive the RQM Klein-Gordon equation, the most basic of the relativistic wave equations. Taking the low-velocity limit \( |\mathbf{v}| << c \) (a standard SR technique) leads to the Schrödinger Equation.
If one has a Relativistic Wave Equation, such as the Klein-Gordon equation, then one has RQM, and thence QM via the low-velocity limit \(|v|<\frac{1}{c}\).

The physical and mathematical properties of QM, usually regarded as axiomatic, are inherent in the Klein-Gordon RWE itself.

QM Principles emerge not from \(\{\text{QM Axioms } + \text{SR } \rightarrow \text{RQM}\}\), but from \(\{\text{SR } + \text{Empirical Facts } \rightarrow \text{RQM}\}\).

The result is a paradigm shift from the idea of \(\{\text{SR and QM as separate theories}\}\) to \(\{\text{QM derived from SR}\}\) – leading to a new interpretation of QM:

*The SRQM or [SR → QM] Interpretation.*

\[
\text{GR } \rightarrow \text{ (low-mass limit } = \text{ curvature } \sim 0 \text{ limit) } \rightarrow \text{ SR}
\]

\[
\text{SR } \rightarrow \text{ (+ a few empirical facts) } \rightarrow \text{ RQM}
\]

\[
\text{RQM } \rightarrow \text{ (low-velocity limit } \{ |v|<\frac{1}{c} \} \) \rightarrow \text{ QM}
\]

The results of this analysis will be facilitated by the use of SR 4-Vectors
# SRQM 4-Vector Path to QM

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Definition Component Notation</th>
<th>Unites</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position</td>
<td>( \mathbf{R} = R^\mu = (ct, r) )</td>
<td>Time, Space - <em>when &amp; where</em></td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( \mathbf{U} = U^\mu = \gamma(c, \mathbf{u}) )</td>
<td>Lorentz Gamma * (c, Velocity) - <em>nothing faster than c</em></td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( \mathbf{P} = P^\mu = \left( \frac{E}{c}, \mathbf{p} \right) = (mc, \mathbf{p}) )</td>
<td>Mass:Energy, Momentum - used in 4-Momenta Conservation</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( \mathbf{K} = K^\mu = \left( \frac{\omega}{c}, \mathbf{k} \right) = \left( \frac{\omega}{c}, \mathbf{\hat{n}} / \nu_{phase} \right) )</td>
<td>Ang. Frequency, WaveNumber - used in Relativistic Doppler Shift</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial = \partial^\mu = \left( \frac{\partial}{c}, -\nabla \right) = \left( \frac{\partial}{\partial c t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) )</td>
<td>Temporal Partial, Spatial Partial - used in SR Continuity Eqns., ProperTime - eg. ( \partial \cdot \mathbf{A} = 0 ) means ( \mathbf{A} ) is conserved</td>
</tr>
</tbody>
</table>

All of these are standard SR 4-Vectors, which can be found and used in a totally relativistic context, with no mention or need of QM. I want to emphasize that these objects are ALL relativistic in origin.
## SRQM 4-Vector Invariants

<table>
<thead>
<tr>
<th>4-Vector</th>
<th>Lorentz Invariant</th>
<th>What it means in SR...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position</td>
<td>( \mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct_0)^2 = (ct)^2 )</td>
<td>SR Invariant Interval</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( \mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2 )</td>
<td>Events move into future at magnitude c</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( \mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2 )</td>
<td>Einstein Mass:Energy Relation</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( \mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_0/c)^2 )</td>
<td>Dispersion Invariance Relation</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = (\partial_t/c)^2 )</td>
<td>The d'Alembert Operator</td>
</tr>
</tbody>
</table>

All 4-Vectors have invariant magnitudes, found by taking the scalar product of the 4-Vector with itself. Quite often a simple expression can be found by examining the case when the spatial part is zero. This is usually found when the 3-velocity is zero. The temporal part is then specified by its “rest” value.

For example: \( \mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2 = (m_0c)^2 \)

\[ E = \sqrt{(E_0)^2 + \mathbf{p} \cdot \mathbf{p} c^2} \text{, from above relation} \]

\[ E = \gamma E_0 \text{, using } \{ \gamma = 1/\sqrt{1-\beta^2} = \sqrt{1+\gamma^2\beta^2} \} \text{ and } \{ \beta = v/c \} \]

meaning the relativistic energy \( E \) is equal to the relative gamma factor \( \gamma \) * the rest energy \( E_0 \).
SR + A few empirical facts: SRQM Overview

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>SI Dimensional Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position ( \mathbf{R} = (ct,\mathbf{r}) ); alt. ( \mathbf{X} = (ct,\mathbf{x}) )</td>
<td>( \mathbf{R} = \text{&lt;Event&gt;}; \text{alt.} \ \mathbf{X} )</td>
<td>[m]</td>
</tr>
<tr>
<td>4-Velocity ( \mathbf{U} = \gamma(c,\mathbf{u}) )</td>
<td>( \mathbf{U} = d\mathbf{R}/d\tau )</td>
<td>[m/s]</td>
</tr>
<tr>
<td>4-Momentum ( \mathbf{P} = (E/c,\mathbf{p}) = (mc,\mathbf{p}) )</td>
<td>( \mathbf{P} = m_0\mathbf{U} )</td>
<td>[kg·m/s]</td>
</tr>
<tr>
<td>4-WaveVector ( \mathbf{K} = (\omega/c,\mathbf{k}) )</td>
<td>( \mathbf{K} = \mathbf{P}/\hbar )</td>
<td>[{rad}/m]</td>
</tr>
<tr>
<td>4-Gradient ( \partial = (\partial/c,-\nabla) )</td>
<td>( \partial = -i\mathbf{K} )</td>
<td>[1/m]</td>
</tr>
</tbody>
</table>

The Axioms of SR, which are actually GR limiting-cases, lead us to the use of Minkowski Space and Physical 4-Vectors, which are elements of Minkowski Space (4D SpaceTime).

Empirical Observation leads us to the transformation relations between the components of these SR 4-Vectors, and to the chain of relations between the 4-Vectors themselves.

These relations all turn out to be Lorentz Invariant Constants, whose values are measured empirically.

The combination of these SR objects and their relations is enough to derive RQM.
SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR, although technically SR is itself the low-curvature limiting-case of GR

\{c, \tau, m_0, \hbar, i\}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors:

- **4-Position**: \( R = (ct, r) \) = <Event>  \( (R \cdot R) = (ct)^2 \)
- **4-Velocity**: \( U = \gamma(c, u) \) = \( (U \cdot \partial)R = (d/d\tau)R = dR/d\tau \)  \( (U \cdot U) = (c)^2 \)
- **4-Momentum**: \( P = (E/c, p) \) = \( m_0U \)  \( (P \cdot P) = (m_0c)^2 \)
- **4-WaveVector**: \( K = (\omega/c, k) \) = \( P/\hbar \)  \( (K \cdot K) = (m_0c/\hbar)^2 \)
- **4-Gradient**: \( \partial = (\partial/c, -\nabla) \) = -i\( K \)  \( (\partial \cdot \partial) = -(m_0c/\hbar)^2 = KG Eqn → RQM → QM \)

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn, and thence to QM via the low-velocity limit \( |v| << c \), giving the Schrödinger Eqn. The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM Diagram:
RoadMap of SR (4-Vectors)

4-Gradient
\( \partial = (\partial/c, -\nabla) \)

4-WaveVector
\( K = (\omega/c, k) \)

4-Position
\( R = (ct, r) = \langle \text{Event} \rangle \)

4-Momentum
\( P = (mc, p) = (E/c, p) \)

4-Velocity
\( U = \gamma(c, u) \)

Trace\[T_{\mu\nu}\] = \( \eta_{\mu\nu} T^{\mu\nu} = T \)
\( V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = (V^0)^2 - \vec{v} \cdot \vec{v} = (\vec{v}_0)^2 \)

Lorentz Scalar

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson
SRQM Diagram:
RoadMap of SR (Connections)

4-Gradient: $\partial = (\partial / c, -\vec{V})$

4-Momentum: $P = (mc, p) = (E/c, p)$

4-WaveVector: $K = (\omega/c, k)$

4-Position: $R = (ct, r) = \langle \text{Event} \rangle$

4-Velocity: $U = \gamma(c, u)$

Minkowski Metric: $\delta_{\mu\nu} = \eta_{\mu\nu}$

Lorentz Transform: $\delta[R^\nu] = \Lambda^\nu_{\nu}$

ProperTime: $U \cdot \partial = \gamma \frac{d}{dt}$

Space Time Dim: $\partial \cdot R = 4$

Derivative: $\partial \nu [R^\mu] = \Lambda^\nu_{\nu}$

Trace: $\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T$

Lorentz Scalar: $V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = (V^0)^2 - \vec{V} \cdot \vec{V} = (v^0)^2$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $V^\mu$, or $T^\mu_\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = V = (\vec{V})$

SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_\mu, \vec{v})$

SR 4-Scalar
Lorentz Scalar

Hamilton-Jacobi:
Hamiltonian $P_T = -\partial[S]$

Plane-Waves:
$K_T = -\partial[\Phi]$

SR Phase:
$-K \cdot R = \Phi_{\text{phase}}$

SR Action:
$-P \cdot R = S_{\text{action}}$

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $V^\mu$, or $T^\mu_\nu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = V = (\vec{V})$

SR 4-CoVector
(0,1)-Tensor $V_\mu = (v_\mu, \vec{v})$

SR 4-Scalar
Lorentz Scalar

Trace:
$\text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T$

Lorentz Scalar:
$V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = (V^0)^2 - \vec{V} \cdot \vec{V} = (v^0)^2$
SRQM Diagram: RoadMap of SR (Free Particle)

A Tensor Study of Physical 4-Vectors

4-Gradient = Alteration of SR <Events>
SR SpaceTime Dimension = 4
SR SpaceTime Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves

Minkowski Metric
\( \partial [R^\nu] = \eta^{\mu\nu} \)

Lorentz Transform
\( \partial [R^\nu] = \Lambda^\nu_\mu \)

Space Time Dim
\( \partial R = 4 \)

ProperTime Derivative
\( U \cdot \partial = d/d\tau = \gamma d/dt \)

4-WaveVector
\( K = (\omega/c, k) \)

Plane-Waves
\( K_T = -\partial[\Phi] \)

SR Phase
\( -\partial[\Phi_{\text{phase,free}}] = K \)

4-Velocity
\( U = \gamma(c, u) \)

SR Action
\( -P \cdot R = S_{\text{action,free}} \)

4-Momentum
\( P = (mc, p) = (E/c, p) \)

SR Proper Time Derivative
\( U \cdot \partial = \gamma d/d\tau = d/dt \)

SR Particle <Events> have 4-Momentum = Substantiation mass:energy & 3-momentum

SR Wave <Events> have 4-WaveVector = Substantiation oscillations proportional to mass:energy & 3-momentum

SR SpaceTime Dimension = 4
SR SpaceTime Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves

\( V \cdot V = V^n \eta_{\alpha\beta} V^\alpha = (V^0)^2 - \mathbf{V} \cdot \mathbf{V} = (V^0)^2 \)

\( = \) Lorentz Scalar

Trace[\( T^{\nu\rho} \)] = \( \eta_{\mu\rho} T^{\mu\nu} = T^\nu = T \)

SciRealm.org
John B. Wilson
SRQM Diagram: RoadMap of SR (Free Particle) with Magnitudes

4-Gradient = Alteration of SR <Events>
SR SpaceTime Dimension = 4
SR SpaceTime Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves

4-Gradient
∂ = (∂t/c - ∇)

∂t/c = (d/c)^2 - ∇ · ∇

d'Alembertian Free Particle Wave Equation

SR Wave <Events> have 4-WaveVector = Substantiation
oscillations proportional to mass:energy & 3-momentum

SR 4-Tensor
(2,0)-Tensor T^μν
(1,1)-Tensor T^μν or T^νμ
(0,2)-Tensor T_{μν}

SR 4-Vector
(2,0)-Tensor V^μ = V = (v^0, v)

SR 4-CoVector
(1,0)-Tensor V_μ = V_μ = (v_0, v)

SR 4-Scalar
(0,0)-Tensor S Lorentz Scalar

K · K = (ω/c)^2 - k · k
= (ω/c)^2

SR Particle <Events> have 4-Momentum = Substantiation
mass:energy & 3-momentum

P · P = (E/c)^2 - p · p
= (m/c)^2 = (E/c)^2

SR SpaceTime Metric
\[ \delta^{\mu\nu}[R^n] = \eta^{\mu\nu} \]

Lorentz Transform
\[\Delta R = 4 \text{ Space Time Dim} \]

SR Phase
\[ -\partial[\Phi, \text{phase,free}] = K \]
\[ -\partial[\Phi, \text{phase}] = K_{\tau} \]

Plane-Waves
\[ K_{\tau} = -\partial[\Phi] \]

Hamilton-Jacobi
\[ P_{t, \text{free}} = -\partial[S] \]

Wave Velocity
\[ v_{\text{group}} = c \]
\[ v_{\text{phase}} = c \]

SR Action
\[ -\partial[S, \text{action,free}] = P \]
\[ -\partial[S, \text{action}] = P_{\tau} \]

SR Proper Time Derivative
\[ \gamma_{dt} [..] = \gamma \frac{d}{d\tau} [..] \]

SR Proper Time
\[ \gamma \frac{d}{d\tau} [..] = \gamma \frac{d}{dt} [..] \]

SR Wave <Events> have 4-Velocity = Motion in SR SpaceTime as both particles & waves

4-Position
R = (ct, r) = \langle Event \rangle

4-Momentum
P = (mc, p) = (E/c, p)

4-WaveVector
K = (ω/c, k)

4-Gradient
\[ \partial = (\partial_t/c, -\nabla) \]

Minkowski Transform
\[ \delta^{\mu\nu}[R^n] = \eta^{\mu\nu} \]

SR Proper Time
\[ U \cdot \partial = \gamma \frac{d}{d\tau} \]

\[ U \cdot \partial = \gamma (c^2 - u \cdot u) = (c)^2 \]

\[ U \cdot U = \gamma (c^2 - u \cdot u) \]

SR SpaceTime Metric
\[ \delta^{\mu\nu}[R^n] = \eta^{\mu\nu} \]

SR Lorentz Transforms
\[ \text{SR Phase} \]
\[ -K \cdot R = \Phi_{\text{phase,free}} \]
\[ \text{SR Action} \]
\[ -P \cdot R = S_{\text{action,free}} \]

SR Lorentz Transforms
\[ \text{SR Action} \]
\[ -P \cdot R = S_{\text{action}} \]

SR Proper Time Derivative
\[ \gamma_{dt} [..] = \gamma \frac{d}{d\tau} [..] \]

SR Proper Time
\[ \gamma \frac{d}{d\tau} [..] = \gamma \frac{d}{dt} [..] \]

SR Wave <Events> have 4-Velocity = Motion in SR SpaceTime as both particles & waves

4-Position
R = (ct, r) = \langle Event \rangle

4-Momentum
P = (mc, p) = (E/c, p)

4-WaveVector
K = (ω/c, k)

SR Wave <Events> have 4-WaveVector = Substantiation
oscillations proportional to mass:energy & 3-momentum

SR Particle <Events> have 4-Momentum = Substantiation
mass:energy & 3-momentum

SR Wave <Events> have 4-WaveVector = Substantiation
oscillations proportional to mass:energy & 3-momentum

SR Particle <Events> have 4-Momentum = Substantiation
mass:energy & 3-momentum
SRQM Diagram: RoadMap of SR (EM Potential)

4-Gradient: Alteration of SR <Events>
- SR SpaceTime Dimension=4
- SR SpaceTime Metric
- SR Lorentz Transforms
- SR Action → 4-Momentum
- SR Phase → 4-WaveVector

A Tensor Study of Physical 4-Vectors

SR Proper Time

SR Wave <Events> have 4-WaveVector=Substantiation mass:energy & 3-momentum

SR Particle <Events> have 4-Momentum=Substantiation mass:energy & 3-momentum

SR Wave <Events> oscillations proportional to 4-WaveVector

SR Phase

SR Lorentz Transforms

SR SpaceTime Metric

SR SpaceTime Dimension=4

4-Gradient= \(\partial = (\partial / c)^2 - \nabla \cdot \nabla\)

