The nonlinear Schrödinger equation with infinite mass wave

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Abstract

The Schrödinger equation with the logarithmic nonlinear term $-b(\ln|\Psi|^2)\Psi$ is derived by the natural generalization of the hydrodynamical model of quantum mechanics. The nonlinear term appears to be logically necessary because it enables explanation of the infinite mass limit of the wave function. The article is the modified version of the articles by author (Pardy, 1993; 2001).

1 Introduction

Many authors have suggested that the quantum mechanics based on linear Schrödinger equation is only an approximation of some more nonlinear theory with the nonlinear Schrödinger equation. The motivation for considerring the nonlinear quations is to get some more nonstandard solution in order to get the better understanding of the synergism of wave and particle. The article is the modified articles by author (Pardy, 1993; 2001).

The ambicious program to create nonlinear wave mechanics was elaborated by de Broglie (1960) and his group. Bialynicki-Birula and Mycielski (1976) considered the generalized Schrödinger equation with the additional term $F(|\Psi|^2)\Psi$ where F is some arbitrary function which they later specified to $-b(\ln|\Psi|^2)$, $b > 0$. The nonlinear term was selected by assuming the factorization of the wave function for the composed system.

The most attractive feature of the logarithmic nonlinearity is the existence of the lower energy bound and validity of Planck's relation $E = \hbar\omega$. At the same time the Born interpretation of the wave function cannot be changed. In this theory the estimation of b was given by the relation $b < 4 \times 10^{-10} eV$ following from the agreement between theory and the observed $2S - 2P$ Lamb shift in hydrogen. This implies an upper bound to the electron soliton spatial width of 10 μm.

Shimony (1979) proposed an experiment which is based on idea that a phase shift occurs when an absorber is moved from one point to another along the path of one of the coherent split beams in a neutron interferometer. In case of the logarithmic nonlinearity Shull at al. (1980) performed the experiment with a two-crystal interferometer. They searched for a phase shift when an attenuator was moved along the neutron propagation direction in one arm of the interferometer. A sheet of Cd, 0.086 mm thick, was used for the absorber.They obtained the upper bound on b of $3.4 \times 10^{-13} eV$ which is more than three orders of magnitude smaller than the bound estimated by Bialynicky-Birula and Mycielski (1976).

The best upper limit on b has been reported by Gähler, Klein and Zeilinger (1981) who has been searched for variations in the free space propagation of neutrons. 20 Å neutrons were difracted from an abrupt highly absorbing knife edge at the object position. By comparing the experimental results with the solution to the ordinary Schrödinger equation they were able to get the limit $b < 3 \times 10^{-15} eV$, which corresponds to an alectron soliton width of 3 mm. The similar results was obtained by the same group from diffraction a 100 μm boron wire.

To our knowledge the Mössbauer effect was not used to determine the constant b although this effect allows to measure energy losses smaller than $10^{-15} eV$. Similarly the Josephson effect has been not applied for the determination of the constant b.

We see that the constant b is very small, nevertheless we cannot it neglect a priori, because we do not know its role in the future physics. The corresponding analogon is the Planck constant which is also very small, however, it plays the fundamental role in physics.

The goal of this article is to give the new derivation of the logarithmic nonlinearity, to find the solution of the nonlinear Schrödinger equation of the one-dimensional case and to show that in the mass limit $m \to \infty$ we get exactly the delta-function behavior of the probability of finding the particle at point x. It means that there exists the classical motion of a particle with sufficient big mass. The nonlinearity of the Schrödinger equation also solves the collaps of the wave function and the Schrödinger cat paradox. We will start from the hydrodynamical formulation of quantum mechanics. The mathematical generalization of the Euler hydrodynamical equations leads automatically to the logarithmic term with $b > 0$.

2 The derivation of the nonlinear Schrödinger equation

We respect here the so called Dirac heuristical principle (Pais, 1986) according to which it is useful to postulate some mathematical requirement in order to get the true information about nature. While the mathematical assumption is intuitive, the consequences have the physical interpretation, or, in other words they are physically meaningfull. In derivation of the logarithmic nonlinearity we use just the Dirac method.

According to Madelung (1926), Bohm and Vigier (1954), Wilhelm (1970), Rosen (1974) and others, the original Schrödinger equation can be transformed into the hydrodynamical system of equations by using the so called Madelung ansatz:

$$\Psi = \sqrt{n} e^{\frac{i}{\hbar} S}, \tag{1}$$

where n is interpreted as the density of particles and S is the classical action for $\hbar \to 0$. The mass density is defined by relation $\varrho = nm$ where m is mass of a particle.

