Refutation of Scott's existence axiom in sheaf theory

Abstract: In two non-tautologous equations we refute Scott's existence axiom in sheaf theory. Therefore it does not follow that automating free logic is supported by using "modern proof assistants and theorem provers for classical higher-order logic", such as the showcased tools Isabelle/HOL, Sledgehammer, and Nitpik. These conjectures form a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ▽, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Benzmüller, C.; Scott, D.S. (2019). Automating free logic in HOL, with an experimental application in category theory. ocrbilu.uni.lu/bitstream/10993/37593/1/article.pdf

Remark 0: We evaluate two formulas from the table below as signature axioms of existence (E_{ii}) in sheaf theory from Scott (1979).

Table 1. Stepwise evolution of Scott's axiom system for category theory from partial monoids. The axiom names are motivated as follows: S stands for strictness, E for existence, A for associativity, C for codomain, D for Domain. The free variables x, y, z range over the raw domain D. The quantifiers in Axioms Sets I and II are free logic quantifiers, that is, they range over the domain E of existing objects.

E(x·y)←(Ex∧Ey∧(∃z.z·z∼=z∧x·z∼=x∧z·y∼=y)) (4.7.1)

LET %p, q, r, s: E, x, y, z.

(%p&(q&r))<(((%p&q)&(%p&r))&(((%s& s)=(s&(q&s)))=((q&(s&r))=r))) ;
FFFF FFCT **FFFF FFFF** ; (4.1.7.2)

The left-to-right direction of existence axiom E_{ii} is implied.

E(x·y)→(Ex∧Ey∧(∃z.z·z∼=z∧x·z∼=x∧z·y∼=y)) (4.7.1)

Remark 4.7.1: If Eq. 4.1.7.2 as rendered with right-to-left direction is *not* tautologous, then there is no reason to expect a left-to-right direction to be implied as a theorem.

(%p&(q&r))>(((%p&q)&(%p&r))&(((%s& s)=(s&(q&s)))=((q&(s&r))=r))) ;
TTTT TTN**F** TTTT TTTT ; (4.7.2)

In two non-tautologous equations we refute Scott's existence axiom in sheaf theory. Therefore it does not follow that automating free logic is supported by using "modern proof assistants and theorem provers for classical higher-order logic", such as the showcased tools Isabelle/HOL, Sledgehammer, and Nitpik.