\(\partial \Phi_{\text{phase,free}} = K\)

\(-\partial [\Phi_{\text{phase}}] = K'\)

Plane-Waves \(K' = -\partial [\Phi]\)

4-WaveVector \(K = (\omega / c, k)\)

\(\mathbf{K} \cdot \mathbf{K} = (\omega / c)^2 - \mathbf{k} \cdot \mathbf{k}\)

\(\mathbf{K} \cdot \mathbf{K} = (\omega / c)^2\)

SR Particle <Events>

SR Action

\(-\partial[S_{\text{action,free}}] = P\)

\(-\partial[S_{\text{action}}] = P_T\)

Hamilton-Jacobi \(P_T = -\partial S\)

Wave Velocity \(v_{\text{wave}} = c^2\)

Einstein \(E = mc^2\)

4-Vector SRQM

4-EMVectorPotential \(A=(\phi / c, a)\)

4-PotentialMomentum \(Q=(V/c, q)=q(\phi / c, a)\)

4-TotalMomentum

\(P_T = (E/c, p_T) = ((E+qp)/c, p+qa)\)

Trace[T^μν] = \(\eta_μν T^μν = T_0^0 = T\)

\(V \cdot V = V^μ V_μ = [(V^0)^2 - V \cdot V] = (V_0^2) = \text{Lorentz Scalar}\)
<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>Discoverer</th>
<th>Physics</th>
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<tr>
<td>4-Position</td>
<td>( R = &lt;\text{Event}&gt; )</td>
<td>Newton+ Einstein</td>
<td>([ t \ &amp; \ r ]) Time &amp; Space Dimensions  ([ R=(ct,r) ]) SpaceTime</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( U = dR/d\tau )</td>
<td>Newton Einstein</td>
<td>([ v=dr/dt ]) Calculus of motion ([ U=\gamma(c,u)=dR/d\tau ] Gamma &amp; Proper Time</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( P = m_0 U )</td>
<td>Newton Einstein</td>
<td>([ p=mv ]) Classical Mechanics ([ P=(E/c,p)=m_0 U ] SR Mechanics</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( K = P/\hbar )</td>
<td>Planck Einstein</td>
<td>([ \hbar ]) Thermal Distribution ([ E=\hbar\nu=\hbar\omega ] Photoelectric Effect ((\hbar=h/2\pi)) ([ p=\hbar k ] Matter Waves</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial = -iK )</td>
<td>Schrödinger</td>
<td>([ \omega=i\partial, \ k=-i\nabla ] (SR) Wave Mechanics</td>
</tr>
</tbody>
</table>

1. The SR 4-Vectors and their components are related to each other via constants
2. We have not taken any 4-vector relation as axiomatic, the constants come from experiment.
3. \( c, \tau, m_0, \hbar \) come from physical experiments, \((-i)\) comes from the general mathematics of waves
The SRQM 4-Vector Relations Explained

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>What it means in SRQM...</th>
<th>Lorentz Invariant</th>
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<tbody>
<tr>
<td>4-Position $\mathbf{R} = (ct, \mathbf{r})$</td>
<td>$\mathbf{R} = \langle\text{Event}\rangle$</td>
<td>SpaceTime as Unified Concept</td>
<td>$c = \text{LightSpeed}$</td>
</tr>
<tr>
<td>4-Velocity $\mathbf{U} = \gamma(c, \mathbf{u})$</td>
<td>$\mathbf{U} = \frac{d\mathbf{R}}{d\tau}$</td>
<td>Velocity is ProperTime Derivative</td>
<td>$\tau = t_0 = \text{ProperTime}$</td>
</tr>
<tr>
<td>4-Momentum $\mathbf{P} = (E/c, \mathbf{p})$</td>
<td>$\mathbf{P} = m_0 \mathbf{U}$</td>
<td>Mass:Energy-Momentum Equivalence</td>
<td>$m_0 = \text{RestMass}$</td>
</tr>
<tr>
<td>4-WaveVector $\mathbf{K} = (\omega/c, \mathbf{k})$</td>
<td>$\mathbf{K} = \frac{\mathbf{P}}{\hbar}$</td>
<td>Wave-Particle Duality</td>
<td>$\hbar = \text{UniversalAction}$</td>
</tr>
<tr>
<td>4-Gradient $\partial = \left(\frac{\partial}{c}, -\nabla\right)$</td>
<td>$\partial = -i\mathbf{K}$</td>
<td>Unitary Evolution, Operator Formalism</td>
<td>$i = \text{ComplexSpace}$</td>
</tr>
</tbody>
</table>

Three old-paradigm QM Axioms: Particle-Wave Duality $[(\mathbf{P})=\hbar(\mathbf{K})]$, Unitary Evolution $[\partial=(-i)\mathbf{K}]$, Operator Formalism $[(\partial)=i\mathbf{K}]$ are actually just empirically-found constant relations between known SR 4-Vectors. Note that these constants are in fact all Lorentz Scalar Invariants.

Minkowski Space and 4-Vectors also lead to idea of Lorentz Invariance. A Lorentz Invariant is a quantity that always has the same value, independent of the motion of inertial observers. Lorentz Invariants can typically be derived using the scalar product relation.

$\mathbf{U}\cdot\mathbf{U} = c^2$, $\mathbf{U}\cdot\partial = d/d\tau$, $\mathbf{P}\cdot\mathbf{U} = m_0 c^2$, etc.

A very important Lorentz invariant is the Proper Time $\tau$, which is defined as the time displacement between two points on a worldline that is at rest wrt. an observer. It is used in the relations between 4-Position $\mathbf{R}$, 4-Velocity $\mathbf{U} = d\mathbf{R}/d\tau$, and 4-Acceleration $\mathbf{A} = d\mathbf{U}/d\tau$. 

SR → QM

A Tensor Study
of Physical 4-Vectors

SciRealm.org
John B. Wilson
SRQM: The SR Path to RQM  
Follow the Invariants...

<table>
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<th>SR 4-Vector</th>
<th>Lorentz Invariant</th>
<th>What it means in SRQM...</th>
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<tbody>
<tr>
<td>4-Position</td>
<td>$\mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct)^2$</td>
<td>SR Invariant Interval</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>$\mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2$</td>
<td>Events move into future at magnitude $c$</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>$\mathbf{P} \cdot \mathbf{P} = (m_0c)^2$</td>
<td>Einstein Mass:Energy Relation</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>$\mathbf{K} \cdot \mathbf{K} = (m_0c/\hbar)^2 = (\omega_0/c)^2$</td>
<td>Matter-Wave Dispersion Relation</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>$\partial \cdot \partial = (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2$</td>
<td>The Klein-Gordon Equation → RQM!</td>
</tr>
</tbody>
</table>

$\mathbf{U} = \frac{d\mathbf{R}}{d\tau}$

Remember, everything after 4-Velocity was just a constant times the last 4-vector, and the Invariant Magnitude of the 4-Velocity is itself a constant

$\mathbf{P} = m_0 \mathbf{U}, \quad \mathbf{K} = \mathbf{P}/\hbar, \quad \partial = -i\mathbf{K}$, so e.g. $\mathbf{P} \cdot \mathbf{P} = m_0 \mathbf{U} \cdot m_0 \mathbf{U} = m_0^2 \mathbf{U} \cdot \mathbf{U} = (m_0c)^2$

The last equation is the Klein-Gordon RQM Equation, which we have just derived without invoking any QM axioms, only SR plus a few empirical facts.
SRQM: Some Basic 4-Vectors, 4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

4-Velocity
\[ \mathbf{U} = \gamma (\mathbf{c}, \mathbf{u}) \]

4-Momentum
\[ \mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p}) \]
\[ \mathbf{P} = -\nabla [S_{\text{action,free}}] \]

4-WaveVector
\[ \mathbf{K} = \left( \frac{\omega}{c}, \mathbf{k} \right) = \left( \frac{\omega}{c}, \frac{\omega}{c} \hat{n}/v \right) \]
\[ \mathbf{K} = -\nabla [\Phi_{\text{phase,plane}}] \]

Treating motion like a particle
Moving particles have a 4-Velocity
4-Momentum is the negative 4-Gradient of the SR Action (S)

Treating motion like a wave
Moving waves have a 4-Velocity
4-WaveVector is the negative 4-Gradient of the SR Phase (\( \Phi \))

\[ \mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 \]

**Note** This is the Phase (\( \Phi \)) for a single plane-wave.
Generally WavePhase is for the 4-TotalWaveVector \( \mathbf{K}_T \) of a system.

See Hamilton-Jacobi Formulation of Mechanics for info on the Lorentz Scalar Invariant SR Action. (P = (E/c, p) = -\( \partial [S_{\text{action,free}}] \))

\{temporal component\} \( E = -\partial /\partial t [S] = -\partial [S] \)

\{spatial component\} \( \mathbf{p} = \nabla [S] \)

**Note** This is the Action \( S_{\text{action}} \) for a free particle.
Generally Action is for the 4-TotalMomentum \( \mathbf{P}_T \) of a system.
SRQM: Wave-Particle Diffraction/Interference Types

The 4-Vector Wave-Particle relation is inherent in all particle types: Einstein-de Broglie \( P = (E/c, p) = \hbar K = \hbar ( \omega/c, k) \).

All waves can diffract: Water waves, gravitational waves, photonic waves of all frequencies, etc. In all cases: experiments using single particles build the diffraction/interference pattern over the course many iterations.

Photon/light Diffraction: Photonic particles diffracted by matter particles. Photons of any frequency encounter a “solid” object or grating. Most often encountered are diffraction gratings and the famous double-slit experiment.

Matter Diffraction: Matter particles diffracted by matter particles. Electrons, neutrons, atoms, small molecules, buckyballs (fullerenes), macromolecules, etc. have been shown to diffract through crystals. Crystals may be solid single pieces or in powder form.

Kapitsa-Dirac Diffraction: Matter particles diffracted by photonic standing waves. Electrons, atoms, super-sonic atom beams have been diffracted from resonant standing waves of light.

Photonic-Photonic Diffraction?: Delbruck scattering. Light-by-light scattering/two-photon physics/gamma-gamma physics. Normally, photons do not interact, but at high enough relative energy, virtual particles can form which allow interaction.
Hold on, aren't you getting the “ℏ” from a QM Axiom?

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>SR Empirical Fact</th>
<th>What it means...</th>
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<td>4-WaveVector</td>
<td>( K = (\omega/c, k) = (\omega/c, \omega \hat{n}/v) )</td>
<td>Wave-Particle Duality</td>
</tr>
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</table>

ℏ is actually an empirically measurable quantity, just like e or c. It can be measured classically from the photoelectric effect, the inverse photoelectric effect, from LED’s (injection electroluminescence), from the Duane-Hunt Law in Bremsstrahlung, Electron Diffraction in crystals, the Watt/Kibble-Balance, etc.

For the LED experiment, one uses several different LED's, each with its own characteristic wavelength. One then makes a chart of wavelength (\( \lambda \)) vs threshold voltage (V) needed to make each individual LED emit. One finds that: \( \lambda = h*c/(eV) \), where e=ElectronCharge and c=LightSpeed. ℏ is found by measuring the slope. Consider this as a blackbox where no assumption about QM is made. However, we know the SR relations \{E = eV\}, and \{\lambda f = c\}. The data force one to conclude that \{E = hf = ℏω\}.

Due to manifest tensor invariance, this means that 4-Momentum \( P = (E/c, p) = ℏK = ℏ(\omega/c, k) = ℏ^*4\)-WaveVector \( K \).

The spatial component (due to De Broglie) follows naturally from the temporal component (due to Einstein) via the nature of 4-Vector mathematics.

This is also derivable from pure SR 4-Vector (Tensor) arguments: \( P = m_0U = (E_0/c^2)U \) and \( K = (\omega/c^2)U \)

Since \( P \) and \( K \) are both Lorentz Scalar proportional to \( U \), then by the rules of tensor mathematics, \( P \) must also be Lorentz Scalar proportional to \( K \). i.e. Tensors obey certain mathematical structures:

Transitivity{if a~b and b~c, then a~c} & Euclideaness: {if a~c and b~c, then a~b} **Not to be confused with the Euclidean Metric**

This invariant proportional constant is empirically measured to be (ℏ) for each known particle type, massive (\( m_0>0 \)) or massless (\( m_0=0 \)):

\[
P = m_0U = (E_0/c^2)U = (E_0/c^2)/(\omega_0/c^2)K = (E_0/\omega_0)K = (\gamma E/\gamma \omega_0)K = (E/\omega)K = (ℏ)K
\]
Hold on, aren't you getting the “K” from a QM Axiom?

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<td>4-WaveVector</td>
<td>$K = (\omega/c, k) = (\omega/c, \omega \hat{n}/v_{\text{phase}}) = (\omega/c^2)U$</td>
<td>Wave-Particle Duality</td>
</tr>
</tbody>
</table>

**K** is a standard SR 4-Vector, used in generating the SR formulae:

**Relativistic Doppler Effect:**

$$\omega_{\text{obs}} = \frac{\omega_{\text{emit}}}{\gamma(1 - \beta \cos[\theta])}, \quad k = \frac{\omega}{c} \text{ for photons}$$

**Relativistic Aberration Effect:**

$$\cos[\theta_{\text{obs}}] = \frac{(\cos[\theta_{\text{emit}}] + |\beta|)}{(1 + |\beta|\cos[\theta_{\text{emit}}])}$$

The 4-WaveVector $K$ can be derived in terms of periodic motion, where families of surfaces move through space as time increases, or alternately, as families of hypersurfaces in SpaceTime, formed by all events passed by the wave surface. The 4-WaveVector is everywhere in the direction of propagation of the wave surfaces.

$$K = -\partial \Phi_{\text{phase}}$$

From this structure, one obtains relativistic/wave optics without ever mentioning QM.
Hold on, aren't you getting the “-i” from a QM Axiom?

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>SR Empirical Fact</th>
<th>What it means...</th>
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<tbody>
<tr>
<td>4-Gradient</td>
<td>$\partial = (\partial_t/c,-\nabla) = -iK$</td>
<td>Unitary Evolution of States Operator Formalism</td>
</tr>
</tbody>
</table>

$[\partial = -iK]$ gives the sub-equations $[\partial_t = -i\omega]$ and $[\nabla = ik]$, and is certainly the main equation that relates QM and SR by allowing Operator Formalism. But, this is a basic equation regarding the general mathematics of plane-waves; not just quantum-waves, but anything that can be mathematically described by plane-waves and superpositions of plane-waves...

This includes purely SR waves, an example of which would be EM plane-waves (i.e. photons)...

$\psi(t,r) = ae^{[i(k \cdot r - \omega t)]}$: Standard mathematical plane-wave equation

$\partial[\psi(t,r)] = \partial[ae^{[i(k \cdot r - \omega t)]}] = (-i\omega)[ae^{[i(k \cdot r - \omega t)]}] = (-i\omega)\psi(t,r)$, or $[\partial_t = -i\omega]$

$\nabla[\psi(t,r)] = \nabla[ae^{[i(k \cdot r - \omega t)]}] = (ik)[ae^{[i(k \cdot r - \omega t)]}] = (ik)\psi(t,r)$, or $[\nabla = ik]$

In the more economical SR notation:

$\partial[\psi(R)] = \partial[ae^{(-iK \cdot R)}] = (-iK)[ae^{(-iK \cdot R)}] = (-iK)\psi(R)$, or $[\partial = -iK]$

This one is more of a mathematical empirical fact, but regardless, it is not axiomatic. It can describe purely SR waves, again without any mention of QM.
Hold on, aren't you getting the “∂” from a QM Axiom?