It is well known that after insertion of the relation (1) into the original Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V\Psi, \tag{2}$$

where V is the potential energy, we get, after separating the real and imaginary parts, the following system of equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V = \frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}} \tag{3}$$

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\mathbf{v}) = 0 \tag{4}$$

with

$$\mathbf{v} = \frac{\nabla S}{m}. \tag{5}$$

Equation (3) is the Hamilton-Jacobi equation with the additional term

$$V_q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}}, \tag{6}$$

which is called the quantum Bohm potential and equation (4) is the continuity equation.

After application of operator $\bigtriangledown$ on eq. (3), it can be cast into the Euler hydrodynamical equation of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{m}\nabla(V + V_q). \tag{7}$$

It is evident that this equation is from the hydrodynamical point of view incomplete as a consequence of the missing term $-\varrho^{-1}\nabla p$ where p is hydrodynamical pressure. We use here this fact just as the crucial point for derivation of the nonlinear Schrödinger equation. We complete the eq. (7) by adding the pressure term and in such a way we get the total Euler equation in the form:

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla(V + V_q) - \nabla F, \tag{8}$$

where

$$\nabla F = \frac{1}{n}\nabla p. \tag{9}$$

The equation (8) can be obtained by the Madelung procedure from the following extended Schrödinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi + F\Psi \tag{10}$$

on the assumption that it is possible to determine F in term of the wave function. From the vector analysis follows that the necessary condition of the existence of F as the solution of the eq. (9) is rot grad F = 0, or,

$$\mathrm{rot}(n^{-1}\nabla p) = 0, \tag{11}$$

which enables to take the linear solution in the form

$$p = -bn = -b|\Psi|^2, \tag{12}$$

where b is some arbitrary constant. We do not consider the more general solution of eq. (11). Then, from eq. (9) i.e. grad $F = \mathbf{a}$ we have:

$$F = \int a_i\, dx_i = -b\int \frac{1}{n}\, dn = -b\ln|\Psi|^2, \tag{13}$$

where we have omitted the additive constant which plays no substantial role in the Schrödinger equation.

Now, we can write the generalized Schrödinger equation which corresponds to the complete Euler equation (8) in the following form:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi - b(\ln|\Psi|^2)\Psi. \tag{14}$$

Let us approach the solving eq. (14).

3 The soliton-wave solution of the nonlinear Schrödinger equation

Let be $c, (\mathrm{Im}\, c = 0), v, k, \omega$ some parameters and let us insert function

$$\Psi(x,t) = cG(x - vt)e^{ikx - i\omega t} \tag{15}$$

into the one-dimensional equation (14) with $V = 0$. Putting the imaginary part of the new equation to zero, we get

$$v = \frac{\hbar k}{m} \tag{16}$$

and for function G we get the folloving nonlinear equation (the symbol $'$ denotes derivation with respect to $\xi = x - vt$):

$$G'' + AG + B(\ln G)G = 0, \tag{17}$$

where

$$A = \frac{2m}{\hbar}\omega - k^2 + \frac{2m}{\hbar^2} b \ln c^2 \tag{18}$$

$$B = \frac{4mb}{\hbar^2}. \tag{19}$$

After multiplication of eq. (17) by G' we get:

$$\frac{1}{2}\left[G'^2\right]' + \frac{A}{2}\left[G^2\right]' + B\left[\frac{G^2}{2}\ln G - \frac{G^2}{4}\right]' = 0, \tag{20}$$

or, after integration

$$G'^2 = -AG^2 - BG^2 \ln G + \frac{B}{2}G^2 + const. \tag{21}$$

If we choose the solution in such a way that $G(\infty) = 0$ and $G'(\infty) = 0$, we get $const. = 0$ and after elementary operations we get the following differential equation to be solved:

$$\frac{dG}{G\sqrt{a - B\ln G}} = d\xi, \tag{22}$$

where

$$a = \frac{B}{2} - A. \tag{23}$$

Eq. (22) can be solved by the elementary integration and the result is

$$G = e^{\frac{a}{B}} e^{-\frac{B}{4}(\xi + d)^2}, \tag{24}$$

where d is some constant.

The corresponding soliton-wave function is evidently in the one-dimensional free particle case of the form:

$$\Psi(x,t) = ce^{\frac{a}{B}} e^{-\frac{B}{4}(x - vt + d)^2} e^{ikx - i\omega t}. \tag{25}$$

4 Normalization and the classical limit

It is not necessary to change the standard probability interpretation of the wave function. It means that the normalization condition in our one-dimensional case is

$$\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1. \tag{26}$$

Using the Gauss integral

$$\int_0^{\infty} e^{-\lambda^2 x^2} \, dx = \frac{\sqrt{\pi}}{2\lambda}, \tag{27}$$

we get with $\lambda = \left(\frac{B}{2}\right)^{\frac{1}{2}}$

$$c^2 e^{\frac{2a}{B}} = \left(\frac{B}{2\pi}\right)^{\frac{1}{2}} \tag{28}$$

and the density probability $\Psi^* \Psi = \delta_m(\xi)$ is of the form (with $d = 0$):

$$\delta_m(\xi) = \sqrt{\frac{m\alpha}{\pi}} e^{-\alpha m \xi^2} \quad ; \quad \alpha = \frac{2b}{\hbar^2}. \tag{29}$$

It may be easy to see that $\delta_m(\xi)$ is the delta-generating function and for $m \to \infty$ is just the Dirac δ-function.

