Refutation of the fixed-point property of self-proving for predicate modal logics

Abstract: The axiom of schema and definition/conjecture of self-prover are refuted, forming a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET   ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ⋗, ⊃, ↠ ; < Not Imply, less than, ∈, ⋖, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≓ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1; (%z<#z) `C` as contingency, ▽, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

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1. Introduction

The propositional modal system GL is obtained from the smallest normal modal logic K by adding the axiom schema

$$\Box(\Box A \rightarrow A) \rightarrow \Box A. \tag{1.1}$$

LET p, q: A, B.

#(#p>p)>#p ;   `CTCT CTCT CTCT CTCT`   (1.2)

Remark 1.2: Eq. 1.2 as rendered is not tautologous, thereby refuting the propositional modal system GL as claimed by the authors below.

The modal system GL is well known as *the logic of provability*, since it has the connection with arithmetical theories, for instance, Peano Arithmetic PA [per Solovay].

6. Formulas having a fixed-point in QGL

Definition 6.3 (Self-provers). An L″-formula A is said to be a self-prover if

$$\mathrm{QGL} \vdash A \rightarrow \Box A. \tag{6.3.1}$$

p>#p ;   `TNTN TNTN TNTN TNTN`   (6.3.2)

Remark 6.3.2: Eq. 6.3.2 is *not* tautologous, hence refuting self-provers as defined.

Lemma 6.4. The Boolean constant ⊤ and L″-formulas of the form □A are self-provers. Moreover, the set of self-provers is closed under ∧, ∨, ∃. Consequently, every Σ-formula is a self-prover.

Proof. Since QGL $\vdash \top\to\Box\top$ and QGL $\vdash \Box A\to\Box\Box A$, $\Box\top$ and $\Box A$ are self-provers.
Suppose that A and B are self-provers. (6.0.1.1)

LET p, q: A, B.

((((p=p)>#(p=p))&(#p>##p))>((#(p=p)&#p)>(p>#p)))>((p=(p>#p))&(q=(q>#q))) ;

FFFN **FFF**N **FFF**N **FFF**N (6.0.1.2)

• Since A and B are self-provers, QGL $\vdash A\wedge B\to\Box A\wedge\Box B$. On the other hand, QGL $\vdash \Box A\wedge\Box B\to\Box(A\wedge B)$. Thus we have QGL $\vdash A\wedge B\to\Box(A\wedge B)$, and hence $A\wedge B$ is a self-prover. (6.4.1.1)

((((((p=p)>#(p=p))&(#p>##p))>((#(p=p)&#p)>(p>#p)))>((p=(p>#p))&(q=(q>#q))))>(((((p>#p)&(q>#q))>((p&q)>(#p&#q)))&(((p>#p)&(q>#q))>((#p&#q)>#(p&q))))>((p&q)>(#p&#q)))) >((p=(p>#p))&(q=(q>#q))) ;

FFFN **FFF**N **FFF**N **FFF**N (6.4.1.2)

Remark 6.4: Eqs. 6.0.1.2 and 6.4.1.2 are *not* tautologous, hence disallowing Lemma 6.4.

The axiom of schema and definition/conjecture of self-prover are refuted.