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<td>4D Gradient Operator</td>
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</table>

\[ \partial = (\partial_t/c, -\nabla) \] is the SR 4-Vector Gradient Operator. It occurs in a purely relativistic context without ever mentioning QM.

\[ \partial \cdot \mathbf{X} = (\partial_t/c, -\nabla) \cdot (ct, \mathbf{x}) = (\partial_t/c[ct] - (-\nabla \cdot \mathbf{x})) = (\partial_t[t] + \nabla \cdot \mathbf{x}) (1)+(3) = 4 \]

The 4-Divergence of the 4-Position \( \partial \cdot \mathbf{X} = \partial^{\mu}\eta_{\mu\nu}X^\nu \) gives the dimensionality of SpaceTime.

\[ \partial[J] = (\partial_t/c, -\nabla)(ct, \mathbf{j}) = (\partial_t/c[ct], -\nabla[J]) = \text{Diag}[1,-1] = \eta^{\mu\nu} \]

The 4-Gradient acting on the 4-Position \( \partial[J] = \partial^{\mu}[X^\nu] \) gives the Minkowski Metric Tensor.

\[ \partial \cdot \mathbf{J} = (\partial_t/c, -\nabla) \cdot (pc, \mathbf{j}) = (\partial_t/c[pc] - (-\nabla \cdot \mathbf{j})) = (\partial_t[p] + \nabla \cdot \mathbf{j}) = 0 \]

The 4-Divergence of the 4-CurrentDensity is equal to 0 for a conserved current. It can be rewritten as \( \partial_t[p] = -\nabla \cdot \mathbf{j} \), which means that the time change of ChargeDensity is balanced by the space change or divergence of CurrentDensity. It is a Continuity Equation, giving local conservation of ChargeDensity. It is related to Noether's Theorem.
Hold on, doesn’t using “∂” in an Equation of Motion presume a QM Axiom?

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<td>4-(Position)Gradient</td>
<td>∂ₚ = ∂ = (∂/c, -∇) = -iK</td>
<td>4D Gradient Operator</td>
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</table>

Klein-Gordon Relativistic Quantum Wave Equation
\[ \partial \cdot \partial [\Psi] = -(\frac{m_o c}{\hbar})^2 [\Psi] = -(\frac{\omega_o}{c})^2 [\Psi] \]

Relativistic Euler-Lagrange Equations
\[ \partial_R [L] = (d/d\tau)\partial_U [L]: \{\text{particle format}\} \]
\[ \partial_\Phi [L] = (\partial_R) \partial_{\partial_R(\Phi)} [L]: \{\text{density format}\} \]

[∂ = (∂/c, -∇)] is the SR 4-Vector (Position)Gradient Operator.
It occurs in a purely relativistic context without ever mentioning QM.
There is a long history of using the gradient operator on classical physics functions, in this case the Lagrangian. And, in fact, it is another area where the same mathematics is used in both classical and quantum contexts.
The QM Schrödinger Relation

\[ P = i\hbar \partial \] 

This is derived from the combination of:

The Einstein-de Broglie Relation

\[ P = \hbar K \]

Complex Plane-Waves

\[ K = i\partial \]

These are the standard QM Schrödinger Relations.

It is this Lorentz Scalar Invariant relation \((i\hbar)\) which connects the 4-Momentum to the 4-Gradient, making it into a QM operator.

Note that these 4-Vectors are already connected in multiple ways in standard SR.

SR 4-Vector

\[ \mathbf{V} = (\mathbf{c}, \mathbf{v}) \]

SR 4-Scalar

\[ S = mc^2 \]

SR 4-EMVectorPotential

\[ \mathbf{A} = (\phi/c, a) \]

SR 4-PotentialMomentum

\[ \mathbf{Q} = (V/c, q) \]

SR 4-TotalMomentum

\[ \mathbf{P}_T = (E/c, p_T) = ((E+q\phi)/c, p+qa) \]
Review of SR 4-Vector Mathematics

4-Gradient \( \partial = (\partial_c, -\nabla) \)
4-Position \( \mathbf{X} = (ct, \mathbf{x}) \)
4-Velocity \( \mathbf{U} = \gamma(c, \mathbf{u}) \)
4-Momentum \( \mathbf{P} = (E/c, \mathbf{p}) = (E_0/c^2)\mathbf{U} \)
4-WaveVector \( \mathbf{K} = (\omega/c, \mathbf{k}) = (\omega_0/c^2)\mathbf{U} \)

\( \partial \cdot \mathbf{X} = (\partial_c, -\nabla) \cdot (ct, \mathbf{x}) = (\partial_c \cdot ct - (\nabla \cdot \mathbf{x})) = 1 - (-3) = 4: \)
\( \mathbf{U} \cdot \partial = \gamma(c, \mathbf{u}) \cdot (\partial_c, -\nabla) = \gamma(\partial + \mathbf{u} \cdot \nabla) = \gamma(d/dt) = d/d\tau: \)
\( \partial[\mathbf{X}] = (\partial_c, -\nabla)(ct, \mathbf{x}) = (\partial_c \cdot ct, -\nabla[\mathbf{x}]) = \text{Diag}[1, -1] = \eta^{iv}: \)
\( \partial[\mathbf{K}] = (\partial_c, -\nabla)(\omega/c, \mathbf{k}) = (\partial_c \cdot \omega/c, -\nabla[\mathbf{k}]) = [[0]] \)
\( \mathbf{K} \cdot \mathbf{X} = (\omega/c, \mathbf{k}) \cdot (ct, \mathbf{x}) = (\omega t - \mathbf{k} \cdot \mathbf{x}) = \phi: \)
\( \partial[\mathbf{K} \cdot \mathbf{X}] = \partial[\mathbf{K}] \cdot \mathbf{X} + \mathbf{K} \cdot \partial[\mathbf{X}] = \mathbf{K} = -\partial[\phi]: \)

\( (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}] = ((\partial_c)^2 - \nabla \cdot \nabla)(\omega t - \mathbf{k} \cdot \mathbf{x}) = 0 \)
\( (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}] = \partial \cdot (\partial[\mathbf{K} \cdot \mathbf{X}]) = \partial \cdot \mathbf{K} = 0: \)

let \( f = ae^{b(k \cdot x)}: \)
then \( \partial[f] = (-i\mathbf{K})ae^{-i(k \cdot x)} = (-i\mathbf{K})f: \quad (\partial = -i\mathbf{K}) \)
and \( \partial \cdot \partial[f] = (-i)^2(k \cdot k)f = -(\omega_0/c)^2f: \)
\( (\partial \cdot \partial) = (\partial_c)^2 - \nabla \cdot \nabla = -(\omega_0/c)^2: \)

Dimensionality of SpaceTime
Derivative wrt. ProperTime is Lorentz Scalar
The Minkowski Metric
Phase of SR Wave
Neg 4-Gradient of Phase gives 4-WaveVector
Wave Continuity Equation, No sources or sinks
Standard mathematical plane-waves if \( \{b = -i\} \)
Unitary Evolution, Operator Formalism
The Klein-Gordon Equation \( \rightarrow \) RQM

Note that no QM Axioms are assumed: This is all just pure SR 4-vector (tensor) manipulation
Review of SR 4-Vector Mathematics

Klein-Gordon Equation: \( \partial \cdot \partial = (\partial/\gamma c)^2 - \nabla \cdot \nabla = -(m_0 c/\hbar)^2 = -(\omega_0/c)^2 = -(1/\Lambda_c)^2 \)

Let \( X_T = (ct + c\Delta t, x) \), then \( \partial[X_T] = (\partial/c, -\nabla)(ct + c\Delta t, x) = \text{Diag}[1,-I_(3)] = \partial[X] = \eta^{\mu\nu} \)
so \( \partial[X_T] = \partial[X] \) and \( \partial[K] = [[0]] \)

let \( f = ae^{-i(K \cdot X_T)} \), the time translated version

\[
\begin{align*}
& (\partial \cdot \partial)[f] \\
& \partial \cdot (\partial[f]) \\
& \partial \cdot (\partial[e^{-i(K \cdot X_T)}]) \\
& \partial \cdot (e^{-i(K \cdot X_T)} \partial[-i(K \cdot X_T)]) \\
& -i \partial \cdot (f \partial[K \cdot X_T]) \\
& -i \partial[f] \partial[K \cdot X_T] + \Psi(\partial \cdot \partial)[K \cdot X_T]) \\
& (-i)^2 f(\partial[K \cdot X_T])^2 + 0 \\
& (-i)^2 f(\partial[K] \cdot X_T + K \cdot \partial[X_T])^2 \\
& (-i)^2 f(0 + K \cdot \partial[X])^2 \\
& (-i)^2 f(K)^2 \\
& -(K \cdot K)f \\
& -((\omega_0/c)^2) f
\end{align*}
\]
What does the Klein-Gordon Equation give us?... **A lot of RQM!**

Relativistic Quantum Wave Equation: \( \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = -(m_0 c / \hbar)^2 = (im_0 c / \hbar)^2 = -(c / \omega_0)^2 \)

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles (Scalars)
Factoring the KG Eqn leads to the RQM Dirac Equation for spin=1/2 particles (Spinors)
Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Taking the low-velocity-limit of the KG leads to the standard QM non-relativistic Schrödinger Eqn, for spin=0
Taking the low-velocity-limit of the Dirac leads to the standard QM non-relativistic Pauli Eqn, for spin=1/2

Setting RestMass \( \{m_0 \rightarrow 0\} \) leads to the RQM Free Wave, Weyl, and Free Maxwell Eqns

In all of these cases, the equations can be modified to work with various potentials by using more
SR 4-Vectors, and more empirically found relations between them, e.g. the Minimal Coupling Relations:
4-TotalMomentum \( P_{\text{tot}} = P + qA \), where \( P \) is the particle 4-Momentum, \( (q) \) is a charge, and \( A \) is a 4-VectorPotential, typically the 4-EMVectorPotential.

Also note that generating QM from RQM (via a low-energy limit) is much more natural than attempting to “relativize or
generalize” a given NRQM equation. Facts assumed from a non-relativistic equation may or may not be applicable to
a relativistic one, whereas the relativistic facts are still true in the low-velocity limiting-cases. This leads to the idea
that QM is an approximation only of a more general RQM, just as SR is an approximation only of GR.
## Relativistic Quantum Wave Eqns.

| Spin-(Statistics) | Relativistic Light-like Mass = 0 | Relativistic Matter-like Mass > 0 | Non-Relativistic Limit (|v|<<c) Mass >0 | Field Representation |
|-------------------|----------------------------------|-----------------------------------|---------------------------------|----------------------|
| Bose-Einstein=n   | Free Wave (0-(Boson))            | Klein-Gordon (1/2-(Fermion))      | Schrödinger (1-(Boson))         | Scalar               |
| Fermi-Dirac=n/2   | N-G Bosons                      | Higgs Bosons, maybe Axions        | Common NRQM Systems             | (0-Tensor)           |
|                   | (\(\partial\cdot\partial\))\(\Psi = 0\) | (\(\partial\cdot\partial + (m_c^2c^2)\))\(\Psi = 0\) | (\(i\hbar\partial_i + [\hbar^\gamma/2m_c, V]\))\(\Psi = 0\) | \(\Psi[K_{\lambda}X^\mu]\) |
|                   |                                 | with minimal coupling             | with minimal coupling            | \(\Psi[\Phi]\)       |
|                   |                                 | \((-\hbar^2/m_c)\partial_i\partial_i\Psi + m_c^3\Psi\Psi\) | \((-\hbar^2/m_c)\partial_i\partial_i\Psi + m_c^3\Psi\Psi\) |                     |

### 1-(Boson)

#### Maxwell
- Photons/Gluons
  - (\(\partial\cdot\partial\))\(\mathbf{A} = 0\) free
  - (\(\partial\cdot\partial\))\(\mathbf{A} = \mu_0\mathbf{J}\) w current src where (\(\partial\cdot\mathbf{A}\)) = 0
  - (\(\partial\cdot\partial\))\(\mathbf{A} = \mu_0e\mathbf{E}_\mu\mathbf{E}_{\nu}\) QED

#### Proca
- Force Bosons
  - (\(\partial\cdot\partial + (m_c^2c^2)\))\(\mathbf{A} = 0\) where (\(\partial\cdot\mathbf{A}\)) = 0
  - (\(\partial^\mu(\partial^\nu\mathbf{A} - \partial^\nu\mathbf{A}^*) + (m_c^2c^2)\mathbf{A}^*\)) = 0

### 0-(Boson)

#### Free Wave
- N-G Bosons
  - (\(\partial\cdot\partial\))\(\Psi = 0\)

#### Klein-Gordon
- Higgs Bosons, maybe Axions
  - (\(\partial\cdot\partial + (m_c^2c^2)\))\(\Psi = 0\)
  - with minimal coupling
  - \((-\hbar^2/m_c)\partial_i\partial_i\Psi + m_c^3\Psi\Psi\) = 0

### 1/2-(Fermion)

#### Weyl
- Idealized Matter Neutinos
  - (\(\sigma\cdot\partial\))\(\Psi = 0\)
  - factored to Right & Left Spinors
    - (\(\sigma\cdot\partial\))\(\Psi_R = 0\), (\(\sigma\cdot\partial\))\(\Psi_L = 0\)

#### Dirac
- Matter Leptons/Quarks
  - (\(i\gamma\cdot\partial - m_c^2c\))\(\Psi = 0\)
  - with minimal coupling
    - \((-\hbar^2/m_c)\partial_i\partial_i\Psi + m_c^3\Psi\Psi\) = 0

### 1-(Fermion)

#### Pauli
- Common NRQM Systems w Spin
  - (\(i\hbar\partial_i - [\sigma\cdot(p^\gamma)]/2m_c\))\(\Psi = 0\)
  - with minimal coupling
    - \((-\hbar^2/m_c)\partial_i\partial_i\Psi + m_c^3\Psi\Psi\) = 0

#### Proca
- Force Bosons
  - (\(\partial\cdot\partial + (m_c^2c^2)\))\(\mathbf{A} = 0\) where (\(\partial\cdot\mathbf{A}\)) = 0
  - (\(\partial^\mu(\partial^\nu\mathbf{A} - \partial^\nu\mathbf{A}^*) + (m_c^2c^2)\mathbf{A}^*\)) = 0

#### Maxwell
- Photons
  - (\(\partial\cdot\partial\))\(\mathbf{A} = 0\) free
  - (\(\partial\cdot\partial\))\(\mathbf{A} = \mu_0\mathbf{J}\) w current src where (\(\partial\cdot\mathbf{A}\)) = 0
  - (\(\partial\cdot\partial\))\(\mathbf{A} = \mu_0e\mathbf{E}_\mu\mathbf{E}_{\nu}\) QED

#### Proca
- Force Bosons
  - (\(\partial\cdot\partial + (m_c^2c^2)\))\(\mathbf{A} = 0\) where (\(\partial\cdot\mathbf{A}\)) = 0
  - (\(\partial^\mu(\partial^\nu\mathbf{A} - \partial^\nu\mathbf{A}^*) + (m_c^2c^2)\mathbf{A}^*\)) = 0
Klein-Gordon Equation: \( \partial^2 - (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_o c^2/\hbar)^2 \)

Since the 4-vectors are related by constants, we can go back to the 4-Momentum description:

\[ (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_o c^2/\hbar)^2 \]
\[ (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (m_o c^2)^2 \]
\[ E^2 - c^2 \mathbf{p} \cdot \mathbf{p} - (m_o c^2)^2 = 0 \]

Factoring: \[ [ E - c \mathbf{a} \cdot \mathbf{p} - \beta (m_o c^2) ] [ E + c \mathbf{a} \cdot \mathbf{p} + \beta (m_o c^2) ] = 0 \]

\( E \) & \( \mathbf{p} \) are quantum operators,
\( \mathbf{a} \) & \( \beta \) are matrices which must obey \( \mathbf{a} \beta = -\beta \mathbf{a}, \mathbf{a} \beta = -\mathbf{a} \beta, \mathbf{a}^2 = \beta^2 = I \)

The left hand term can be set to 0 by itself, giving...
\[ [ E - c \mathbf{a} \cdot \mathbf{p} - \beta (m_o c^2) ] = 0 \], which is one form of the Dirac equation

Remember: \( \mathbf{P}^\mu = (p^0, \mathbf{p}) = (E/c, \mathbf{p}) \) and \( \mathbf{a}^\mu = (\alpha^0, \mathbf{a}) \) where \( \alpha^0 = I_2 \)

\[ [ E - c \mathbf{a} \cdot \mathbf{p} - \beta (m_o c^2) ] = [ c\alpha^0 p^0 - c \mathbf{a} \cdot \mathbf{p} - \beta (m_o c^2) ] = [ c\alpha^0 p^\mu - \beta (m_o c^2) ] = 0 \]
\[ [ \alpha^\mu p_\mu - \beta (m_o c) ] = [ i\hbar \alpha^\mu \partial_\mu - \beta (m_o c) ] = 0 \]
\( \alpha^\mu \partial_\mu = -\beta (im_o c/\hbar) \)