It means that the motion of a particle with sufficiently big mass m is strongly localized and in other words it means that the motion of this particle is the classical one. Such behaviour of a particle cannot be obtained in the standard quantum mechanics because the plane wave

$$e^{ikx - i\omega t} \tag{30}$$

corresponds to the free particle with no possibility of localization for $m \to \infty$.

Let us still remark that coefficient c^2 is real and positive number because it is a result of the solution of eq. (28) which can be transformed into equation ($x = c^2$)

$$x^{1-r} = const. \tag{31}$$

Let us remark that the principle of superposition is in our theory broken. If φ_1 and φ_2 are two different solution of the nonlinear Schrödinger equation then the linear combination $\varphi = a\varphi_1 + b\varphi_2$ where a and b are the arbitrary constants is not the solution of the same equation because of its nonlinearity. In other words the original principle of superposition of the standard quantum mechanics is broken. The consequence of the breaking of the principle of superposition is the resolution of the Schrödinger cat paradox (Glauber, 1986).

5 Discussion

We have seen that the introduction of the logarithmic nonlinearity in the Schrödinger equation was logically supported by the fact that the nonlinear Schrödinger equation gives results which are physically meaningful. We have obtained the correct mass limit of the wave function.

The further strong point of the nonlinear Schrödinger equation (14) is the result (16) which is equivalent to the famous de Broglie relation

$$\lambda = \frac{h}{p} \tag{32}$$

because of $\lambda = 2\pi/k = 2\pi(\hbar/mv) = 2\pi(h/2\pi)(1/p)$ and it means that de Broglie relation is involved in this form of the nonlinear quantum mechanics.

The nonlinear equation (14) has also the normalized plane-wave solution

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{ikx - i\omega t}. \tag{33}$$

After insertion of eq. (33) into eq. (14), we get the following dispersion relation:

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + b\ln(2\pi), \tag{34}$$

from which the relations follows:

$$\hbar\omega = b\ln(2\pi); \quad k = 0 \tag{35}$$

and

$$k = \pm i\sqrt{\frac{2m}{\hbar^2} b\ln(2\pi)}; \quad \omega = 0. \tag{36}$$

It is no easy to give the physical interpretation of eqs. (35) and (36) and so we cannot say that the plane-solution of the nonlinear Schrödinger equation is physically meaningful. Only the soliton-wave solution of the nonlinear Schrödinger equation can be taken as relevant. Only this solution is suitable for the physical verification. The possible new tests of the nonlinear quantum mechanics are discussed in the author article (Pardy, 1994).

The generalization to the motion of particle in the electromagnetic field with potentials $\varphi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ can be performed by the standard transformation

$$\frac{\hbar}{i}\nabla \to \frac{\hbar}{i}\nabla - \left(\frac{e}{c}\right)\mathbf{A}(\mathbf{x}, t) \tag{37}$$

and adding the scalar potential energy $\varphi(x, t)$ in the Schrödinger equation for the free particles. According to (Bialynicky-Birula et al., 1976) the solution of the equation in this case can be taken in the form

$$\Psi(\mathbf{x}, t) = e^{\frac{i}{\hbar}S}G(\mathbf{x} - \mathbf{u}(t)), \tag{38}$$

where function G is necessary to determine. In the similar form the problem was yet solved (Barut, 1990).

Kamesberger and Zeilinger (1998) have given the numerical solution of the original Schrödinger equation and this equation with the nonlinear term $-b(\ln|\Psi|^2)\Psi$ in order to visualize the spreading of the diffractive waves. When comparing the evolution patterns of the nonlinear case with the linear one, one notices that the maxima are more pronounced in the nonlinear solution. It can be understood as a mechanism compressing the wave maxima spatially. In the quantitative comparison of the both cases this enhacement of the maxima and minima can be seen very clearly.

Although we have given reasons for the introducing of the nonlinear Schrödinger equation it is obvious that only the crucial experiments can establish the physical and not only logical necessity of such equation. In case that the nonlinear Schrödinger equation will be confirmed by experiment, then it can be expected that it will influence other parts of theoretical physics.

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