Transforming from Pauli Spinor (2 component) to Dirac Spinor (4 component) form:
Dirac Equation: \( (\gamma^\mu \partial_\mu) \psi = -(im_o c/\hbar) \psi \)

Thus, the Dirac Eqn is guaranteed by construction to be one solution of the KG Eqn

The KG Equation is at the heart of all the various relativistic wave equations, which differ based on mass and spin values, but all of them respect \( E^2 - c^2 \mathbf{p} \cdot \mathbf{p} - (m_o c^2)^2 = 0 \)
Relativistic Quantum Wave Equation: \( \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = -(m_0 c / \hbar)^2 = (im_0 c / \hbar)^2 = -(\omega_0 / c)^2 \)
\( \partial \cdot \partial = -(m_0 c / \hbar)^2 \)

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles \{Higgs\} (4-Scalars)
Factoring the KG Eqn leads to the RQM Dirac Equation for spin=1/2 particles (4-Spinors)
Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Setting RestMass \( \{m_0 \rightarrow 0\} \) leads to the:
RQM Free Wave (4-Scalar massless)
RQM Weyl (4-Spinor massless)
Free Maxwell Eqns (4-Vector massless)

So, the same Relativistic Quantum Wave Equation is simply applied to different SR Tensorial Quantum Fields
See Mathematical_formulation_of_the_Standard_Model at Wikipedia:

4-Scalar (massive)  
Higgs Field \( \varphi \)  
[\( \partial \cdot \partial = -(m_0 c / \hbar)^2 \)\( \varphi \)]  
Free Field Eqn→Klein-Gordon Eqn  
\( \partial \cdot \varphi = -(m_0 c / \hbar)^2 \varphi \)

4-Vector (massive)  
Weak Field \( Z^\mu \), \( W^\mu \)  
[\( \partial \cdot \partial = -(m_0 c / \hbar)^2 \)\( Z^\mu \)]  
Free Field Eqn→Proca Eqn  
\( \partial \cdot Z^\mu = -(m_0 c / \hbar)^2 Z^\mu \)

4-Vector (massless \( m_0 = 0 \))  
Photon Field \( A^\mu \)  
[\( \partial \cdot \partial = 0 \)\( A^\mu \)]  
Free Field Eqn→EM Wave Eqn  
\( \partial \cdot A^\mu = 0 \)

4-Spinor (massive)  
Fermion Field \( \psi \)  
[\( \gamma \cdot \partial = -im_0 c / \hbar \)\( \psi \)]  
Free Field Eqn→Dirac Eqn  
\( \gamma \cdot \psi = -(im_0 c / \hbar) \psi \)

*The Fermion field is a special case, the Dirac Gamma Matrices \( \gamma^\mu \) and 4-Spinor field \( \psi \) work together to preserve Lorentz Invariance.
In relativistic quantum mechanics and quantum field theory, the Bargmann–Wigner equations describe free particles of arbitrary spin $j$, an integer for bosons ($j = 1, 2, 3 ...$) or half-integer for fermions ($j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} ...$). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields.

Bargmann–Wigner equations: 

$$(-\gamma_{\mu} P_{\mu} + mc)_{\alpha\alpha'_{1}...\alpha'_{r...\alpha_{2j}}} \psi_{\alpha_{1}...\alpha_{r}...\alpha_{2j}} = 0$$

In relativistic quantum mechanics and quantum field theory, the Joos–Weinberg equation is a relativistic wave equations applicable to free particles of arbitrary spin $j$, an integer for bosons ($j = 1, 2, 3 ...$) or half-integer for fermions ($j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} ...$). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields. The spin quantum number is usually denoted by $s$ in quantum mechanics, however in this context $j$ is more typical in the literature.

Joos–Weinberg equation: 

$$[\gamma^{\mu_{1}\mu_{2}...\mu_{2j}} P_{\mu_{1}} P_{\mu_{2}} ... P_{\mu_{2j}} + (mc)^{2j}] \Psi = 0$$

The primary difference appears to be the expansion in either the wavefunctions for (BW) or the Dirac Gamma’s for (JW)

For both of these: A state or quantum field in such a representation would satisfy no field equation except the Klein-Gordon equation.

Yet another form is the Duffin-Kemmer-Petiau Equation vs Dirac Equation

DKP Eqn {spin 0 or 1}: $(i\hbar \gamma^a \partial_a - m_o c)\Psi = 0$, with $\beta^a$ as the DKP matrices

Dirac Eqn (spin $\frac{1}{2}$): $(i\hbar \gamma^a \partial_a - m_o c)\Psi = 0$, with $\gamma^a$ as the Dirac Gamma matrices
## A few more SR 4-Vectors

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<th>Definition</th>
<th>Units</th>
</tr>
</thead>
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<td>4-Position</td>
<td>( \mathbf{R} = (ct, \mathbf{r}); \text{ alt. } \mathbf{X} = (ct, \mathbf{x}) )</td>
<td>Time, Space</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( \mathbf{U} = \gamma (c, \mathbf{u}) )</td>
<td>Gamma, Velocity</td>
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<tr>
<td>4-Momentum</td>
<td>( \mathbf{P} = (E/c, \mathbf{p}) = (mc, \mathbf{p}) )</td>
<td>Energy:Mass, Momentum</td>
</tr>
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<td>4-WaveVector</td>
<td>( \mathbf{K} = (\omega/c, \mathbf{k}) = (\omega/c, \omega \hat{n}/v_{\text{phase}}) )</td>
<td>Frequency, WaveNumber</td>
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<td>( \partial = (\partial t/c, -\nabla) )</td>
<td>Temporal Partial, Space Partial</td>
</tr>
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<td>4-VectorPotential</td>
<td>( \mathbf{A} = (\varphi/c, \mathbf{a}) )</td>
<td>Scalar Potential, Vector Potential</td>
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<td>4-TotalMomentum</td>
<td>( \mathbf{P}_{\text{tot}} = (E/c+q\varphi/c, \mathbf{p}+qa) )</td>
<td>Energy-Momentum inc. EM fields</td>
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<td>4-CurrentDensity</td>
<td>( \mathbf{J} = (c\rho, \mathbf{j}) = q\mathbf{J}_{\text{prob}} )</td>
<td>Charge Density, Current Density</td>
</tr>
<tr>
<td>4-ProbabiltyCurrentDensity</td>
<td>( \mathbf{J}<em>{\text{prob}} = (c\rho</em>{\text{prob}}, \mathbf{j}_{\text{prob}}) )</td>
<td>QM Probability (Density, Current Density)</td>
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*can have complex values*
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<td>SpaceTime as Single United Concept</td>
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<td>$\mathbf{U} = \frac{d\mathbf{R}}{d\tau}$</td>
<td>Velocity is Proper Time Derivative</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>$\mathbf{P} = m_o \mathbf{U} = \left(\frac{E_o}{c^2}\right)\mathbf{U}$</td>
<td>Mass-Energy-Momentum Equivalence</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>$\mathbf{K} = \mathbf{P}/\hbar = \left(\frac{\omega_o}{c^2}\right)\mathbf{U}$</td>
<td>Wave-Particle Duality</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>$\partial = -i\mathbf{K}$</td>
<td>Unitary Evolution of States</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Operator Formalism, Complex Waves</td>
</tr>
<tr>
<td>4-VectorPotential</td>
<td>$\mathbf{A} = (\varphi/c,\mathbf{a}) = \left(\frac{\varphi_o}{c^2}\right)\mathbf{U}$</td>
<td>Potential Fields...</td>
</tr>
<tr>
<td>4-TotalMomentum</td>
<td>$\mathbf{P}_{\text{tot}} = \mathbf{P} + q\mathbf{A}$</td>
<td>Energy-Momentum inc. Potential Fields</td>
</tr>
<tr>
<td>4-TotalWaveVector</td>
<td>$\mathbf{K}_{\text{tot}} = \mathbf{K} + \left(q/\hbar\right)\mathbf{A}$</td>
<td>Freq-WaveNum inc. Potential Fields</td>
</tr>
<tr>
<td>4-CurrentDensity</td>
<td>$\mathbf{J} = \rho_o \mathbf{U} = q\mathbf{J}_{\text{prob}}$</td>
<td>ChargeDensity-CurrentDensity Equivalence</td>
</tr>
<tr>
<td></td>
<td>$\partial \cdot \mathbf{J} = 0$</td>
<td>CurrentDensity is conserved</td>
</tr>
<tr>
<td>4-Probability CurrentDensity</td>
<td>$\mathbf{J}<em>{\text{prob}} = (c\mathbf{p}</em>{\text{prob}},\mathbf{j}_{\text{prob}})$</td>
<td>QM Probability from SR</td>
</tr>
<tr>
<td></td>
<td>$\partial \cdot \mathbf{J}_{\text{prob}} = 0$</td>
<td>Probability Worldlines are conserved</td>
</tr>
</tbody>
</table>
Minimal Coupling = Potential Interaction

Klein-Gordon Eqn $\rightarrow$ Schrödinger Eqn

\[ P = P + Q = P + qA \]

Minimal Coupling: Total = Dynamic + Charge Coupled to 4-(EM)Vector Potential

\[ K = i\partial \]

Complex Plane-Waves

\[ P = \hbar K \]

Einstein-de Broglie QM Relations

\[ P = i\hbar \partial \]

Schrödinger Relations

\[ P = (E/c, p) = P_T - qA = (E_T/c - q\phi/c, p_T - qa) \]

\[ = \hbar K = i\hbar \partial \]

\[ \delta \cdot \partial = (\partial/c)^2 - \nabla^2 = (m_o c/\hbar)^2 \]

\[ \delta \cdot \partial = (\partial/c)^2 - \nabla^2 = -(m_o c/\hbar)^2 \]

\[ \vec{P} \cdot \vec{P} = (E/c)^2 - p^2 = (m_o c)^2 \]

Relativistic

\[ E^2 = (m_o c)^2 + c^2 p^2 \]

\[ E \sim [(m_o c)^2 + p^2/2m_o] \]

Relativistic with Minimal Coupling

\[ (E_T - q\phi)^2 = (m_o c)^2 + c^2 (p_T - qa)^2 \]

\[ (E_T - q\phi) \sim [(m_o c)^2 + (p_T - qa)^2/2m_o] \]

Relativistic with Minimal Coupling

\[ (i\hbar \partial_T - q\phi)^2 = (m_o c)^2 + c^2 (-i\hbar \nabla_T - qa)^2 \]

\[ (i\hbar \partial_T - q\phi) \sim [(m_o c)^2 + (-i\hbar \nabla_T - qa)^2/2m_o] \]

Relativistic with Minimal Coupling

\[ (i\hbar \partial_T) \sim [q\phi + (m_o c)^2 + (i\hbar \nabla_T + qa)^2/2m_o] \]

\[ (i\hbar \partial_T) \sim [V + (i\hbar \nabla_T + qa)^2/2m_o] \]

\[ (i\hbar \partial_T) \sim [V - (i\hbar \nabla_T)^2/2m_o] \]

Typically the 3-vector potential \( \vec{a} \sim 0 \) in many situations

\[ (i\hbar \partial_T) \Psi > \sim [V - (i\hbar \nabla_T)^2/2m_o] \Psi > \]

Schrödinger NRQM Wave Equation (non-relativistic QM)

\[ \Psi > \sim [V - (i\hbar \nabla_T)^2/2m_o] \Psi > : \]

The better statement is that the Schrödinger Eqn is the limiting low-velocity case of the more general KG Eqn, not that the KG Eqn is the relativistic generalization of the Schrödinger Eqn
Once one has a **Relativistic Wave Eqn...**

Klein-Gordon Equation:  $\partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2$

Once we have derived a RWE, what does it imply?

The KG Eqn. was derived from the physics of SR plus a few empirical facts. It is a 2\textsuperscript{nd} order, linear, wave PDE that pertains to physical objects of reality from SR.

Just being a linear wave PDE implies all the mathematical techniques that have been discovered to solve such equations generally: Hilbert Space, Superpositions, $<$Bra$|,|Ket>$ notation, wavevectors, wavefunctions, etc. These things are from mathematics in general, not only and specifically from an Axiom of QM.

Therefore, if one has a physical RWE, it implies the mathematics of waves, the formalism of the mathematics, and thus the mathematical Principles and Formalism of QM. Again, QM Axioms are not required – they emerge from the physics and math...
Once one has a Relativistic Wave Eqn... Examine Photon Polarization

From the Wikipedia page on [Photon Polarization]

Photon polarization is the quantum mechanical description of the classical polarized sinusoidal plane electromagnetic wave. An individual photon can be described as having right or left circular polarization, or a superposition of the two. Equivalently, a photon can be described as having horizontal or vertical linear polarization, or a superposition of the two.

The description of photon polarization contains many of the physical concepts and much of the mathematical machinery of more involved quantum descriptions and forms a fundamental basis for an understanding of more complicated quantum phenomena. Much of the mathematical machinery of quantum mechanics, such as state vectors, probability amplitudes, unitary operators, and Hermitian operators, emerge naturally from the classical Maxwell's equations in the description. The quantum polarization state vector for the photon, for instance, is identical with the Jones vector, usually used to describe the polarization of a classical wave. Unitary operators emerge from the classical requirement of the conservation of energy of a classical wave propagating through lossless media that alter the polarization state of the wave. Hermitian operators then follow for infinitesimal transformations of a classical polarization state.

Many of the implications of the mathematical machinery are easily verified experimentally. In fact, many of the experiments can be performed with two pairs (or one broken pair) of polaroid sunglasses.

The connection with quantum mechanics is made through the identification of a minimum packet size, called a photon, for energy in the electromagnetic field. The identification is based on the theories of Planck and the interpretation of those theories by Einstein. The correspondence principle then allows the identification of momentum and angular momentum (called spin), as well as energy, with the photon.
Principle of Superposition:
From the mathematics of waves

Klein-Gordon Equation: \( \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = -(m_0 c / \hbar)^2 = -(\omega_0 / c)^2 \)

The Extended Superposition Principle for Linear Equations

Suppose that the non-homogeneous equation, where \( L \) is linear, is solved by some particular \( u_p \).
Suppose that the associated homogeneous problem is solved by a sequence of \( u_i \).
\[ L(u_p) = C; \quad L(u_0) = 0, \quad L(u_1) = 0, \quad L(u_2) = 0 \ldots \]
Then \( u_p \) plus any linear combination of the \( u_n \) satisfies the original non-homogeneous equation:
\[ L(u_p + \sum a_n u_n) = C, \]
where \( a_n \) is a sequence of (possibly complex) constants and the sum is arbitrary.

Note that there is no mention of partial differentiation. Indeed, it's true for any linear equation, algebraic or integro-partial differential-whatever.

QM superposition is not axiomatic, it emerges from the mathematics of the Linear PDE.
Klein-Gordon obeys

Principle of Superposition

Klein-Gordon Equation: \( \partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_o c^2/h)^2 = -(\omega_o/c)^2 \)

\( \mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_o/c)^2 \): The particular solution (w rest mass)

\( \mathbf{K}_n \cdot \mathbf{K}_n = (\omega_n/c)^2 - \mathbf{k}_n \cdot \mathbf{k}_n = 0 \): The homogenous solution for a (virtual photon?) microstate \( n \)

Note that \( \mathbf{K}_n \cdot \mathbf{K}_n = 0 \) is a null 4-vector (photonic)

Let \( \Psi_p = A e^{-i(\mathbf{K} \cdot \mathbf{X})} \), then \( \partial \cdot \partial \Psi_p = (-i)^2(\mathbf{K} \cdot \mathbf{K})\Psi_p = -(\omega_o/c)^2\Psi_p \)

which is the Klein-Gordon Equation, the particular solution...

Let \( \Psi_n = A_n e^{-i(\mathbf{K}_n \cdot \mathbf{X})} \), then \( \partial \cdot \partial \Psi_n = (-i)^2(\mathbf{K}_n \cdot \mathbf{K}_n)\Psi_n = (0)\Psi_n \)

which is the Klein-Gordon Equation homogeneous solution for a microstate \( n \)

We may take \( \Psi = \Psi_p + \sum_n \Psi_n \)

Hence, the Principle of Superposition is not required as an QM Axiom, it follows from SR and our empirical facts which lead to the Klein-Gordon Equation. The Klein-Gordon equation is a linear wave PDE, which has overall solutions which can be the complex linear sums of individual solutions – i.e. it obeys the Principle of Superposition.

This is not an axiom – it is a general mathematical property of linear PDE's.

This property continues over as well to the limiting case \( \{ |\mathbf{v}| << c \} \) of the Schrödinger Equation.
QM Hilbert Space: From the mathematics of waves

Klein-Gordon Equation: \( \partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla = -(m_0c/\hbar)^2 \)

Hilbert Space (HS) representation:
- if \( |\Psi\rangle \in \text{HS} \), then \( c|\Psi\rangle \in \text{HS} \), where \( c \) is complex number
- if \( |\Psi_1\rangle \) and \( |\Psi_2\rangle \in \text{HS} \), then \( |\Psi_1\rangle + |\Psi_2\rangle \in \text{HS} \)
- if \( |\Psi\rangle = c_1|\Psi_1\rangle + c_2|\Psi_2\rangle \), then \( \langle \Phi | \Psi \rangle = c_1\langle \Phi | \Psi_1 \rangle + c_2\langle \Phi | \Psi_2 \rangle \) and \( \langle \Psi | = c_1^* \langle \Psi_1 | + c_2^* \langle \Psi_2 | \)
- \( \langle \Phi | \Psi \rangle = \langle \Psi | \Phi \rangle \)
- \( \langle \Psi | \Psi \rangle \geq 0 \)
- if \( \langle \Psi | \Psi \rangle = 0 \), then \( |\Psi\rangle = 0 \)
- etc.

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. They are indispensable tools in the theories of partial differential equations, Fourier analysis, signal processing, heat transfer, ergodic theory, and Quantum Mechanics.

The QM Hilbert Space emerges from the fact that the KG Equation is a linear wave PDE – Hilbert spaces as solutions to PDE's are a purely mathematical phenomenon – no QM Axiom is required.

Likewise, this introduces the \( \langle \text{bra}|\text{ket}\rangle \) notation, wavevectors, wavefunctions, etc.

Note:

One can use Hilbert Space descriptions of Classical Mechanics using the Koopman-von Neumann formulation. One can not use Hilbert Space descriptions of Quantum Mechanics by using the Phase Space formulation of QM.
Standard QM Canonical Commutation Relation: $[x^j, p^k] = i\hbar \delta^{jk}$

The Standard QM Canonical Commutation Relation is simply an axiom in standard QM. It is just given, with no explanation. You just had to accept it.

I always found that unsatisfactory.

There are at least 4 parts to it:

Where does the commutation ($[ , ]$) come from?  
Where does the imaginary constant ($i$) come from?  
Where does the Planck constant ($\hbar$) come from?  
Where does the Kronecker Delta ($\delta^{jk}$) come from?

See the next page for SR enlightenment...
Let (f) be an arbitrary SR function 
X[t] = Xf, \quad \partial f = \partial [t] 
X, function or not, has no effect on (f) 
\partial = \partial \] is definitely an SR function:operator

\[
X(\partial f) = [X \partial f] = [X \partial]f = \partial [Xf] - X[\partial f] = \partial [X]f 
\]
Recognize this as a commutation relation 
[ \partial , X ] f = \partial [X] f

\[
\partial , X ] = \partial [X] \\
= \partial \mu [\partial \nu X^\nu] \\
= (\partial /c, \nabla X)(ct,x) \\
= (\partial /c, -\partial X, -\partial X, \partial X)((ct, x, y, z)] \\
= \text{Diag}(1, -1, -1, -1) = \text{Diag}(1, -\delta^k) \\
= \eta^{\mu\nu} = \text{Minkowski Metric}
\]

\[[\partial^\mu, X^\nu] = \eta^{\mu\nu} \quad \text{Tensor form: true for all observers} \]
\[[\partial^\mu, X^\nu] = i\hbar \eta^{\mu\nu} \quad \text{Independently true from empirical constants (i), (\hbar)} \]
\[[p^\mu, x^\nu] = -i\hbar \delta^\mu \nu \]
\[[0^\mu, x^\nu] = [E/c, ct] = [E, t] = i\hbar 
\]

\[[t, E] = -i\hbar 
\]

\{P = \hbar K\} and \{K = i\hbar\} are empirical SR relations
SRQM Study: 4-Position and 4-Gradient

SR: Minkowski Metric
\[ \partial[R] = \partial^\mu R^\nu = \eta^{\mu\nu} \]
→ Diag\[\{1, -1, -1, -1\} \]
= Diag\[\{1, -1, -1, -1\} \]
= Diag\[\{1, -\delta^{ik}\} \]
(in Cartesian form)
"Particle Physics" Convention
\{\eta_{\mu\nu}\} = 1/\{\eta^{\mu\nu}\}
Tr[\eta^{\mu\nu}] = 4
\eta_\mu^\nu = \delta_\mu^\nu

SR: Lorentz Transform
\[ \partial_t[R^{\nu}] = \partial R^{\nu}/\partial R^t = \Lambda^{\nu}_{\mu} \]
\[ \Lambda^\nu_{\mu} = \eta^\nu_{\mu} = \delta^\nu_{\mu} \]
\[ \eta_{\mu\nu} \Lambda^\mu_{\alpha} \Lambda^\alpha_{\nu} = \eta_{\alpha\beta} \] (Det[\Lambda])^2 = 1
\[ \text{Det}[\Lambda] = \pm 1 \]
\[ \Lambda_{\nu\mu} = (\Lambda^{-1})^\mu_{\nu} \]
\[ \Lambda_{\nu\mu} \Lambda^{\mu\nu} = 4 \]

SRQM: Non-Zero Commutation
\[ [\partial, R] = [\partial^\mu, R^\nu] \]
= \partial_\mu R^\nu = \partial^\mu R^\nu

SRQM: Tensor Zero Exterior Product
\[ \partial^\mu R^\nu - \partial^\nu R^\mu = \eta^{\mu\nu} - \eta^{\nu\mu} = 0^{\mu\nu} \]

SRQM: Wave Equation
\[ \partial \partial = (\partial/c)^2 \nabla \nabla = (\partial/c)^2 \]

SR 4-Tensor
(2,0)-Tensor T^{\mu\nu}
(1,1)-Tensor V^\mu = V = (v^0, v^i)
SR 4-Vector
(0,1)-Tensor V_\mu = (v_0, -v^i)
SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]
\[ dR = (cdt, dr) \]

4-Position
\[ R = (ct, r) \]

SR → QM

Invariant Interval
\[ R \cdot R = (ct)^2 - r \cdot r = (ct)^2 \]

4-Gradient
\[ \partial = \partial^t/c, \nabla = (\partial_x, \partial_y, \partial_z) \]

4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]
\[ dR = (cdt, dr) \]

4-Position
\[ R = (ct, r) \]
Heisenberg Uncertainty Principle: Viewed from SRQM

Heisenberg Uncertainty Principle:

\[ \sigma_A^2 \sigma_B^2 \geq \frac{1}{2} \langle [A, B] \rangle \]

arises from the non-commuting nature of certain operators.

The commutator is \([A, B] = AB - BA\), where \(A\) & \(B\) are functional “measurement” operators. The Operator Formalism arose naturally from our SR \(\rightarrow\) QM path: \(\partial = -i\mathbf{K}\).

The generalized uncertainty relation:

\[ \sigma_f^2 \sigma_g^2 = \frac{1}{2} \langle [F, G] \rangle \]

The Cauchy–Schwarz inequality asserts that (for all vectors \(f\) and \(g\) of an inner product space, with either real or complex numbers):

\[ \sigma_f^2 \sigma_g^2 = \| f \langle f | g \rangle \|^2 \]

But first, let's back up a bit; Using standard complex number math, we have:

\[ z = a + ib \]
\[ z^* = a - ib \]
\[ \text{Re}(z) = a = (z + z^*)/(2) \]
\[ \text{Im}(z) = b = (z - z^*)/(2i) \]

Next, generically, based on the rules of a complex inner product space we can arbitrarily assign:

\[ z = \langle f | g \rangle, \quad z^* = \langle g | f \rangle \]

Which allows us to write:

\[ \| f \langle f | g \rangle \|^2 = \| \langle f | f \rangle + \langle g | f \rangle \|^2 \]

We can also note that:

\[ | f \rangle = | F \rangle | \Psi \rangle \text{ and } | g \rangle = | G \rangle | \Psi \rangle \]

Thus,

\[ | f \langle g | \|^2 = [(\langle \Psi | F^* G | \Psi \rangle + (\langle \Psi | G^* F | \Psi \rangle)/2)]^2 + [(\langle \Psi | F^* G | \Psi \rangle - (\langle \Psi | G^* F | \Psi \rangle)/2i)]^2 \]

For Hermetian Operators...

\[ F^* = +F, \quad G^* = +G \]

For Anti-Hermetian (Skew-Hermetian) Operators...

\[ F^* = -F, \quad G^* = -G \]

Assuming that \(F\) and \(G\) are either both Hermetian, or both anti-Hermetian...

\[ | f \langle g | \|^2 = [(\langle \Psi | (\pm)FG | \Psi \rangle + (\langle \Psi | (\pm)GF | \Psi \rangle)/2)]^2 + [(\langle \Psi | (\pm)FG | \Psi \rangle - (\langle \Psi | (\pm)GF | \Psi \rangle)/2i)]^2 \]

We can write this in commutator and anti-commutator notation...

\[ | f \langle g | \|^2 = [(\langle \Psi | {F,G} | \Psi \rangle + (\langle \Psi | i[F,G] | \Psi \rangle)/2)]^2 + [(\langle \Psi | i[F,G] | \Psi \rangle - (\langle \Psi | {F,G} | \Psi \rangle)/2i)]^2 \]

Due to the squares, the \((\pm)\)'s go away, and we can also multiply the commutator by an \((i^2)\)

\[ | f \langle g | \|^2 = [(\langle \Psi | {F,G} | \Psi \rangle + (\langle \Psi | i[F,G] | \Psi \rangle)/2) + [(\langle \Psi | i[F,G] | \Psi \rangle - (\langle \Psi | {F,G} | \Psi \rangle)/2i)]^2 \]

The Cauchy–Schwarz inequality again...

\[ \sigma_f^2 \sigma_g^2 = [(\langle f | f \rangle + \langle g | f \rangle) + \langle f | g \rangle - \langle g | f \rangle) + \langle f | g \rangle - \langle g | f \rangle)]^2 \]

Taking the root:

\[ \sigma_f^2 \sigma_g^2 = (1/2) | i[F,G] | \]

Which is what we had for the generalized Uncertainty Relation.

*Note* This is not a QM axiom - This is just pure math. At this stage we already see the hints of commutation and anti-commutation. It is true generally, whether applying to a physical or purely mathematical situation.
Heisenberg Uncertainty Principle: Simultaneous vs Sequential

Heisenberg Uncertainty \( \{ \sigma^2_A \sigma^2_B \geq (1/2)|<[A,B]>| \} \) arises from the non-commuting nature of certain operators.

\[ [\partial^\mu, X^\nu] = \partial[X] = \eta^{\mu\nu} = \text{Minkowski Metric} \]

\[ [P^\mu, X^\nu] = [i\hbar \partial^\mu, X^\nu] = i\hbar [\partial^\mu, X^\nu] = i\hbar \eta^{\mu\nu} \]

Consider the following:
Operator A acts on System \(|\Psi>\) at SR Event A: \( A|\Psi> \rightarrow |\Psi>'\)
Operator B acts on System \(|\Psi'>\) at SR Event B: \( B|\Psi'> \rightarrow |\Psi''>\)
or \( BA|\Psi> = B|\Psi'> = |\Psi''>\)

If measurement Events A & B are space-like separated, then there are observers who can see \( \{ \text{A before B, A simultaneous with B, A after B} \} \), which of course does not match the quantum description of how Operators act on Kets.

If Events A & B are time-like separated, then all observers will always see A before B. This does match how the operators act on Kets, and also matches how \(|\Psi>\) would be evolving along its worldline, starting out as \(|\Psi>\), getting hit with operator A at Event A to become \(|\Psi'>\), then getting hit with operator B at Event B to become \(|\Psi''>\).

The Uncertainty Relation here does NOT refer to simultaneous (space-like separated) measurements, it refers to sequential (time-like separated) measurements. This removes the need for ideas about the particles not having simultaneous properties. There are simply no “simultaneous measurements” of non-zero commuting properties on an individual system, a single worldline – they are sequential, and the first measurement places the system in such a state that the outcome of the second measurement will be altered wrt. if the order of the operations had been reversed.
The Pauli Exclusion Principle is a result of the empirical fact that nature uses identical particles, and this combined with the Spin-Statistics theorem from SR, leads to an exclusion principle for fermions (anti-symmetric, Fermi-Dirac statistics) and an aggregation principle for bosons (symmetric, Bose-Einstein statistics). The Spin-Statistics Theorem is related as well to the CPT Theorem.

For large numbers and/or mixed states these both tend to the Maxwell-Boltzmann statistics. In the \{kT>>(\varepsilon-\mu)\} limit, Bose-Einstein reduces to Rayleigh-Jeans. The commutation relations here are based on space-like separation particle exchanges, unlike the time-like separation for measurement operator exchanges in the Uncertainty Principle.
Complex 4-vectors are simply 4-Vectors where the components may be complex-valued

\[ \mathbf{A} = A^\mu = (a^0, a^1, a^2, a^3) \rightarrow (a^t, a^x, a^y, a^z) \]

\[ \mathbf{B} = B^\mu = (b^0, b^1, b^2, b^3) \rightarrow (b^t, b^x, b^y, b^z) \]

Examples of 4-Vectors with complex components are the 4-Polarization and the 4-ProbabilityCurrentDensity

Minkowski Metric \( g^{\mu \nu} \rightarrow \eta^{\mu \nu} = \eta_{\mu \nu} \rightarrow \text{Diag}[1,-1,-1,-1] = \text{Diag}[1,-I(3)] \),
which is the {curvature~0 limit = low-mass limit} of the GR metric \( g^{\mu \nu} \).

Applying the Metric to raise or lower an index also applies a complex-conjugation *

Scalar Product = Lorentz Invariant → Same value for all inertial observers
\[ \mathbf{A} \cdot \mathbf{B} = \eta_{\mu \nu} A^\mu B^\nu = A_v^* B^v = A^\mu B^*_\mu = (a^0\ast b^0 - a^\ast \cdot b) \] using the Einstein summation convention

This reverts to the usual rules for real components
However, it does imply that \( \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \)
SRQM: CPT Theorem

Phase Connection, Lorentz Invariance

The Phase is a Lorentz Scalar Invariant – all observers must agree on its value.

The gamma factor (which defines the particle’s worldline at each point.

Note that for matter particles operations only to the particle’s 4-Momentum

The 4-Position

It is only the combination of all three ops: (C,P,T), or pairs of singles: (CC),(PP),(TT)

that leave the Unit-Temporal 4-Vector, and thus the Phase, Invariant.

CPT Theorem

Theorem

they all remain null 4-vectors

They all remain temporal 4-vectors

It's a null 4-vector

It's a temporal 4-vector

They are all temporal 4-vectors

They all remain null 4-vectors

They all remain spatial 4-vectors

It's a null 4-vector

It's a temporal 4-vector

They are all spatial 4-vectors

They all remain null 4-vectors

Trace[T^n] = n^αT^α = T = T

V ∙ V = V^2n^αV^α = (V^0)^2 = Lorentz Scalar
SRQM: CPT Theorem
(Charge) vs (Parity) vs (Time)

Classical SR Time-Reversal neglects spin and charge.
SRQM includes these effects.
Then one gets (CC), (PP), (TT), & (CPT) transforms
all leading back to the Identity (I).

Identity and Space-Parity are Unitary
Time-Reversal and Charge-Conjugation are Anti-Unitary.
SRQM Transforms: Venn Diagram

Poincaré = Lorentz + Translations

(10) (6) (4)

4-Vector SRQM Interpretation of QM

SciRealm.org

John B. Wilson

<table>
<thead>
<tr>
<th>M_{01}</th>
<th>M_{02}</th>
<th>M_{03}</th>
</tr>
</thead>
<tbody>
<tr>
<td>P^0</td>
<td>P^1</td>
<td>P^2</td>
</tr>
<tr>
<td>P^3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4-AngularMomentum $M^\alpha = X^\mu \wedge P^\nu = X^\mu P^\nu \wedge X^\nu P^\mu$

= Generator of Lorentz Transformations (6)

= \{ $\Lambda^\mu = R^\mu = \Lambda^\nu = B^\nu$, Rotations (3) \}

4-LinearMomentum $P^\mu$

= Generator of Translation Transformations (4)

= \{ $\Delta X^\mu = (c\Delta t, 0)$ Time (1) \} or \{ $\Delta X^\mu = (0, \Delta x)$ Space (3) \}

Det[$\Lambda^\mu$] = +1 for Proper Lorentz Transforms

Det[$\Lambda^\mu$] = -1 for Improper Lorentz Transforms

Lorentz Matrices can be generated by a matrix $M$ with $\text{Tr}[M]=0$ which gives:

$\Lambda^\mu = e^M$ or $(e^M)$

Lorentz Transform

$\Lambda^\mu \rightarrow \Lambda^\nu = e^M$

Discrete

Continuous

Lorentz Transform $\Lambda^\mu \rightarrow \Lambda^\nu$

4-Tensor \{mixed type-(1,1)\}

| Time-reversal $\Lambda^\mu \rightarrow T^\nu$ |
| Space parity $\Lambda^\mu \rightarrow P^\nu$ |
| Charge-Conjugation $\Lambda^\mu \rightarrow C^\nu$ |

SpatialFlipCombos $\Lambda^\mu \rightarrow \tilde{\Lambda}^\nu$

Identity $I_{4(0)}$

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation

Lie group of all affine isometries of SR:Minkowski TimeSpace (preserve quadratic form $\eta_{\mu\nu}$)

General Linear,Affine Transform $X^\mu = \Lambda^\mu_{\nu}X^\nu + \Delta X^\mu$

with $\det(\Lambda^\mu_{\nu}) = \pm 1$

($(6+4=10)$)

Discrete

Continuous

Rotation $\Lambda^\mu \rightarrow R^\nu$

Identity $I_{4(0)}$

SpatialFlipCombos $\Lambda^\mu \rightarrow \Lambda^\nu = \Delta^\nu$

Charge-Conjugation $\Lambda^\mu \rightarrow C^\nu$

Parity-Inversion $\Lambda^\mu \rightarrow P^\nu$

R → -R*, q → -q

Rest frame $\Delta X^\nu = (0, 0)$

Spatial $\Delta X^\nu \rightarrow (0, \Delta x)$

Temporal $\Delta X^\nu \rightarrow (c\Delta t, 0)$

Isotropy (same all directions)

Homogeneity (same all points)

$\Lambda^\mu_{\nu} \Lambda^\nu_{\lambda} = \delta^\mu_{\lambda}$

Rotation $J_i \rightarrow -J_i$, $R \rightarrow -R$

Feynman-Stueckelberg Interpretation

Amusingly, Inhomogeneous Lorentz adds homogeneity.
The Hermitian Generators that lead to translations and rotations via unitary operators in QM...

These all ultimately come from the Poincaré Invariance → Lorentz Invariance that is at the heart of SR and Minkowski Space.

Infintesimal Unitary Transformation
\[ \hat{U}_t(\hat{G}) = \mathbb{I} + i\varepsilon \hat{G} \]

Finite Unitary Transformation
\[ \hat{U}_\alpha(\hat{G}) = e^{i\alpha \hat{G}} \]

let \( \hat{G} = \frac{P}{\hbar} = \mathbf{K} \)
let \( \alpha = \Delta x \)

\[ \hat{U}_{\Delta x}(P/h)\Psi(X) = e^{i(\Delta x \cdot P/h)\Psi(X)} = e^{i(-\Delta x \cdot \partial)\Psi(X)} = \Psi(X - \Delta x) \]

Time component: \( \hat{U}_{\Delta t}(P/h)\Psi(ct) = e^{i(\Delta t E/h)\Psi(ct)} = e^{i(-\Delta t \partial_t)\Psi(ct)} = \Psi(ct - c\Delta t) = c\Psi(t - \Delta t) \)

Space component: \( \hat{U}_{\Delta x}(p/h)\Psi(x) = e^{i(\Delta x \cdot p/h)\Psi(x)} = e^{i(\Delta x \cdot \nabla)\Psi(x)} = \Psi(x + \Delta x) \)

By Noether's Theorem, this leads to \( \partial \cdot K = 0 \)

We had already calculated
\[ (\partial \cdot \partial)[K \cdot X] = ((\partial / c)^2 - \nabla \cdot \nabla)(\omega t - k \cdot x) = 0 \]
\[ (\partial \cdot \partial)[K \cdot X] = \partial \cdot (\partial [K \cdot X]) = \partial \cdot K = 0 \]

Poincaré Invariance also gives the Casimir invariants of mass and spin, and ultimately leads to the spin-statistics theorem of RQM.
QM Correspondence Principle: Analogous to the GR and SR limits

Basically, the old school QM Correspondence Principle says that QM should give the same results as classical physics in the realm of large quantum systems, i.e. where macroscopic behavior overwhelms quantum effects. Perhaps a better way to state it is when the change of system by a single quantum has a negligible effect on the overall state.

There is a way to derive this limit, by using Hamilton-Jacobi Theory:

\( (i\hbar\partial_t)|\Psi> \sim [ V - (\hbar\nabla_T)^2/2m_o ]|\Psi> \) : The Schrödinger NRQM Equation for a point particle (non-relativistic QM)

Examine solutions of form \( \Psi = \Psi_o e^{iS/\hbar} \), where S is the QM Action

\( \partial_t[\Psi] = (i/\hbar)\Psi\partial_t[S] \) and \( \partial_x[\Psi] = (i/\hbar)\Psi\partial_x[S] \) and \( \nabla^2[\Psi] = (i/\hbar)\Psi\nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2 \)

\( (i)(i/\hbar)\Psi\partial_t[S] = V\Psi - ((i\hbar/2m_o)(i/\hbar)\Psi\nabla^2[S] - (\Psi/2m_o)(\nabla[S])^2) \)

\( (i)(i)\Psi\partial_x[S] = V\Psi - ((i\hbar/2m_o)\Psi\nabla^2[S] - (\Psi/2m_o)(\nabla[S])^2) \)

\( \partial_t[S] = -V + (i\hbar/2m_o)\nabla^2[S] - (1/2m_o)(\nabla[S])^2 \)

\( \partial_x[S] + [V+(1/2m_o)(\nabla[S])^2] = (i\hbar/2m_o)\nabla^2[S] : \) Quantum Single Particle Hamilton-Jacobi

\( \partial_t[S] + [V+(1/2m_o)(\nabla[S])^2] = 0 : \) Classical Single Particle Hamilton-Jacobi

Thus, the classical limiting case is:

\( \nabla^2[\Phi] \ll (\nabla[\Phi])^2 \)

\( \hbar\nabla^2[S] \ll (\nabla[S])^2 \)

\( \hbar\nabla \cdot p \ll (p \cdot p) \)

\( (pA)\nabla \cdot p \ll (p \cdot p) \)
QM Correspondence Principle: Analogous to the GR and SR limits

\[ \dot{S} + \frac{V}{2m_0}(\nabla S)^2 = \frac{i\hbar}{2m_0}(\nabla^2 S) \] : Quantum Single Particle Hamilton-Jacobi

\[ \dot{S} + \frac{V}{2m_0}(\nabla S)^2 = 0 \] : Classical Single Particle Hamilton-Jacobi

Thus, the quantum → classical limiting-case is:

\[ \hbar \nabla^2 [S_{\text{action}}] \ll (\nabla [S_{\text{action}}])^2 \]

\[ \nabla^2 [\Phi_{\text{phase}}] \ll (\nabla [\Phi_{\text{phase}}])^2 \]

\[ \hbar \nabla \cdot \nabla [S_{\text{action}}] \ll (\nabla [S_{\text{action}}])^2 \]

\[ \nabla \cdot \nabla [\Phi_{\text{phase}}] \ll (\nabla [\Phi_{\text{phase}}])^2 \]

\[ \hbar \cdot \nabla \cdot p \ll (p \cdot p) \]

\[ (p \cdot \lambda) \nabla \cdot p \ll (p \cdot p) \]

with

\[ P = (E/c, p) = -\partial[S_{\text{action}}] = -(\partial/c, \nabla)S_{\text{action}} = -(\partial/c, \nabla)S_{\text{action}} \]

\[ K = (\omega/c, k) = -\partial[\Phi_{\text{phase}}] = -(\partial/c, \nabla)\Phi_{\text{phase}} = -(\partial/c, \nabla)\Phi_{\text{phase}} \]

It is analogous to GR → SR in limit of low curvature (low mass), or SR → CM in limit of low velocity \( |v| \ll c \).

It still applies, but is now understood as the same type of limiting-case as these others.

*Note* The commonly seen form of \( (c \to \infty, \hbar \to 0) \) as limits are incorrect!

\( c \) and \( \hbar \) are universal constants – they never change.

If \( c \to \infty \), then photons (light-waves) would have infinite energy \( E = pc \). This is not true classically.

If \( \hbar \to 0 \), then photons (light-waves) would have zero energy \( E = \hbar \omega \). This is not true classically.

Always better to write the SR Classical limit as \( |v| < c \), the QM Classical limit as \( \nabla^2 [\Phi_{\text{phase}}] \ll (\nabla [\Phi_{\text{phase}}])^2 \)

Again, it is more natural to find a limiting-case of a more general system than to try to unite two separate theories which may or may not ultimately be compatible. From logic, there is always the possibility to have a paradox result from combination of arbitrary axioms, whereas deductions from a single true axiom will always give true results.
Consider the following purely mathematical argument (based on Green’s Vector Identity):
\[
\partial \cdot ( f \partial [g] - \partial [f] g ) = f \partial \cdot \partial [g] - \partial \cdot \partial [f] g
\]
with \(f\) and \(g\) as SR Lorentz Scalar functions.

Proof:
\[
\partial \cdot ( f \partial [g] - \partial [f] g ) = \partial \cdot ( f \partial [g] - \partial \cdot \partial [f] g ) = (f \partial \cdot \partial [g] + \partial \cdot \partial [f] g)
\]
We can also multiply this by a Lorentz Invariant Scalar Constant \(s\):
\[
s ( f \partial \cdot \partial [g] - \partial \cdot \partial [f] g ) = s ( f \partial \cdot \partial [g] - \partial \cdot \partial [f] g ) = d\cdot s ( f \partial [g] - \partial [f] g )
\]
Ok, so we have the math that we need…

Now, on to the physics… Start with the Klein-Gordon Eqn.
\[
\partial \cdot \partial [- i m_c c \hbar ] = - (m_c c / \hbar )^2 \partial \cdot \partial + (m_c c / \hbar )^2 = 0
\]
Let it act on SR Lorentz Invariant function \(g\):
\[
\partial \cdot \partial [g] - (m_c c / \hbar )^2 [g] = 0
\]
Then pre-multiply by \(f\):
\[
[f] \partial \cdot \partial [g] + [f] (m_c c / \hbar )^2 [g] = 0
\]
Do similarly with SR Lorentz Invariant function \(f\):
\[
\partial \cdot \partial [f] + (m_c c / \hbar )^2 [f] = 0
\]
Then post-multiply by \(g\):
\[
\partial \cdot \partial [f] [g] + (m_c c / \hbar )^2 [f] [g] = 0
\]
Now, subtract the two equations:
\[
[f] \partial \cdot \partial [g] - \partial \cdot \partial [f] [g] = 0
\]
And as we noted from the mathematical Green’s Vector identity at the start…
\[
[f] \partial \cdot \partial [g] - \partial \cdot \partial [f] [g] = \partial \cdot ( f \partial [g] - \partial [f] g ) = 0
\]
Therefore,
\[
s \partial \cdot ( f \partial [g] - \partial [f] g ) = 0
\]
\[
\partial \cdot s ( f \partial [g] - \partial [f] g ) = 0
\]
Thus, there is a conserved current 4-Vector, \(J_{\text{prob}} = s ( f \partial [g] - \partial [f] g )\), for which \(\partial \cdot J_{\text{prob}} = 0\), and which also solves the Klein-Gordon equation.

Let’s choose as before \((\partial = -iK)\) with a plane wave function \(f = ae^{-i(K\cdot X)} = \psi\), and choose \(g = f^* = ae^{i(K\cdot X)} = \psi^*\) as its complex conjugate.

At this point, I am going to choose \(s = (i\hbar / 2m_c)\), which is Lorentz Scalar Invariant, in order to make the probability have dimensionless units and be normalized to unity in the rest case.
4-Vector Quantum Probability

4-ProbabilityFlux, Klein-Gordon RQM Eqn

A Tensor Study of Physical 4-Vectors

SR → QM

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SciRealm.org

4-Vector SRQM Interpretation of QM

SR 4-Vector

(2,0)-Tensor \( T^{iv} \)

(1,1)-Tensor \( T^v \)

or \( T^s \)

(0,2)-Tensor \( T_{vx} \)

4-Gradient

\( \delta \delta \sim (\delta \cdot \cdot c) \cdot V \cdot V \)

d’Alembertian

\( \delta \delta \sim -(m_c c/ h)^2 \)

Klein-Gordon

Complex Plane-waves

\( \mathbf{K} = \mathbf{i} \mathbf{G} \)

Examine the temporal component, the Relativistic Probability Density

\[ \rho_{\text{prob}} = \langle \psi* \rangle \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \]

Assume wave solution in following general form:

\[ \psi = A f [k] e(-i\omega t) \]

then

\[ \partial [\psi] = (\omega) A f [k] e(-i\omega t) \]

\[ \partial [\psi*] = (\omega) A f [k]* e(\omega t) \]

\[ \rho_{\text{prob}} = \langle \omega^2/m_c c^2 \rangle (\psi^* \cdot \partial [\psi] - \partial [\psi^*] \cdot \psi) \]

\[ \rho_{\text{prob}} = \langle \psi \cdot \partial^2 \psi - \partial^2 \psi^* \cdot \psi \rangle \]

\[ \rho_{\text{prob}} = \langle \psi^* \cdot \partial^2 \psi - \partial^2 \psi^* \cdot \psi \rangle \]

Fimally, multiply by charge (q) to get standard SR EM

\[ 4-\text{CurrentDensity} = 4-\text{ChargeFlux} = J = (c\rho, j) = q J_{\text{prob}} = q(c\rho_{\text{prob}}, j_{\text{prob}}) \]

Existing SR Rules

Quantum Principles
4-Vector Quantum Probability

4-ProbabilityFlux, Klein-Gordon RQM Eqn with Minimal Coupling

4-ProbabilityCurrentDensity, a.k.a. 4-ProbabilityFlux

J_{prob} = (c \rho_{prob} j_{prob}) = (i \hbar/2m)(\psi^* \partial \psi - \partial \psi^* \psi) = (\rho_{prob}) U = (\rho_{prob}) \gamma (c, u) = (\gamma p_{prob}) (c, u) = (\rho_{prob})(c, u)

with 4-Divergence of Probability \{ \partial J_{prob} = 0 \} by construction via Green's Vector Identity and the Klein-Gordon RQM Eqn.

If we include minimal coupling:

J_{prob} = (c \rho_{prob} j_{prob}) = (i \hbar/2m)(\psi^* \partial \psi - \partial \psi^* \psi) + (q/m_c)(\psi^* \psi) A

Start at \mathbf{J} 

Follow past Born Rule

Follow past \( (q/m) (\psi^* \psi) A \)

Now have the additional factor:

Follow past Born Rule to get to \( J \)

Start at \( J \)

If we include minimal coupling:

J_{prob} = (c \rho_{prob} j_{prob}) = (i \hbar/2m)(\psi^* \partial \psi - \partial \psi^* \psi)

Follow past Born Rule to get to \( U \)

Follow past Born Rule to get to \( U \)

Now have the additional factor:

\( + (q/m_c)(\psi^* \psi) A \)

An alternate way would be to take \( A \) to \( U \) via the direct route:

\( + (c^2/\hbar)(\psi^* \psi) A \)

which would lead to a term like

\( \rho_{max} \rightarrow (\gamma)(\psi^* \psi) + (\gamma)(\psi^*(\psi^* \psi)) = (\gamma)(1 + \psi^*(\psi^* \psi)) \)

with potential due to particle \( (\psi_u) \) typically much less than the potential due to the whole field \( (\psi_n) \),

\( (\psi_u) \ll (\psi_n) \)

Conservation of EM Field

\( \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0 \)

4-Gradient

\( \partial = (\partial/c - \nabla) \)

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\( (\psi_u) \ll (\psi_n) \)

Conservation of EM Field

\( \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0 \)

4-Gradient

\( \partial = (\partial/c - \nabla) \)
4-Vector Quantum Probability
Newtonian Limit

4-ProbabilityCurrentDensity \( \mathbf{J}_{\text{prob}} = (c \rho_{\text{prob}}, \mathbf{j}_{\text{prob}}) = (i \hbar/2m_c)(\psi^* \partial[\psi] - \partial[\psi^*] \psi) + (q/m_c)(\psi^* \psi) \mathbf{A} \)

Examine the temporal component:
\( \rho_{\text{prob}} = (i \hbar/2m_c)^2(\psi^* \partial[\psi] - \partial[\psi^*] \psi) + (q/m_c)(\psi^* \psi)(\varphi/c^2) \)
\( \rho_{\text{prob}} \rightarrow (\gamma)(\psi^* \psi) + (\gamma)(q \varphi/m_c c^2)(\psi^* \psi) = (\gamma)[1 + q \varphi_c E_0](\psi^* \psi) \)

Typically, the particle EM potential energy \((q \varphi_c)\) is much less than the particle rest energy \((E_0)\), else it could generate new particles. So, take \((q \varphi_c << E_0)\), which gives the EM factor \((q \varphi_c/E_0) \sim 0\)

Now, taking the low-velocity limit \((\gamma \rightarrow 1)\), \( \rho_{\text{prob}} = \gamma[1 + ~0](\psi^* \psi) \), \( \rho_{\text{prob}} \rightarrow (\psi^* \psi) = (\rho_{\text{prob}}^0) \) for \(|v|<<c\)

The Standard Born Probability Interpretation, \((\psi^* \psi) = (\rho_{\text{prob}}^0)\), only applies in the low-potential-energy & low-velocity limit

This is why the \{non-positive-definite\} probabilities and \(|\text{probabilities}| > 1\) in the RQM Klein-Gordon equation gave physicists fits, and is the reason why one must regard the probabilities as charge conservation instead.

The original definition from SR is Continuity of Worldlines, \( \partial \cdot \mathbf{J}_{\text{prob}} = 0 \), for which all is good and well in the RQM version.
The definition says there are no external sources or sinks of probability = conservation of probability.

The Born idea that \((\rho_{\text{prob}}^0) \rightarrow \text{Sum}[(\psi^* \psi)] = 1\) is just the Low-Velocity QM limit.
Only the non-EM rest version \((\rho_{\text{prob}}^0) = \text{Sum}[(\psi^* \psi)] = 1\) is true.
It is not a fundamental axiom, it is an emergent property which is valid only in the NRQM limit

We now multiply by charge \((q)\) to instead get a
4-"Charge"CurrentDensity \( \mathbf{J} = (c \rho, \mathbf{j}) = q \mathbf{J}_{\text{prob}} = q(c \rho_{\text{prob}}, \mathbf{j}_{\text{prob}}) \), which is the standard SR EM 4-CurrentDensity
A Tensor Study

1. $(1,1)$-Tensor $T_{\mu\nu}$

$$ P_{\mu} = (m_c, p_{\mu}) = (E_c, p_{\mu}) $$

2. $(2,0)$-Tensor $V^\mu = V = (\nu, v)$

3. $(0,1)$-Tensor $V_\nu = (v_\nu, v)$

4. $(0,0)$-Tensor $S$

SciRealm.org John B. Wilson

**SR 4-Vector Study:**

**The QM Compton Effect**

**Compton Scattering Derivation:** Compton Effect

$$ P \cdot P = (m_{c})^2 \text{ generally } \rightarrow 0 \text{ for photons } (m_{c} = 0) $$

$$ P_{\mu} \cdot P_{\nu} = T_{\mu\nu} = (h \gamma/c) (\hat{e} \cdot \gamma/c + \hat{n} \cdot \gamma/c) $$

$$ P_{\mu} \cdot P_{\nu} = T_{\mu\nu} = (h \gamma/c) (\hat{e} \cdot \gamma/c + \hat{n} \cdot \gamma/c) $$

$$ K = (\omega/c, n/\lambda) $$

$$ \Delta \lambda = (\lambda' - \lambda) = (h/m_c)(1 - \cos[\theta]) $$

The Compton Effect: Compton Scattering

$$ \lambda_c = (h/m_c)(1 - \cos[\theta]) $$

$$ \lambda_c = (h/m_c)(1 - \cos[\theta]) $$

Calculates the wavelength shift of a photon scattering from an electron (ignoring spin)

$E/\gamma = E/\gamma_{\text{photon}} = E/\gamma_{\text{photon}} = \hbar K_{\text{photon}} = (\omega/c)(1, n) = \text{null} \quad \{\omega \lambda = \nu \lambda = \text{c} \} \text{ for photons}$

4-Gradient $\partial = (\partial/c, -\vec{V})$

SpaceTime $\partial R = 4$ Dimension

Wave Velocity $V_{\text{phase}} = c^2$

Energy: Mass $E = mc^2$

Compton Scattering $P = (mc, p) = (E/c, p)$

4-Momentum $P = (E/c, p)$

4-TotalMomentum $P = (E/c, p)$

Initial

$P_{\mu} = (m_c, p_{\mu}) = (E_c, p_{\mu})$

Final

$P_{\mu} = (m_c, p_{\mu}) = (E_c, p_{\mu})$

4-WaveVector $K = (\omega/c, n/\lambda)$

$\Delta \lambda = (\lambda' - \lambda) = (h/m_c)(1 - \cos[\theta])$

$\lambda_c = (h/m_c)(1 - \cos[\theta])$

$\lambda_c = (h/m_c)(1 - \cos[\theta])$

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Energy: Mass $E = mc^2$

Compton Scattering $P = (mc, p) = (E/c, p)$

4-Momentum $P = (E/c, p)$

4-TotalMomentum $P = (E/c, p)$

Initial

$P_{\mu} = (m_c, p_{\mu}) = (E_c, p_{\mu})$

Final

$P_{\mu} = (m_c, p_{\mu}) = (E_c, p_{\mu})$
SRQM 4-Vector Study:
The QM Aharonov-Bohm Effect

**QM Potential** \( \Delta \Phi = -\frac{(q/\hbar)}{\text{path}} \mathbf{A} \cdot d\mathbf{X} \)

**Electric AB effect:** \( \Delta \Phi = -(q/\hbar) \int_\text{path} (\varphi dt - \mathbf{a} \cdot d\mathbf{X}) \)

**Magnetic AB effect:** \( \Delta \Phi = -(q/\hbar) \int_\text{path} (\mathbf{A} \cdot d\mathbf{X}) \)

Proves that the 4-Vector Potential \( \mathbf{A} \) is more fundamental than \( \mathbf{e} \) and \( \mathbf{b} \) fields, which are just components of the Faraday EM Tensor.
**SRQM 4-Vector Study:**

**The QM Josephson Junction Effect = SuperCurrent**

**EM 4-VectorPotential** \( A = -(\frac{\hbar}{q}) \partial [\Delta \Phi] \)

**Josephson Effect**

The EM 4-VectorPotential gives the Aharonov-Bohm Effect.

Phase \( \Phi_{pot} = -(q/\hbar) A \cdot X \)

Rearrange the equation a bit: \( -[\hbar/q] \Delta \Phi_{pot} = A \cdot \Delta X \)

Assume that \( (d/dt [A] \cdot \Delta X) = 0 \)

Which explains Josephson Effect criteria:

\( \Delta X \sim 0: \text{small gap} \)

\( d/dt [A] \cdot \Delta X \sim \text{orthogonal: ??} \)

Take the temporal part:

EM ScalarPotential \( \Phi = -(\hbar/q) (\partial_t [\Delta \Phi]) \); \( \omega = (q/\hbar) \phi \)

If the charge \( (q) \) is a Cooper-electron-pair: \( q = -2e \)

Voltage \( V(t) = \phi(t) = (\hbar/2e)(\partial_t \partial_l [\Delta \Phi]) \); \( \text{AngFreq} \omega = -2eV/\hbar \)

This is the superconducting phase evolution equation of the Josephson Effect.

\( (h/2e) \) is defined to be the Magnetic Flux Quantum \( \Phi_0 \)
SRQM Symmetries:
Hamilton-Jacobi vs Relativistic Action
Josephson vs Aharonov-Bohm
Differential (4-Vector) vs Integral (4-Scalar)

**Notice the Symmetry:**

**SR Hamilton-Jacobi Equation**

\[
\mathbf{P}_T = -\partial[\Delta S_{\text{action}}] + \mathbf{q} \mathbf{A} = -\partial[\Delta S_{\text{action}}] - \partial[\hbar \Delta \Phi_{\text{phase,dyn}}] + (\hbar) \Delta \Phi_{\text{phase,pot}}
\]

**Josephson Junction Relation**

\[
\mathbf{A} = -\left(\frac{\hbar}{2e}\right) \partial[\Delta \Phi_{\text{potential}}] \\
= -\left(\frac{1}{q}\right) \partial[\Delta S_{\text{act,pot}}] \\
= \mathbf{Q}/q
\]

**Technically, the standard Josephson Junction uses just the temporal part \{ \mathbf{A} = (\varphi/c, \mathbf{a}) \} & Cooper-pair-electrons \{ q = -2e \}**

giving \( V(t) = \varphi = (\hbar/2e) \partial/\partial t[\Delta \Phi_{\text{pot}}] \).
There should be a spatial part as well.

**SR Action Equation**

\[
\Delta S_{\text{action}} = -\int_{\text{path}} \mathbf{P}_T \cdot d\mathbf{X}
\]

\[
\Delta S_{\text{action}} = -\int_{\text{path}} (\mathbf{P} + q \mathbf{A}) \cdot d\mathbf{X}
\]

\[
\hbar \Delta \Phi_{\text{phase,dyn}} + \hbar \Delta \Phi_{\text{phase,pot}} = -\int_{\text{path}} (\mathbf{P} + q \mathbf{A}) \cdot d\mathbf{X}
\]

**Aharonov-Bohm Relation**

\[
\Delta \Phi_{\text{potential}} = -\left(\frac{q}{\hbar}\right) \int_{\text{path}} \mathbf{A} \cdot d\mathbf{X}
\]

\[
= -\left(\frac{1}{\hbar}\right) \int_{\text{path}} \mathbf{Q} \cdot d\mathbf{X}
\]

\[
= \Delta S_{\text{act,pot}}/\hbar
\]
The h Connection

\[ P = hK: \text{Basic Einstein-de Broglie} \]

\[ P + Q = P + Q \]

\[ P + Q = h(K_{\text{dyn}} + hK_{\text{pot}}) \]

\[ P + Q = h(K'_{\text{dyn}} + K'_{\text{pot}}) \]

Sum over n particles: \[ P_T = \sum_n (P + Q), K_T = \sum_n (K_{\text{dyn}} + K_{\text{pot}}) \]

The SR Hamilton-Jacobi Equation

\[ \{\text{SR Hamilton-Jacobi}\} = \frac{P}{S} - S \cdot \frac{P}{S} \]

\[ \text{Sum over } n \text{ particles: } P_T = \sum_n (P + Q), K_T = \sum_n (K_{\text{dyn}} + K_{\text{pot}}) \]

\[ P_T = hK_T \]

\[ \{\text{SR Hamilton-Jacobi}\} = h\{\text{QM Complex Plane-Waves}\} \]

The SR Hamilton-Jacobi Equation, and the QM idea of Complex Plane-Waves, are related by a simple constant (h) relation.

The h Connection

\[ P = hK: \text{Basic Einstein-de Broglie} \]

\[ P + Q = P + Q \]

\[ P + Q = h(K_{\text{dyn}} + hK_{\text{pot}}) \]

\[ P + Q = h(K'_{\text{dyn}} + K'_{\text{pot}}) \]

Sum over n particles: \[ P_T = \sum_n (P + Q), K_T = \sum_n (K_{\text{dyn}} + K_{\text{pot}}) \]

The SR Hamilton-Jacobi Equation, and the QM idea of Complex Plane-Waves, are related by a simple constant (h) relation.
SRQM 4-Vector Study: Dimensionless Physical Objects

Dimensionless Physical Objects

There are a number of dimensionless physical objects in SR that can be constructed from Physical 4-Vectors. Most are 4-Scalars, but there are few 4-Vector and 4-Tensors. 

- **ΔX=4**: SpaceTime Dimension
- **T**= 1: Lorentz Scalar “Magnitude” of the 4-UnitTemporal
- **S**= 0: Lorentz Scalar of 4-UnitTemporal with 4-UnitSpatial
- **K**= (w t-k x) = -Φ: Phase of an SR Wave
  
  **(P-O)** = (E c/k o T o); 4-Momentum with 4-InverseMomentum used in statistical mechanics particle distributions
  
  F(state) ~ e^a e^-(K·X)

- α = (1/4πε o)(e^2/hc) = (μ o/4π)(e^2/hc): Fine Structure Constant constructed from Lorentz 4-Scalars, which are themselves constructed from 4-Vectors via the Lorentz Scalar Product. 
  
  ex. h=(P·X)\((K·X)\); q=(E·X)/(A·X) →e for electron; c=(T·U)

- **q** = (\((∂/∂)[A·X])/(J·X)\) when (\((∂/∂)\))=0

- \((v)\): Dirac Gamma Matrix (“4-Vector”)
- \((a)\): Pauli Spin Matrix (“4-Vector”)

Components are matrices of numbers, not just numbers
SRQM: QM Axioms Unnecessary

QM Principles emerge from SR

QM is derivable from SR plus a few empirical facts – the “QM Axioms” aren't necessary. These properties are either empirically measured or are emergent from SR properties...

3 “QM Axioms” are really just empirical constant relations between purely SR 4-Vectors:
- Particle-Wave Duality \[ (\mathbf{P}) = \hbar (\mathbf{K}) \]
- Unitary Evolution \[ [\hat{\alpha} = (-i)\mathbf{K}] \]
- Operator Formalism \[ [(\partial) = -i\mathbf{K}] \]

2 “QM Axioms” are just the result of the Klein-Gordon Equation being a linear wave PDE:
- Hilbert Space Representation (\langle \text{bra}|,|\text{ket}\rangle, wavefunctions, etc.) & The Principle of Superposition

3 “QM Axioms” are a property of the Minkowski Metric and the empirical fact of Operator Formalism:
- The Canonical Commutation Relation
- The Heisenberg Uncertainty Principle (time-like-separated measurement exchange)
- The Pauli Exclusion Principle (space-like-separated particle exchange)

1 “QM Axiom” only holds in the NRQM case:
- The Born QM Probability Interpretation – Not applicable to RQM, use Conservation of Worldlines instead

1 “QM Axiom” is really just another level of limiting cases, just like SR → CM in limit of low velocity:
- The QM Correspondence Principle (QM → CM in limit of \( \nabla^2[\phi] \ll (\nabla[\phi])^2 \))
The SRQM interpretation fits fairly well with Carlo Rovelli's Relational QM interpretation:

Relational QM treats the state of a quantum system as being observer-dependent, that is, the QM State is the relation between the observer and the system. This is inspired by the key idea behind Special Relativity, that the details of an observation depend on the reference frame of the observer.

All systems are quantum systems: no artificial Copenhagen dichotomy between classical/macroscopic/conscious objects and quantum objects.

The QM States reflect the observers' information about a quantum system. Wave function "collapse" is informational – not physical. A particle always knows it's complete properties. An observer has at best only partial information about the particle's properties.

No Spooky Action at a Distance. When a measurement is done locally on an entangled system, it is only the partial information about the distant entangled state that "changes/becomes-available-instantaneously". There is no superluminal signal. Measuring/physically-changing the local particle does not physically change the distant particle.

ex. Place two identical-except-for-color marbles into a box, close lid, and shake. Without looking, pick one marble at random and place it into another box. Send that box very far away. After receiving signal of the far box arrival at a distant point, open the near box and look at the marble. You now instantaneously know the far marble’s color as well. The information did not come by signal. You already had the possibilities (partial knowledge). Looking at the near marble color simply reduced the partial knowledge of both marble's color to complete knowledge of both marbles' color. No signal was required, superluminal or otherwise.

ex. The quantum version of the same experiment uses the spin of entangled particles. When measured on the same axis, one will always be spin-up, the other will be spin-down. It is conceptually analogous. Entanglement is only about correlations of system that interacted in the past and are determined by conservation laws.
SRQM Interpretation:

Interpretation of EPR-Bell Experiment

Einstein and Bohr can both be “right” about EPR:
Per Einstein: The QM State measured is not a “complete” description, just one observer’s point-of-view.
Per Bohr: The QM State measured is a “complete” description, it’s all that a single observer can get.

The point is that many observers can all see the “same” system, but see different facets of it. But a single measurement is the maximal information that a single observer can get without re-interacting with the system, which of course changes the system in general. Remember, the Heisenberg Uncertainty comes from non-zero commutation properties which *require separate measurement arrangements*. The properties of a particle are always there. Properties define particles. We as observers simply have only partial information about them.

Relativistic QM, being derived from SR, should be local – The low-velocity limit to QM may give unexpected anomalous results if taken out of context, or out of the applicable validity range, such as with velocity addition \( v_{12} = v_1 + v_2 \), where the correct formula should be the relativistic velocity composition \( v_{12} = \frac{(v_1 + v_2)}{\sqrt{1 + \frac{v_1 v_2}{c^2}}} \)

These ideas lead to the conclusion that the wavefunction is just one observer’s state of information about a physical system, not the state of the physical system itself. The “collapse” of the wavefunction is simply the change in an observer’s information about a system brought about by a measurement or, in the case of EPR, an inference about the physical state.

EPR doesn’t break Heisenberg because measurements are made on different particles. The happy fact is that those particles interacted and became correlated in the causal past. The EPR-Bell experiments prove that it is possible to maintain those correlations over long distances. It does not prove superluminal signaling.
We should not be surprised by the “quantum” probabilities being correct instead of “classical” in the EPR and Bell Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

Examples

* The limit of $\hbar \rightarrow 0$ \{Fallacy\}: $\hbar$ is a Lorentz Scalar Invariant and Fundamental Physical Constant. It never becomes 0. \{Fact\}

* The classical commutator being zero $[p^i, x^j] = 0$ \{Fallacy\}: $[P^\mu, X^\nu] = i\hbar \eta^{\mu\nu}$; $[p^0, x^0] = [E/c, ct] = [E, t] = i\hbar$; Again, it never becomes 0 \{Fact\}

* Using Maxwell-Boltzmann (distinguishable) statistics for counting probabilities of (indistinguishable) quantum states \{Fallacy\}: Must use Fermi-Dirac statistics for Fermions: Spin=$(n+1/2)$; Bose-Einstein statistics for Bosons: Spin=$(n)$ \{Fact\}

* Using sums of classical probabilities on quantum states \{Fallacy\}: Must use sums of quantum probability-amplitudes \{Fact\}

* Ignoring phase cross-terms and interference effects in calculations \{Fallacy\}: Quantum systems and entanglement require phase cross-terms \{Fact\}

* Assuming that one can simultaneously “measure” non-commuting properties at a single spacetime event \{Fallacy\}: Particle properties always exist. However, non-commuting ones require separate measurement arrangements to get information about the properties. The required measurement arrangements on a single particle/worldline are at best sequential events, where the temporal order plays a role; \{Fact\}

However, EPR allows one to “infer (not measure)” the other property of a particle by the separate measurement of an entangled partner. \{Fact\}

This does not break Heisenberg Uncertainty, which is about the order of operations (measurement events) on a single worldline. \{Fact\}

In the entangled case, both/all of the entangled partners share common past-causal entanglement events, typically due to a conservation law. \{Fact\}

Information is not transmitted at FTL. The particles simply carried their normal respective “correlated” properties (no hidden variables) with them. \{Fact\}

* Assuming that QM is a generalization of CM, or that classical probabilities apply to QM \{Fallacy\}: CM is a limiting-case of QM for when changes in a system by a few quanta have a negligible effect on the whole/overall system. \{Fact\}
We should not be surprised by the "quantum" probabilities being correct instead of "classical" in the EPR and Bell Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

{from Wikipedia}

No-Communication Theorem/No-Signaling:
A no-go theorem from quantum information theory which states that, during measurement of an entangled quantum state, it is not possible for one observer, by making a measurement of a subsystem of the total state, to communicate information to another observer. The theorem shows that quantum correlations do not lead to what could be referred to as "spooky communication at a distance". SRQM: There is no FTL signaling.

No-Teleportation Theorem:
The no-teleportation theorem stems from the Heisenberg uncertainty principle and the EPR paradox: although a qubit $|\psi\rangle$ can be imagined to be a specific direction on the Bloch sphere, that direction cannot be measured precisely, for the general case $|\psi\rangle$. The no-teleportation theorem is implied by the no-cloning theorem.

SRQM: Ket states are informational, not physical.

No-Cloning Theorem:
In physics, the no-cloning theorem states that it is impossible to create an identical copy of an arbitrary unknown quantum state. This no-go theorem of quantum mechanics proves the impossibility of a simple perfect non-disturbing measurement scheme. The no-cloning theorem is normally stated and proven for pure states; the no-broadcast theorem generalizes this result to mixed states. SRQM: Measurements are arrangements of particles that interact with a subject particle.

No-Broadcast Theorem:
Since quantum states cannot be copied in general, they cannot be broadcast. Here, the word "broadcast" is used in the sense of conveying the state to two or more recipients. For multiple recipients to each receive the state, there must be, in some sense, a way of duplicating the state. The no-broadcast theorem generalizes the no-cloning theorem for mixed states. The no-cloning theorem says that it is impossible to create two copies of an unknown state given a single copy of the state. SRQM: Conservation of worldlines.

No-Deleting Theorem:
In physics, the no-deleting theorem of quantum information theory is a no-go theorem which states that, in general, given two copies of some arbitrary quantum state, it is impossible to delete one of the copies. It is a time-reversed dual to the no-cloning theorem, which states that arbitrary states cannot be copied. SRQM: Conservation of worldlines.

No-Hiding Theorem:
The no-hiding theorem is the ultimate proof of the conservation of quantum information. The importance of the no-hiding theorem is that it proves the conservation of wave function in quantum theory. SRQM: Conservation of worldlines. RQM wavefunctions are Lorentz Scalars (spin=0), Spinors (spin=1/2), 4-Vectors (spin=1), all of which are Lorentz Invariant.
We should not be surprised by the “quantum” probabilities being correct instead of “classical” probabilities in the EPR/Bell-Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

Quantum information (qubits) differs strongly from classical information, epitomized by the bit, in many striking and unfamiliar ways. Among these are the following:

- A unit of quantum information is the qubit. Unlike classical digital states (which are discrete), a qubit is continuous-valued, describable by a direction on the Bloch sphere. Despite being continuously valued in this way, a qubit is the smallest possible unit of quantum information, as despite the qubit state being continuously-valued, it is impossible to measure the value precisely.
- A qubit cannot be (wholly) converted into classical bits; that is, it cannot be "read". This is the no-teleportation theorem.
- Despite the awkwardly-named no-teleportation theorem, qubits can be moved from one physical particle to another, by means of quantum teleportation. That is, qubits can be transported, independently of the underlying physical particle. SRQM: Ket states are informational, not physical.
- An arbitrary qubit can neither be copied, nor destroyed. This is the content of the no cloning theorem and the no-deleting theorem. SRQM: Conservation of worldlines.
- Although a single qubit can be transported from place to place (e.g. via quantum teleportation), it cannot be delivered to multiple recipients; this is the no-broadcast theorem, and is essentially implied by the no-cloning theorem. SRQM: Conservation of worldlines.
- Qubits can be changed, by applying linear transformations or quantum gates to them, to alter their state. While classical gates correspond to the familiar operations of Boolean logic, quantum gates are physical unitary operators that in the case of qubits correspond to rotations of the Bloch sphere.
- Due to the volatility of quantum systems and the impossibility of copying states, the storing of quantum information is much more difficult than storing classical information. Nevertheless, with the use of quantum error correction quantum information can still be reliably stored in principle. The existence of quantum error correcting codes has also led to the possibility of fault tolerant quantum computation.
- Classical bits can be encoded into and subsequently retrieved from configurations of qubits, through the use of quantum gates. By itself, a single qubit can convey no more than one bit of accessible classical information about its preparation. This is Holevo's theorem. However, in superdense coding a sender, by acting on one of two entangled qubits, can convey two bits of accessible information about their joint state to a receiver.

Quantum information can be moved about, in a quantum channel, analogous to the concept of a classical communications channel. Quantum messages have a finite size, measured in qubits; quantum channels have a finite channel capacity, measured in qubits per second.
Minkowski still applies in local GR

QM is a local phenomenon

The QM Schrodinger Equation is not fundamental. It is just the low-energy limiting-case of the RQM Klein-Gordon Equation. All of the standard QM Axioms are shown to be empirically measured constants or emergent properties of SR. It is a bad approach to start with NRQM as an axiomatic starting point and try to generalize it to RQM, in the same way that one cannot start with CM and derive SR. Since QM *can* be derived from SR, this partially explains the difficulty of uniting QM with GR:

QM is not a “separate formalism” outside of SR that can be used to “quantize” just anything...

Strictly speaking, the use of the Minkowski space to describe physical systems over finite distances applies only in the SR limit of systems without significant gravitation. In the case of significant gravitation, SpaceTime becomes curved and one must abandon SR in favor of the full theory of GR.

Nevertheless, even in such cases, based on the GR Equivalence Principle, Minkowski space is still a good description in a local region surrounding any point (barring gravitational singularities). More abstractly, we say that in the presence of gravity, SpaceTime is described by a curved 4-dimensional manifold for which the tangent space to any point is a 4-dimensional Minkowski Space. Thus, the structure of Minkowski Space is still essential in the description of GR.

So, even in GR, at the local level things are considered to be Minkowskian:

i.e. SR → QM “lives inside the surface” of this local SpaceTime, GR curves the surface.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM Interpretation: Main Result

QM is derivable from SR!

Hopefully, this interpretation will shed light on why Quantum Gravity has been so elusive. Basically, QM rules of “quantization” don’t apply to GR. They are a manifestation-of/derivation-from SR. Relativity *is* the “Theory of Measurement” that QM has been looking for.

This would explain why no one has been able to produce a successful theory of Quantum Gravity, and why there have been no violations of Lorentz Invariance nor of the Equivalence Principle.

If quantum effects “live” in Minkowski SpaceTime with SR, then GR curvature effects are at a level above the RQM description, and two levels above standard QM. SR+QM are “in” SpaceTime, GR is the “shape” of SpaceTime...

Thus, this treatise explains the following:

• Why GR works so well in it’s realm of applicability {large scale systems}.

• Why QM works so well in it’s realm of applicability {micro scale systems and certain macroscopic systems}. i.e. The tangent space to any point in GR curvature is locally Minkowskian, and thus QM is typically found in small local volumes...

• Why RQM explains more stuff than QM without SR {because QM is just the low-velocity limiting-case of RQM}.

• Why all attempts to “quantize gravity” have failed {essentially, everyone has been trying to put the cart (QM) before the horse (GR)}.

• Why all attempts to modify GR keep conflicting with experimental data {because GR is apparently fundamental}.

• Why QM works perfectly well with SR as RQM but not with GR {because QM is derivable from SR, hence a manifestation of SR rules}.

• How Minkowski Space, 4-Vectors, and Lorentz Invariants play vital roles in RQM, and give the SRQM Interpretation of Quantum Mechanics.
SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR, although technically SR is itself the low-curvature limiting-case of GR

\{c,τ,m₀,ℏ,𝑖\}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors:

<table>
<thead>
<tr>
<th>4-Position</th>
<th>4-Velocity</th>
<th>4-Momentum</th>
<th>4-WaveVector</th>
<th>4-Gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R = (ct,r) )</td>
<td>( U = γ(c,u) )</td>
<td>( P = (E/c,p) )</td>
<td>( K = (ω/c,k) )</td>
<td>( ∂ = (c/ℏ,-∇) )</td>
</tr>
</tbody>
</table>

Related by these SR Lorentz Invariants:

- \( R·R = (ct)^2 \)
- \( U·U = (c)^2 \)
- \( P·P = (m₀c)^2 \)
- \( K·K = (m₀c/ℏ)^2 \)
- \( ∂·∂ = -(m₀c/ℏ)^2 = KG Eqn \rightarrow RQM \rightarrow QM \)

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn, and thence to QM via the low-velocity limit \( \{ |v| << c \} \), giving the Schrödinger Eqn. The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SR → QM

**SRQM Diagram:**

**Special Relativity → Quantum Mechanics**

**RoadMap of SR→QM (EM Potential)**

4-Gradient = Alteration of SR <Events>

SR SpaceTime Dimension=4
SR SpaceTime Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves

SR → QM Klein-Gordon
Relativistic Quantum
Particle in EM Potential
da'Alamberian Wave Equation

\[ \partial^2 \Phi = \frac{1}{c^2} \partial^2_\tau \Phi \]

Limit: \( |v| < c \)

\[ (i \hbar \partial_\tau) \sim [ q \Phi + (m c^2) + (i \hbar \nabla + q a)^2 / (2m_c) ] \]

with potential \( V = q \Phi + (m c^2) \)

Schrödinger QM Equation (EM potential)

\[ \[( S \rightarrow QM )^* \]\]

SR Wave <Events> have 4-WaveVector = Substantiation
oscillations proportional to mass:energy & 3-momentum

\[ K = (\omega/c, k) \]

SR Particle <Events> have 4-Momentum = Substantiation
mass:energy & 3-momentum

\[ P = (mc, p) = (E/c, p) \]

4-Position = Location in SR SpaceTime

\[ R = (ct, r) \]

4-Velocity

\[ U = (\gamma(c^2, u)) \]

4-Momentum

\[ P = (mc, p) = (E/c, p) \]

4-Tensor

\[ A = (\Phi, c, a) \]

EM Faraday

\[ \partial^\alpha A^\nu - \partial^\nu A^\alpha = F^{\alpha\nu} \]

Einstein, de Broglie

\[ \hbar = \frac{\lambda}{c} \]

4-TotalMomentum

\[ P_T = (E/c, p, q) \]

Minimal Coupling

\[ P_T = (P+qA) = (P+qA) \]

4-Scalar

\[ \bar{T} = (0, 1)-Tensor S \]

Lorentz Scalar

\[ T^\nu = (v^\nu, v) \]

4-Vector

\[ V^\nu = (V^\nu, V) \]

Least Squares

\[ T^\nu = (v^\nu, -v) \]

4-Tensor

\[ T^\nu = (\bar{v}, v) \]

Least Squares

\[ T^\nu = (v^\nu, v) \]

4-Vector

\[ V^\nu = (V^\nu, V) \]

Least Squares

\[ T^\nu = (v^\nu, -v) \]
Special Relativity → Quantum Mechanics

The SRQM Interpretation: Links

See also:
http://scirealm.org/SRQM.html (alt discussion)
http://scirealm.org/SRQM-RoadMap.html (main SRQM website)
http://scirealm.org/4Vectors.html (4-Vector study)
http://scirealm.org/SRQM-Tensors.html (Tensor & 4-Vector Calculator)
http://scirealm.org/SciCalculator.html (Complex-capable RPN Calculator)

or Google “SRQM”

http://scirealm.org/SRQM.pdf (this document)
The 4-Vector SRQM Interpretation

QM is derivable from SR!

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

quantum
relativity

SRQM = SciRealm QM?
A happy coincidence... :)